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Hidden truncation circular normal distribution

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Abstract

Many circular distributions are known to be not only asymmetric but also bimodal. Hidden truncation method of generating asymmetric distribution is applied to a bivariate circular distribution to generate an asymmetric circular distribution. While many other existing asymmetric circular distributions can only model an asymmetric data, this new circular model has great flexibility in terms of asymmetry and bi-modality. Some properties of the new model, such as the trigonometric moment generating function, and asymptotic inference about the truncation parameter are presented. Simulation and real data examples are provided at the end to demonstrate the utility of the novel distribution.

Keywords: Asymmetric distribution, circular distribution, circular statistics, conditional distribution, hidden truncation, trigonometric moments and von Mises distribution.

1. Introduction

Circular random variables are found in various areas of research such as biology, medicine, just to name a few. Because of the periodic nature of a circular variable, it is necessary to use a circular distribution to model a circular variable. For various types of circular distribution, including von Mises (VM) or circular normal distribution, the readers can refer to Jammalamadaka and SenGupta (2001). The circular normal distribution, which is symmetric, has been mainly used to model a circular random variable. However, circular distributions are rarely symmetric, i.e. they are usually asymmetric and even multi-modal (Kim, 2011). Therefore, the VM distribution is not suitable to model such a data set. In fact, this is also the case in linear statistical analysis (Arnold and Beaver, 2000; Azzalini, 1986) that the normal distribution is often not suitable. One way to model an asymmetric and multimodal distribution is using a mixture of von Mises distributions (Batschelet, 1981). Another model suitable for an aymmetric and/or multimodal circular distribution is based on nonnegative trigonometric sums (Fernandez-Duran, 2004). Other existing asymmetric and/or bimodal circular distributions are appeared in Jammadamalaka and Kozubowski (2004), Gatto and Jammalamadaka (2007), Umbach and Jammalamadaka (2009) and Kim

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(2011). Jammadamalaka and Kozubowski (2004) propose a model that is the result of wrapping the exponential and Laplace distributions. Gatto and Jammalamadaka (2007) propose a generalization of the von Mises distribution, which is an extension of the von Mises distribution. In Umbach and Jammalamadaka (2009), a method of building asymmetry into circular distributions is discussed. Kim (2011) proposes an exponential family of distributions as a new family of circular distributions, which is absolutely suitable to model any shape of circular distributions. In this paper, a new asymmetric and/or bimodal distribution is proposed using the hidden truncation method, which is illustrated in the following. For other researches on circular variables, such as regression methods, our readers can refer to Kim (2011) and SenGupta and Ugwuowo (2006).

Asymmetric distributions can occur in situations when the observed variables represent a sample that has been truncated with respect to some hidden (or available) covariable (Arnold and Beaver, 2000). Therefore, one can generate an asymmetric density, by beginning with a bivariate density, then obtaining the conditional density given that the covariable is truncated at some point. This method is called the hidden truncation method, and it is well known that the method generates an asymmetric distribution when applied to linear bivariate distributions. In this paper, it is one of our primary interest to show that the hidden truncation method also works for a circular bivariate distribution in generating an asymmetric circular distribution. It is shown that the new circular distribution allows for great flexibility in terms of asymmetry and bimodality.

We apply the hidden truncation method to the circular normal conditionals (CNC) density, and show that the resulting distribution, called the hidden truncation circular normal (HTCN) distribution, is asymmetric. Simulation results have shown that the HTCN distribution can also model a circular distribution that has two modes. After applying the method to several other bivariate circular distributions, such as wrapped bivariate normal distribution, wrapped bivariate cauchy distribution and bivariate cauchy distribution, it is conjectured that the hidden truncation method applied to other bivariate circular distributions produces asymmetric distributions. Trigonometric moment generating function of the new model and asymptotic confidence interval and hypothesis testing for the truncation parameter are presented. Then, the likelihood ratio test is applied to Fisher's bird's nest data (Fisher, 1993) to assess the goodness of fit of the novel distribution. However, in order to invoke the standard asymptotic distribution theory, certain regularity conditions have to be met. Since the null value is a boundary point of the feasible region, we have used a mixture of chi-squared distributions as the asymptotic distribution of test statistics as discussed in Self and Liang (1987). All the numerical computations in this paper are performed using 'optim' function routine R.

