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Multivariate EWMA control charts for monitoring the variance-covariance matrix

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Abstract

We know that the exponentially weighted moving average (EWMA) control charts are sensitive to detecting relatively small shifts. Multivariate EWMA control charts are considered for monitoring of variance-covariance matrix when the distribution of process variables is multivariate normal. The performances of the proposed EWMA control charts are evaluated in term of average run length (ARL). The performance is investigated in three types of shifts in the variance-covariance matrix, that is, the variances, covariances, and variances and covariances are changed respectively. Numerical results show that all multivariate EWMA control charts considered in this paper are effective in detecting several kinds of shifts in the variance-covariance matrix.

Keywords: Average run length, multivariate exponentially weighted moving average control chart, variance-covariance matrix.

1. Introduction

A control chart is very useful in monitoring various production processes. There are many situations in which the simultaneous control of two or more related quality characteristics is necessary. To monitor the product quality in the multivariate production processes, it is more advantageous to use multivariate control charts rather than univariate control charts. Multivariate control charts have been interested in new research and remain an important area of research in statistical quality control.

Control charts are continuously monitoring the production process to detect quickly the changes that may produce any deterioration in the quality of the product. Control charts are becoming increasingly common for the quality of processes to be characterized by multiple variables.

The multivariate control charts of the Shewhart type was introduced by Hotelling (1947). This multivariate Shewhart chart was constructed based on Hotelling's T^2 statistic. Jackson (1959), and Ghare and Torgersen (1968) presented a multivariate Shewhart chart based on Hotelling's T^2 statistic. Other multivariate Shewhart charts are discussed by Alt (1984), Wierda (1994), and Lowry and Montgomery (1995).

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The exponentially weighted moving average (EWMA) control chart was introduced by Roberts (1959). It is well known that the EWMA control charts are much more effective than Shewhart-type charts for detecting small and moderate shifts in process parameters.

The EWMA charts have been developed to detect shifts in the process variability (Hunter, 1968; Robinson and Ho, 1978; Crowder, 1987, 1989; Lucas and Saccucci, 1990; Saccucci and Lucas, 1990). Sweet (1986) suggest using two EWMA control charts, one for detecting mean shifts and the other for detecting shifts of variances. MacGregor and Harris (1993) propose EWMA control charts for controlling vairance that also could be used for individual observations.

The development of multivariate EWMA control charts has concentrated on the problem of monitoring mean vector μ. A multivariate extension of the univariate control chart was studied by Lowry *et al.* (1992), Prabhu and Runger (1997), Reynolds and Kim (2005), and Lim and Cho (2008), Chang and Shin (2009), Cho (2010), and Im and Cho (2009). We study the multivariate EWMA control charts for monitoring the variance-covariance matrix Σ.

2. Notation and assumptions

In working with the multivariate normal distribution, it will be convenient to let σ represent the vector of standard deviations of the p variables. Let μ_0, Σ_0, and σ_0 represent the in-control values for μ, Σ, and σ, respectively. In practice, some of the in-control parameter values would need to be estimated during a Phase I period when process data are collected for purposes of parameter estimation. Here we consider control chart performance in Phase II under the simplifying assumption that the in-control parameter values are known.

Suppose that the process will be monitored by taking a sample of $n \geq p$ independent observation vectors at each sampling point, where the sampling points are d time units apart. Let X_{kij} represent observation j $(j = 1, 2, \cdots, n)$ for variable i $(i = 1, 2, \cdots, p)$ at sampling point k $(k = 1, 2, \cdots)$, and let the corresponding standardized observation be

$$Z_{kij} = (X_{kij} - \mu_{0i})/\sigma_{0i},$$

where μ_{0i} is the ith component of target mean vector μ_0, and σ_{0i} is the ith diagonal component of σ_0. Also let

$$Z_{kj} = (Z_{k1j}, Z_{k2j}, \cdots, Z_{kpj})^{'}, j = 1, 2, \cdots, n.$$

be the vector of standardized observations for observation vector j at sampling point k. Let Σ_Z be the covariance matrix of Z_{kj}, and let Σ_{Z0} be the in-control value of Σ_Z. The in-control distribution of Z_{kij} is standard normal, so Σ_{Z0} is also the in-control correlation matrix of the unstandardized observations.

