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Some Fixed Point Theorems on G-Expansive Mappings

Sushanta Kumar Mohanta
Department of Mathematics, West Bengal State University, Barasat, 24 Parganas (North), West Bengal, Kolkata 700126, India
e-mail : smwbes@yahoo.in

Abstract. In this paper, we have studied some fixed point theorems on complete G-metric spaces.

1. Introduction

The study of metric fixed point theory is very actual today due to enormous applications in many important areas such as mathematical economics, operation research, and approximation theory. In 2006, Mustafa and Sims introduced a new concept of generalized metric space called G-metric space [3]. Recently, Mustafa et.al. $[4-8]$ and Shatanawi [11] studied many fixed point theorems for mappings satisfying various contractive conditions on complete G-metric spaces. In this paper, we prove some fixed point theorems on complete G-metric spaces. Moreover, strength of hypothesis made in Theorem 2.1 [7] has been weighed through an illustrative example.

2. Definitions and preliminaries

In this section, we present some basic definitions and results for G-metric spaces that will be used in the sequel.

Definition 2.1([3]). Let X be a non empty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:
$(G_1)\, G(x, y, z) = 0 \; if \; x = y = z,$
$(G_2)\, 0 < G(x, x, y), for \; all \; x, y \in X, \; with \; x \neq y,$
$(G_3)\, G(x, x, y) \leq G(x, y, z), for \; all \; x, y, z \in X, \; with \; z \neq y,$
$(G_4)\, G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots (symmetry \; in \; all \; three \; variables),$
$(G_5)\, G(x, y, z) \leq G(x, a, a)+G(a, y, z), for \; all \; x, y, z, a \in X, \; (rectangle \; inequality).$

Then the function G is called a generalized metric, or, more specifically a G-

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metric on X, and the pair (X, G) is called a G-metric space.

Example 2.2([3]). Let R be the set of all real numbers. Define $G : R\times R\times R \to R^+$ by

$$G(x, y, z) =| x - y | + | y - z | + | z - x |, \ for \ all \ x, y, z \in R.$$

Then it is clear that (R, G) is a G-metric space.

Proposition 2.3([3]). *Let (X, G) be a G-metric space. Then for any $x, y, z,$ and $a \in X$, it follows that*

(1) $if \ G(x, y, z) = 0 \ then \ x = y = z,$

(2) $G(x, y, z) \le G(x, x, y) + G(x, x, z),$

(3) $G(x, y, y) \le 2G(y, x, x),$

(4) $G(x, y, z) \le G(x, a, z) + G(a, y, z),$

(5) $G(x, y, z) \le \frac{2}{3}\left(G(x, y, a) + G(x, a, z) + G(a, y, z)\right),$

(6) $G(x, y, z) \le G(x, a, a) + G(y, a, a) + G(z, a, a).$

Definition 2.4([3]). Let (X, G) be a G-metric space, let (x_n) be a sequence of points of X, we say that (x_n) is G-convergent to x if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $n_0 \in N$ such that $G(x, x_n, x_m) < \epsilon, \ for \ all \ n, m \ge n_0$. We call x as the limit of the sequence (x_n) and write $x_n \xrightarrow{(G)} x$.

Proposition 2.5([3]). *Let (X, G) be a G-metric space, then the following are equivalent.*

(1) $(x_n) \ is \ G - convergent \ to \ x.$

(2) $G(x_n, x_n, x) \to 0, \ as \ n \to \infty.$

(3) $G(x_n, x, x) \to 0, \ as \ n \to \infty.$

(4) $G(x_n, x_m, x) \to 0, \ as \ n, m \to \infty.$

Definition 2.6([3]). Let (X, G) be a G-metric space, a sequence (x_n) is called G-Cauchy if given $\epsilon > 0$, there is $n_0 \in N$ such that $G(x_n, x_m, x_l) < \epsilon, \ for \ all \ n, m, l \ge n_0$; that is, if $G(x_n, x_m, x_l) \to 0 \ as \ n, m, l \to \infty$.

Proposition 2.7([3]). *In a G-metric space (X, G), the following are equivalent.*

(1) $The \ sequence \ (x_n) \ is \ G - Cauchy.$

(2) $For \ every \ \epsilon > 0, \ there \ exists \ n_0 \in N \ such \ that$

$$G(x_n, x_m, x_m) < \epsilon, \ for \ all \ n, m \ge n_0.$$

Definition 2.8([3]). Let (X, G) and $(X^{'}, G^{'})$ be G-metric spaces and let $f : (X, G) \to (X^{'}, G^{'})$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; $G(a, x, y) < \delta$ *implies* $G^{'}(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 2.9([3]). *Let (X, G) and $(X^{'}, G^{'})$ be G-metric spaces, then a function $f : X \to X^{'}$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 2.10([3]). *Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 2.11([3]). *Every G-metric space (X, G) will define a metric space (X, d_G) by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad for\ all\ x, y \in X.$$

Definition 2.12([3]). A G-metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.13([3]). *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Definition 2.14([3]). A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$, and called non-symmetric if it is not symmetric.

Theorem 2.15([7]). *Let (X, G) be a complete G-metric space and let $T : X \longrightarrow X$ be an onto mapping satisfying the following condition for all $x, y, z \in X$,*

$$G(T(x), T(y), T(z)) \geq a\, G(x, y, z) + b\, G(x, x, T(x)) + c\, G(y, y, T(y)) + d\, G(z, z, T(z))$$

where $a + b + c + d > 1$ and $b + c < 1$. Then T has a fixed point.