Throughout this paper all circular random variables are assumed to take on values in the interval $(-\pi, \pi]$, and as the consequence, the corresponding densities are positive in this interval and zero elsewhere. Density functions in this paper will not include explicit reference to the support of the corresponding densities.

1.1. Hidden Truncation Circular Normal Model

In this section, it is shown how the hidden truncation method is applied to a bivariate circular distribution to produce an asymmetric circular distribution. The circular normal

conditionals (CNC) density is given by

$$f(\theta,\phi) = C \cdot \exp\{a_1\cos\theta + a_2\sin\theta + a_3\cos\phi + a_4\sin\phi + a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi + a_7\sin\theta\cos\phi + a_8\sin\theta\sin\phi\}, \quad (1.1)$$

where C is the normalizing constant, and $a_i \in R$ for i=1,.., 8. Its conditional distributions are given as circular normal distributions, from which the name came. The marginal density of Φ, after integrating out (1.1) with respect to Θ, is given by

$$f(\phi) \propto 2\pi I_0[h(\phi)]\exp\{a_3\cos\phi + a_4\sin\phi\}, \quad (1.2)$$

where $\mathrm{I}_0(h(\phi))$, the modified Bessel function of the first kind of order 0, is given by

$$2\pi\mathrm{I}_0(h(\phi)) = \int_{-\pi}^{\pi}\exp[h(\phi)\cos(\theta - g(\phi))]d\theta =$$

$$\int_{-\pi}^{\pi}\exp[(a_1 + a_5\cos\phi + a_6\sin\phi)\cos\theta + (a_2 + a_7\cos\phi + a_8\sin\phi)\sin\theta]d\theta,$$

and $h(\phi)$ and $g(\phi)$ are given by

$$h(\phi) = \sqrt{[a_1 + a_5\cos\phi + a_6\sin\phi]^2 + [a_2 + a_7\cos\phi + a_8\sin\phi]^2}$$

$$g(\phi) = \arctan\left\{\frac{[a_2 + a_7\cos\phi + a_8\sin\phi]}{[a_1 + a_5\cos\phi + a_6\sin\phi]}\right\}.$$

Next, $f(\theta, \phi > b)$ and $P(\Phi > b)$ for some $b \in (-\pi, \pi)$ are respectively obtained by integrating (1.1) and (1.2) from b to π. Then, the conditional density of Θ given that the distribution of Φ is left-truncated at b is given by

$$f(\theta|\phi > b) = \frac{f(\theta,\phi > b)}{P(\Phi > b)} = \frac{\exp[a_1\cos\theta + a_2\sin\theta]\int_b\exp\left[h_1(\theta)\cos(\phi - h_2(\theta))\right]d\phi}{\int_b\int\exp\left[a_1\cos\theta + a_2\sin\theta + h_1(\theta)\cos(\phi - h_2(\theta))\right]d\theta d\phi}, \quad (1.3)$$

where

$$[h_1(\theta)]^2 = (a_3 + a_5\cos\theta + a_7\sin\theta)^2 + (a_4 + a_6\cos\theta + a_8\sin\theta)^2,$$

$$h_2(\theta) = \arctan\left\{\frac{a_4 + a_6\cos\theta + a_8\sin\theta}{a_3 + a_5\cos\theta + a_7\sin\theta}\right\}.$$

The density in (1.3) is called the HTCN distribution, analogous to the hidden truncation normal density that is obtained by applying the method to the bivariate normal distribution (Arnold and Beaver, 2006). Simulation results show that the HTCN distribution is suitable to model an asymmetric circular data, and even a bimodal circular data in some situations. Various shapes of the HTCN density with $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ ={1,1,0,0,-1,-1,1,-1} and $b = \{-\pi, -\frac{\pi}{3}, \frac{\pi}{8}, \frac{\pi}{4}\}$, are shown in Figure 1.1. An extensive sampling of different parameter values has failed to turn up any distribution with more than 2 modes. It is conjectured that only unimodal and bimodal forms of the densities exist. Some of the properties of the HTCN density are as shown below.