When $n \geq p$, some control statistics used for monitoring Σ are functions of the sample estimates Σ_Z. At sampling point k, let $\widehat{\Sigma}_{Zk}$ be the maximum likelihood estimator of Σ_Z, where the $(i, i^{'})$ element of $\widehat{\Sigma}_{Zk}$ is $\sum_{j=1}^{n} Z_{kij} Z_{ki^{'}j}/n$.

We investigate a number of different control charts based on plotting multivariate EWMA control statistics.

3. Multivariate EWMA control charts

Hotelling (1947) proposed a Shewhart-type control chart for Σ using the statistic Y_k given by (3.1).

$$\sum_{j=1}^{n}(Z_{k1j}, Z_{k2j}, \cdots, Z_{kpj})\Sigma_{Z0}^{-1}(Z_{k1j}, Z_{k2j}, \cdots, Z_{kpj})' \\ = ntr(\widehat{\Sigma}_{Zk}\Sigma_{Z0}^{-1}) = Y_k \tag{3.1}$$

where Σ_0 is assumed to be known. When the process is in control, the distribution of the control statistic Y_k is chi-square with np degrees of freedom.

For constructing multivariate EWMA control charts for monitoring the variance-covariance matrix, we use the statistic Y_k given by (3.1). A multivariate EWMA control chart for monitoring Σ is based on the statistic

$$E_k = (1-\lambda)E_{k-1} + \lambda Y_k, \quad k = 1, 2, \cdots \tag{3.2}$$

where $E_0 = 0$ and λ is weighting or tuning parameter satisfying $0 < \lambda \leq 1$.

Therefore, a multivariate EWMA based on the statistic E_k signals whenever

$$E_k \geq h, \tag{3.3}$$

where a control limit h can be obtained to satisfy a specified in-control ARL. We can use a Markov chain approach, integral equations or simulations to get h value. If the process shifts from Σ_0, then it is necessary to use simulations to get ARL values.

4. Numerical performances and concluding remarks

In this section the following control procedures are compared on the basis of their ARL performances of the Multivariate EWMA control charts with control statistic given by (3.2). The performance of the control charts for monitoring the variance-covariance matrix depends on the value of Σ. The following types of shifts were considered :

(1) variances are changed and covariances are not changed,

(2) covariances are changed and variances are not changed,

(3) variances and covariances are simultaneously changed.

The ARL and h values in the multivariate EWMA control chart based on (3.2) were obtained by using 10,000 runs. Tables 4.1-4.9 give the h and ARL values for $p = 2,\ 4$, $\rho_0 = 0.5$, and $\lambda = 0.05, 0.1, 0.3$. The ARL at $\Sigma = \Sigma_0$ is approximately 800. The parameters h were obtained to give specified ARL when the process is in control.

When variances are changed, the values in the Tables 4.1-4.3 are h and ARL for $\lambda = 0.05, 0.1, 0.3$, respectively. The numbers in each cell from the row named $c = 1.21$ to the row named $c = 4.00$ in Tables 4.1-4.3 are ARLs when one, two,..., p variances are changed respectively. Standard deviations are changed from σ_0 to $\sigma = \sqrt{c}\sigma_0$, for $c = 1.21, 1.44, 1.69, 4$.

As shown in Tables 4.1-4.3, the multivariate EWMA control charts given by (3.2) for monitoring the variance-covariance matrix are effective in detecting only changes of variances in Σ.

When covariances are changed, the values in the Tables 4.4-4.6 are h and ARL for $\lambda = 0.05, 0.1, 0.3$, respectively. Covariances are changed from $\rho_0 = 0.5$ to $\rho = 0.45, 0.35, 0.25, 0.15, 0.05$. As shown in Tables 4.4-4.6, the multivariate EWMA control charts given by (3.2) for monitoring the variance-covariance matrix are effective in detecting only changes of covariances in Σ.

When variances and covariances are simultaneously changed, the values in the Tables 4.7-4.9 are h and ARL for $\lambda = 0.05, 0.1, 0.3$, respectively. The numbers in each cell from the row named $c = 1.21$ to the row named $c = 4.00$ in Tables 4.7-4.8 are ARLs when one, two,..., p variances and p covariances are simultaneously changed respectively. As shown in Tables 4.7-4.9, the multivariate EWMA control charts given by (3.2) for monitoring the variance-covariance matrix are also effective in detecting simultaneously changes of variances and covariances in Σ.