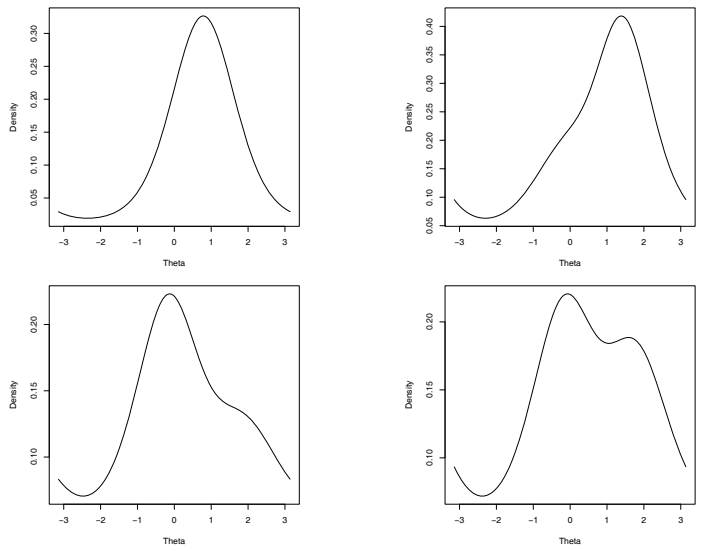

Figure 1.1 Density plots of the hidden truncation circular normal (HTCN) densities with $b = -\pi, -\frac{\pi}{3}, \frac{\pi}{8}, \frac{\pi}{4}$ beginning from the top left and in the clockwise direction

Proposition 1.1 The trigonometric moment generating function is given by

$$E(\exp(ip\theta)) \propto \int_b \exp(iph_2(\phi)) \frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))} d\phi, \quad p = \pm 1, \ldots.$$

Proof: Trigonometric moment generating function for the HTCN distribution is derived as below.

$$\begin{aligned} E(\exp(ip\theta)) &\propto \int_b \int [\cos p\theta + ip \sin\theta] \exp[a_3 \cos\theta + a_4 \sin\theta + a_5 \cos\phi \cos\theta] \\ &\quad \times \exp[a_6 \cos\phi \sin\theta + a_7 \sin\phi \cos\theta + a_8 \sin\phi \sin\theta] d\theta d\phi \\ &\propto \int_b \int [\cos p\theta + ip \sin\theta] \exp[(a_3 + a_5 \cos\phi + a_7 \sin\phi) \cos\theta] \\ &\quad \times \exp[(a_4 + a_6 \cos\phi + a_8 \sin\phi) \sin\theta] d\theta d\phi \\ &\propto \int_b \int [\cos p\theta + ip \sin\theta] \exp[h_1(\phi) \cos(\theta - h_2(\phi))] d\theta d\phi \\ &\propto \int_b \exp[iph_2(\phi)] \frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))} d\phi. \end{aligned}$$

□

Corollary 1.1 The mean direction of HTCN variable is given by

$$\mu = \arctan \left\{ \frac{\int_b \sin h_2(\phi) \frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))} d\phi}{\int_b \cos h_2(\phi) \frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))} d\phi} \right\}.$$

The circular variance of the HTCN distribution is given by

$$E(1-\cos(\theta-\mu)) = 1 - \int_b \cos(\mu - h_2(\phi))\frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))}d\phi.$$

Proof: From the trigonometric moment generating function for the HTCN, we get the first sine and cosine moments as shown below.

$$\begin{aligned} E(\exp(i\theta)) &\propto \int_b \exp[ih_2(\phi)]\frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))}d\phi \propto \int_b (\cos h_2(\phi) + i\sin h_2(\phi))\frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))}d\phi \\ &= \int_b \sin h_2(\phi)\frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))}d\phi + \int_b \cos h_2(\phi)\frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))}d\phi \propto E(\cos\theta) + iE(\sin\theta). \end{aligned}$$

Then, we get the mean direction of Φ as follows.

$$\mu = \arctan\left\{\frac{E(\sin\theta)}{E(\cos\theta)}\right\} = \arctan\left\{\frac{\int_b \sin h_2(\phi)\frac{\mathrm{I}_1(\kappa(\phi))}{\mathrm{I}_0(\kappa(\phi))}d\phi}{\int_b \cos h_2(\phi)\frac{\mathrm{I}_1(\kappa(\phi))}{\mathrm{I}_0(\kappa(\phi))}d\phi}\right\}.$$

Next, the circular variance is derived as follows.

$$\begin{aligned} E(\cos(\theta-\mu)) &\propto \int_b\int \cos(\theta-\mu)\exp[a_3\cos\theta + a_4\sin\theta + a_5\cos\phi\cos\theta] \\ &\quad \times \exp[a_6\cos\phi\sin\theta] + a_7\sin\phi\cos\theta + a_8\sin\phi\sin\theta]d\theta d\phi \\ &= \int_b\int \cos(\theta-\mu)\exp[(a_3 + a_5\cos\phi_7\sin\phi)\cos\theta] \\ &\quad \times \exp[(a_4 + a_6\cos\phi + a_8\sin\phi)\sin\theta]d\theta d\phi \\ &= \int_b\int \cos(\theta-\mu)\exp[h_1(\phi)\cos(\theta - h_2(\phi))]d\theta d\phi \\ &\propto \int_b \cos(\mu - h_2(\phi))\frac{\mathrm{I}_1(h_1(\phi))}{\mathrm{I}_0(h_1(\phi))}d\phi. \end{aligned}$$

□

Suppose we observe n circular data points $\theta_1, \ldots, \theta_n$. The log likelihood of the HTCN distribution is given by

$$\begin{aligned} \log L[\underline{a}, b] &= -n\log C(\underline{a}, b) \\ +a_1\sum_1^n \cos\theta_i + a_2\sum_1^n \sin\theta_i + \sum_1^n \log\left\{\int_b \exp\left[h_1(\theta_i)\cos(\phi - h_2(\theta_i))\right]d\phi\right\}, \end{aligned} \tag{1.4}$$

where $\underline{a} = (a_1, a_2, a_5, a_6, a_7, a_8)'$ and $C(\underline{a}, b)$ is the denominator in (1.3). The first order equations are obtained by using the Leibnitz's rule of differentiation of integrals (Casella and Berger, 2001) and the bounded convergence theorem (Royden, 1988). The MLEs are the solutions of the first order equations, and they can be solved numerically. The information matrix, $I(\underline{a}, b)$, is the inverse of negative of the Hessian matrix of (1.4). For example, its

third diagonal element is given by the inverse of

$$I[3,3] = -E\left\{\frac{\partial^2 \log L(\underline{a},b)}{\partial a_3 \partial a_3}\right\} = -\frac{\partial}{\partial a_3}\left\{\frac{n}{C(\underline{a},b)}\frac{\partial C(\underline{a},b)}{\partial a_3}\right\}$$
$$+\sum_1^n \frac{\partial}{\partial a_3}\left\{\frac{1}{\int_b \exp\left[h_1(b)\cos(\phi - h_2(b))\right]d\phi}\right\}$$
$$\times\left\{\int_b^\pi \exp\left[h_1(\theta_i)\cos(\phi - h_2(\theta_i))\right]\frac{\partial h_1(\theta_i)\cos(\phi - h_2(\theta_i))}{\partial a_3}d\phi\right\}.$$

The regularity conditions for the asymptotic normality of the estimated parameters are met for the likelihood function in (1.4) since the density is an analytic function of a (bounded) circular variable. The asymptotic variance covariance matrix of $(\hat{\underline{a}},\hat{b})'$ is given by $I(\underline{a},b)^{-1}$. So, asymptotically, we have

$$\sqrt{n}((\hat{\underline{a}},\hat{b})' - (\underline{a},b)') \sim N(0, I(\underline{a},b)^{-1}). \tag{1.5}$$

1.2. Asymptotic inference about the truncation parameter

While the skewness of the hidden truncation normal density is determined by the truncated value only, it is determined not only by the truncated value, but also by the signs of double trigonometric terms in the HTCN density. After an extensive sampling of different parameter values, it is also conjectured that 5 parameters, b, a_5, a_6, a_7 and a_8, are related in determining the presence of skewness of the density. If a_5, a_6, a_7 and a_8 are zeros, (1.3) becomes a von Mises density, i.e. symmetric. When there is no truncation, i.e. $b = -\pi$, the density is also a circular normal, i.e. symmetric regardless of the values of the other 4 parameters. When $b > -\pi$, it is asymmetric except when $b = 0$. When $b = 0$, it becomes symmetric bimodal. When $-\pi < b < 0$ and $0 < b < \pi$, the direction of skewness is opposite to each other. For this reason, a test of asymmetry or bimodality of the density can be provided by determining whether there is a truncation or not.