Tables 4.1-4.9 show that all multivariate EWMA control charts considered in this paper are effective in detecting several kinds of shifts in the variance-covariance matrix. But in the Tables 4.1-4.9, the ARL values are not much different for $\lambda = 0.05, 0.1, 0.3$.

Table 4.1 ARL for multivariate EWMA control charts when variances are changed ($\lambda = 0.05$)

$\rho_0 = 0.5$ $\lambda = 0.05$	$p = 2$		$p = 4$
	$n = 2$	$n = 4$	$n = 4$
	$h = 4.6986$	$h = 8.8775$	$h = 17.1282$
$c = 1$	800.0142	800.0830	800.0830
$c = 1.21$	347.7446 199.5582	286.0061 144.6772	395.3211 226.8075 141.1556 94.1885
$c = 1.44$	150.9045 70.6520	113.1614 43.5088	176.4608 71.4338 37.2813 22.5933
$c = 1.69$	75.3854 32.5111	49.9399 17.8798	83.8674 29.0189 12.5963 8.4832
$c = 4.00$	5.9735 2.8788	3.3754 1.6560	4.2764 1.8392 1.3169 1.1326

Table 4.2 ARL for multivariate EWMA control charts when variances are changed ($\lambda = 0.1$)

$\rho_0 = 0.5$ $\lambda = 0.1$	$p = 2$		$p = 4$
	$n = 2$	$n = 4$	$n = 4$
	$h = 5.3980$	$h = 9.7555$	$h = 18.2580$
$c = 1$	800.0830	800.0830	800.0830
			395.7915
$c = 1.21$	344.3180	282.0967	226.1035
	198.6709	147.5938	140.8239
			97.1350
			178.8615
$c = 1.44$	154.5788	110.5971	73.1143
	71.1631	42.9793	38.1706
			22.3568
			84.5061
$c = 1.69$	75.9184	50.6152	28.5394
	31.8922	17.8152	12.6162
			8.4572
			4.2206
$c = 4$	6.0710	3.3389	1.8601
	2.9111	1.65‘2	1.3156
			1.1300

Table 4.3 ARL for multivariate EWMA control charts when variances are changed ($\lambda = 0.3$)

$\rho_0 = 0.5$ $\lambda = 0.3$	$p = 2$		$p = 4$
	$n = 2$	$n = 4$	$n = 4$
	$h = 8.1903$	$h = 13.2662$	$h = 22.7615$
$c = 1$	800.0830	800.0830	800.0830
			392.5439
$c = 1.21$	339.7663	291.7363	224.0184
	198.8770	145.5450	140.7379
			96.0928
			179.7888
$c = 1.44$	152.9800	111.3255	71.9902
	71.0416	43.4007	37.9122
			22.8305
			83.5025
$c = 1.69$	75.9006	50.4200	28.4059
	32.4269	17.5431	12.2888
			8.4714
			4.2671
$c = 4.00$	6.0403	3.3626	1.8471
	2.9246	1.6549	1.3122
			1.1239

Table 4.4 ARL for multivariate EWMA control charts when covariances are changed ($\lambda = 0.05$)

$\rho_0 = 0.5$ $\lambda = 0.05$	$p = 2$		$p = 4$
	$n = 2$	$n = 4$	$n = 4$
	$h = 4.6986$	$h = 8.8775$	$h = 17.1282$
$\rho = 0.5$	800.0142	799.9887	800.1258
$\rho = 0.45$	589.9066	566.4876	377.0370
$\rho = 0.35$	316.6366	279.2700	110.4036
$\rho = 0.25$	176.6587	138.9008	42.7011
$\rho = 0.15$	108.1872	71.2605	20.9380
$\rho = 0.05$	72.6904	50.4704	12.0682

Table 4.5 ARL for multivariate EWMA controls chart when covariances are changed ($\lambda = 0.1$)

$\rho_0 = 0.5$, $\lambda = 0.1$	$p = 2$		$p = 4$
	$n = 2$	$n = 4$	$n = 4$
	$h = 5.3980$	$h = 9.7555$	$h = 17.1282$
$\rho = 0.5$	799.9645	799.9020	800.0820
$\rho = 0.45$	598.1236	565.7006	382.9879
$\rho = 0.35$	313.3510	274.7477	109.2216
$\rho = 0.25$	180.4640	142.7244	43.2916
$\rho = 0.15$	108.8055	81.1860	21.0628
$\rho = 0.05$	71.8636	50.4704	12.3642

Table 4.6 ARL for multivariate EWMA control charts when covariances are changed ($\lambda = 0.3$)

$\rho_0 = 0.5$, $\lambda = 0.3$	$p = 2$		$p = 4$
	$n = 2$	$n = 4$	$n = 4$
	$h = 8.1903$	$h = 13.2662$	$h = 22.7615$
$\rho = 0.5$	799.9889	799.9158	800.1380
$\rho = 0.45$	590.9664	560.3408	377.1021
$\rho = 0.35$	317.9438	273.9652	109.2003
$\rho = 0.25$	182.4872	140.1832	42.2236
$\rho = 0.15$	107.7375	79.4267	20.8749
$\rho = 0.05$	71.2405	50.1246	11.9735

Table 4.7 ARL for multivariate EWMA control charts when variances and covariances are changed ($\lambda = 0.05$)

$\rho_0 = 0.5$, $\lambda = 0.05$		$p = 2$		$p = 4$
		$n = 2$	$n = 4$	$n = 4$
		$h = 4.6986$	$h = 8.8775$	$h = 17.1282$
$c = 1$		800.0032	800.2011	800.2011
$c = 1.21$	$\rho = 0.4$	192.3537 116.5407	158.5621 84.5779	111.6488 70.2721 47.6992 34.7748
	$\rho = 0.3$	115.0910 74.9759	89.5070 51.3570	29.3045 29.3045 20.8874 16.1489
$c = 1.44$	$\rho = 0.4$	97.2219 46.2967	68.6802 28.9353	61.7443 29.0628 16.6873 10.8411
	$\rho = 0.3$	63.5825 32.3112	44.5883 19.4080	26.1292 14.2505 9.0122 6.2505
$c = 4.00$	$\rho = 0.4$	5.4124 2.5832	3.0649 1.5509	3.3569 1.5954 1.1966 1.0819
	$\rho = 0.3$	4.9336 2.4482	2.8274 1.4732	2.7296 1.4450 1.1533 1.0496

Table 4.8 ARL for multivariate EWMA control charts when variances and covariances are changed ($\lambda = 0.1$)

$\rho_0 = 0.5$ $\lambda = 0.1$		$p = 2$		$p = 4$
		$n = 2$	$n = 4$	$n = 4$
		$h = 5.3980$	$h = 9.7555$	$h = 18.2580$
$c = 1$		800.0032	800.2011	800.2011
$c = 1.21$	$\rho = 0.4$	206.0894 124.8640	162.0527 85.7969	110.6742 70.9878 48.5659 35.2827
	$\rho = 0.3$	124.8409 78.4954	89.2529 51.7762	42.5713 28.7698 21.4159 16.0013
$c = 1.44$	$\rho = 0.4$	103.8395 48.9727	69.0140 28.8839	62.2538 29.1244 16.7269 10.6993
	$\rho = 0.3$	68.1848 34.4964	43.3188 19.5606	26.8357 14.4030 8.9106 6.2212
$c = 4.00$	$\rho = 0.4$	5.4911 2.6538	3.0632 1.5453	3.3860 1.6314 1.2070 1.0785
	$\rho = 0.3$	5.0974 2.4886	2.7951 1.4800	2.7121 1.4393 1.1426 1.0527

Table 4.9 ARL for multivariate EWMA control charts when variances and covariances are changed ($\lambda = 0.3$)