Suppose n circular data points $\theta_1, ..., \theta_n$ represent a sample from a HTCN distribution. Unless all of a_5, a_6, a_7 and a_8 are zeros, as mentioned in the above, the distribution is asymmetric only when $b \neq -\pi$. Using the result in (1.5), asymptotic $100(1-\alpha)\%$ confidence interval is given by

$$[\max(\hat{b} - z_{\frac{\alpha}{2}}\sqrt{I^{-1}[9,9]}, -\pi), \min(\hat{b} - z_{1-\frac{\alpha}{2}}\sqrt{I^{-1}[9,9]}, \pi)],$$

where $z_{\frac{\alpha}{2}}$ represents the value in the standard normal distribution whose right tail area is equal to $\frac{\alpha}{2}$. Next, suppose a hypothesis testing for the truncation parameter at α significant level is performed. The null and alternative hypotheses are given by

$$H_0 : \ b = -\pi \quad \text{vs.} \quad H_1 : \ \pi > b > -\pi,$$

and the test statistic is given by

$$\frac{\hat{b}}{\sqrt{I^{-1}[9,9]}}.$$

The critical value is z_α. Interval estimation and hypothesis testing for the other parameters in (1.3) can be established likewise.

2. Simulation example

In the following, it is shown that the distribution of simulated values from a HTCN density are asymmetric and/or bimodal. Since the marginal distribution is not in a form of a familiar univariate circular distribution, the rejection method is employed to generate values of Φ. ϕ's are generated one by one until a value greater than $\frac{\pi}{4}$ is observed, where $\frac{\pi}{4}$ is the chosen hidden truncation point. Once a value for Φ greater than $\frac{\pi}{4}$ is observed, it is proceeded to sample a value from the conditional vonMises distribution, which is given by

$$f(\theta|\phi) = \frac{\exp\left[h_1(\phi)\cos(\theta - h_2(\phi))\right]}{\int \exp\left[h_1(\phi)\cos(\theta - h_2(\phi))\right] d\theta}, \tag{2.1}$$

where $h_1(\phi)$ and $h_2(\phi)$ are the same as in (1.3). In such a way, 38 observations of Θ are collected, which represent a sample from the HTCN density shown in (1.3), with the values of parameters $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$'= (1,1.2,2,2.1,3,-1,1,3)'. In Figure 2.1, simulated data sets are plotted using the rose diagram. They evidently illustrate that the hidden truncation method produces an asymmetric and/or a bimodal circular distribution.

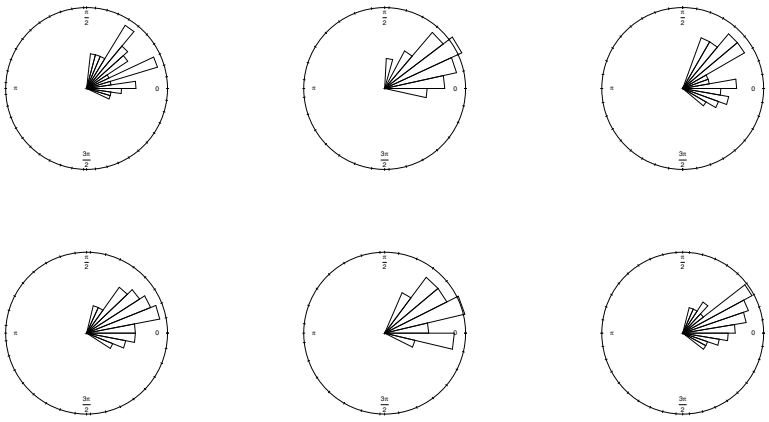

Figure 2.1 Rose diagrams of samples from the HTCN distribution

3. Real data example

In this section, we illustrate the goodness of fit of the new distribution using the real life example found in Fisher (1993). The data set refers to measurements on the orientation of the nests of 50 noisy scrub birds along the bank of a creek bed. In Figure 3.1, the

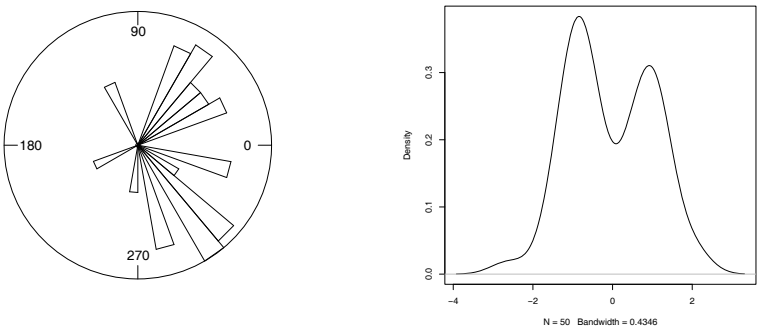

Figure 3.1 Plots of 50 orientations of the nests of the noisy scrub birds

smoothed histogram and the rose diagram of the 50 orientations exhibit that the distribution is asymmetric and bimodal. The likelihood ratio test is applied to the data set in order to compare the goodness of fit between two models: one assuming some truncation, i.e. $-\pi < b < \pi$, and the other with no truncation, i.e. $b = -\pi$, where the latter represents the von Mises distribution that is shown in (2.1). Since the null value in our test, i.e. $b = -\pi$, is a boundary point of the feasible region of the parameter space, instead of invoking the standard asymptotic distribution theory, we consider a mixture of chi-squared distribution as the asymptotic distribution of test statistic, which is the special case 5 of Theorem 3 (Self and Liang,1987) that is a generalization of the results found Chernoff (1954). Here, we write the special case 5 of Theorem 3. Readers can find more special cases in Self and Liang (1987) and a proof in Chernoff (1954) Let λ_n denote the likelihood ratio statistic. The asymptotic distribution of -2ln λ_n is given by a 50:50 mixture of χ^2_0 and χ^2_1, where χ^2_0 has a point mass at 0, i.e. $\chi^2_0 \equiv 0$ (Self and Liang, 1987).

The maximum likelihood estimation result for both HTCN densities without truncation ($b = -\pi$) and with some truncation ($-\pi < b < \pi$) are shown in Table 3.1. Our value of the likelihood ratio test statistic for the hypotheses of no truncation versus some truncation is given by

$$-2 \ln \lambda_n = -2 \times (106.9 - 165.5) = 117.2,$$

which is larger than $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \chi^2_{0.001,1} = \frac{1}{2} \cdot 10.83 = 5.42$. Thus, it is concluded that the truncation parameter estimate $\hat{b}(= -1.034)$ is statistically highly significantly different from $-\pi$ at the 0.001 level, providing strong evidence that the new distribution fits much better than a von Mises distribution for the data set in this example, which is asymmetric and bimodal.

Table 3.1 ML estimates for 50 orientations of bird's nest

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8		LogL
$b = -\pi$	-7.4	6.1	3.6	-3.2	-1.7	3.9	-0.5	-0.3		165.5
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	b	LogL
$-\pi < b < \pi$	1.4	1.8	0.8	-2.1	2.0	4.3	-5.1	-5.1	-1.9	106.9

4. Conclusion

In this paper, the hidden truncation method is applied to the circular normal conditionals density in order to propose an asymmetric density called the HTCN. It is shown that the new density is suitable to model asymmetric and/or bimodal distribution, while many asymmetric circular distributions can only model an asymmetric distribution. Using simulations and real data example, the utility of the novel distribution has been demonstrated in this paper. After trying several other bivariate circular distributions, it is conjectured that the hidden truncation method applied to other bivariate circular distributions produces asymmetric circular distributions. Although the new model provides as much flexibility as the generalized von-Mises or few other asymmetric/bimodal distributions, it is concluded that these distributions are typically not practical due to their large number of parameters and the complexity involved. Nevertheless, it is feasible to estimate the models, as demonstrated in this paper, provided that adequate computational facilities are available to handle the appropriate numerical approximation methods.

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