$\rho_0 = 0.5$ $\lambda = 0.3$		$p = 2$		$p = 4$
		$n = 2$	$n = 4$	$n = 4$
		$h = 8.1903$	$h = 13.2662$	$h = 22.7615$
$c = 1$		800.0032	800.2011	800.2011
$c = 1.21$	$\rho = 0.4$	207.0155 126.4167	159.2849 83.9290	112.4995 70.5829 48.3336 34.8184
	$\rho = 0.3$	125.2643 78.5746	89.2553 51.9738	41.8801 28.9826 20.8339 15.9049
$c = 1.44$	$\rho = 0.4$	102.0828 48.7680	68.6715 23.2388	26.2645 14.3416 8.8931 6.1815
	$\rho = 0.3$	69.2215 33.7246	44.1189 19.4736	26.2645 14.3416 8.8931 6.1815
$c = 4.00$	$\rho = 0.4$	5.5649 2.6920	3.0485 1.5528	3.3298 1.6057 1.2046 1.0745
	$\rho = 0.3$	5.0342 2.4331	2.8289 1.4714	2.7644 1.4469 1.1445 1.0510

References

Alt, F. B. (1984). Multivariate control charts. In *Encyclopedia of Statistical Sciences*, edited by S. Kotz and N. L. Johnson, John Wiley, New York.

Chang, D. J. and Shin, J. K. (2009). Variable sampling interval control charts for variance-covariance matrix. *Journal of the Korean Data & Information Science Society*, **21,** 999-1008.

Cho, G. Y. (2010). Multivariate Shewhart control charts with variable sampling intervals. *Journal of the Korean Data & Information Science Society*, **21,** 999-1008.

Crowder, S. V (1987). A simple method for studying run length distributions of exponentially weighted moving average control charts. *Technometircs*, **29**, 401-407.

Crowder, S. V (1989). Design of exponentially weighted moving average schemes. *Journal of Quality Technology*, **21**, 155-162.

Ghare, P. H. and Torgerson, P. E. (1968). The multicharacteristic control chart. *Journal of Industrial Engineering*, **19**, 269-272.

Hotelling, H. (1947). *Multivariate quality control, techniques of statistical analysis*, McGraw-Hill, New York, 111-184.

Hunter, J. S. (1968). The exponentially weighted moving average. *Journal of Quality Technology*, **18**, 203-210.

Im, C. D. and Cho, G. Y. (2009). Multiparameter CUSUM charts with variable sampling intervals. *Journal of the Korean Data & Information Science Society*, **20,** 593-599.

Jackson, J. S. (1959). Quality control methods for several related variables. *Technometrics*, **1**, 359-377.

Lim, C and Cho, G. Y. (2008). A new EWMA control chart for monitoring the covariance matrix of bivariate processes. *Journal of the Korean Data & Information Science Society*, **19**, 677-683.

Lowry, C. A. and Montgomery, D. C. (1995). A review of multivariate control charts. *IIE Transactions*, **27**, 800-810.

Lowry, C. A., Woodall, W. H., Champ, C. W. and Rigdon, S. E. (1992). A multivariate exponentially weighted moving average control chart. *Technometrics*, **34**, 46-53.

Lucas, J. M. and Saccucci, M. S. (1990). Exponentially weighted moving average control schemes: Properties and enhancements. *Technometrics*, **32**, 1-12.

MacGregor, J. F. and Jarris, T. J. (1993). The exponentially weighted moving variance. *Journal of Quality Technology*, **25**, 106-118.

Reynolds, M. R., Jr. and Kim, G. (2005). Multivariate monitoring of the mean vector using sequential sampling. *Journal of Quality Technology*, **37**, 149-162.

Roberts, S. W. (1959). Control chart tests based on geometric moving averages. *Technometrics*, **1**, 239-250.

Robinson, P. B. and Ho, T. Y. (1978). Average run length of geometric moving average charts by numerical methods. *Technometrics*, **20**, 85-93.

Prabhu, S. S. and Runger, G. C. (1997). Designing a multivariate EWMA control chart. *Journal of Quality Technology*, **29**, 8-15.

Saccucci, M. S. and Lucas, J. M. (1990). Average run lengths for exponentially weighted moving average control schemes using the Markov chain approach. *Jounal of Quality Technology*, **22,** 154-162.

Sweet, A. L. (1986). Control chart using coupled exponentially weighted moving averages. *IIE Transactions* **18**, 26-33.

Wierda, S. J. (1994). multivariate statistical process control - recent results and directions for future research. *Statistica Neerlandica*, **48,** 147-168.