3. Main results

We begin with the following definition.

Definition 3.1([7]). Let (X, G) be a G-metric space and T be a self mapping on X. Then T is called G-expansive mapping if there exists a constant $c > 1$ such that for all $x, y, z \in X$, we have

$$G(T(x), T(y), T(z)) \geq c\, G(x, y, z).$$

Before presenting our results, we state the following theorem.

Theorem 3.2([7]). *Let (X, G) be a complete G-metric space. If there exists a constant $c > 1$ and a surjective self mapping T on X, such that for all $x, y, z \in X$*

$$G(T(x), T(y), T(z)) \geq c\, G(x, y, z), \tag{3.1}$$

then T has a unique fixed point.

The following example shows that T is onto in Theorem 3.2 can not be relaxed.

Example 3.3. Let R be the set of all real numbers. Define $G : R \times R \times R \to R^+$ by

$$G(x, y, z) = \mid x - y \mid + \mid y - z \mid + \mid z - x \mid, \text{ for all } x, y, z \in R.$$

Then it is clear that (R, G) is a G-metric space.

Let $T : R \longrightarrow R$ be a mapping defined as follows:

$$T(x) = \begin{cases} 2x - 1 & \text{for } x \leq 0, \\ 2x + 1 & \text{for } x > 0. \end{cases}$$

We have

$$G(T(x), T(y), T(z)) = \mid T(x) - T(y) \mid + \mid T(y) - T(z) \mid + \mid T(z) - T(x) \mid, \tag{3.2}$$

for all $x, y, z \in R$.

Now,

$$\begin{aligned} \mid T(x) - T(y) \mid &= \mid 2x - 1 - 2y - 1 \mid \text{ for } x \leq 0 \text{ and } y > 0 \\ &= 2 \mid y + 1 - x \mid \text{ for } x \leq 0 \text{ and } y > 0 \\ &\geq 2 \mid y - x \mid \text{ for } x \leq 0 \text{ and } y > 0 \\ &= 2 \mid x - y \mid \text{ for } x \leq 0 \text{ and } y > 0. \end{aligned}$$

Also,

$$\mid T(x) - T(y) \mid = 2 \mid x - y \mid \text{ for } x, y \leq 0$$

and

$$\mid T(x) - T(y) \mid = 2 \mid x - y \mid \text{ for } x, y > 0.$$

Thus,

$$\mid T(x) - T(y) \mid \geq 2 \mid x - y \mid \text{ for } x, y \in R.$$

The argument similar to that used above establishes

$$\mid T(y) - T(z) \mid \geq 2 \mid y - z \mid \text{ for } y, z \in R$$

and

$$\mid T(z) - T(x) \mid \geq 2 \mid z - x \mid \text{ for } z, x \in R.$$

So, (3.2) becomes

$$G(T(x), T(y), T(z)) \geq 2 \mid x-y \mid +2 \mid y-z \mid +2 \mid z-x \mid = 2\,G(x,y,z),$$

for all $x, y, z \in R$, which implies that T satisfies Condition (3.1) in Theorem 3.2. Again, for all $x, y \in R$,

$$\begin{aligned} d_G(x,y) &= G(x,y,y) + G(y,x,x) \\ &= 2 \mid x-y \mid +2 \mid x-y \mid \\ &= 4 \mid x-y \mid . \end{aligned}$$

Clearly, (X, d_G) is complete and hence by Proposition 2.13, (X, G) is a complete G-metric space.
But T does not have a fixed point, since it is not onto.

Theorem 3.4. *Let (X, G) be a complete G-metric space, and let $T : X \longrightarrow X$ be an onto mapping satisfying*

$$\begin{aligned} &G(T(x), T(y), T(z)) \\ &\geq a\,G(x,y,z) + b\,G(x,x,T(x)) + c\,G(y,y,T(y)) + d\,G(z,z,T(z)) \end{aligned} \tag{3.3}$$

for all $x, y, z \in X$, where $a, b, c, d \geq 0$ with $a+b+c+d > 1$ and $b < 1$. Then T has fixed points in X.

Proof. Let $x_0 \in X$ be arbitrary. Since T is onto, there is an element $x_1 \in X$ satisfying $x_1 \in T^{-1}(x_0)$. By the same way, we can find $x_n \in T^{-1}(x_{n-1})$ for $n = 2, 3, 4, \cdots$. If $x_{m-1} = x_m$ for some m, then $x_m \in T^{-1}(x_{m-1})$ implies $T(x_m) = x_{m-1} = x_m$ and so x_m is a fixed point of T.
Without loss of generality, we can suppose that $x_{n-1} \neq x_n$ for every n. From (3.3), we have

$$\begin{aligned} G(x_{n-1}, x_n, x_n) &= G(T(x_n), T(x_{n+1}), T(x_{n+1})) \\ &\geq a\,G(x_n, x_{n+1}, x_{n+1}) + b\,G(x_n, x_n, x_{n-1}) \\ &\quad + c\,G(x_{n+1}, x_{n+1}, x_n) + d\,G(x_{n+1}, x_{n+1}, x_n). \end{aligned}$$

So, it must be the case that

$$(1-b)\,G(x_{n-1}, x_n, x_n) \geq (a+c+d)\,G(x_n, x_{n+1}, x_{n+1}). \tag{3.4}$$

If $(a+c+d) = 0$, then $a+b+c+d = b < 1$, which is a contradiction since $a+b+c+d > 1$.
Hence $a+c+d \neq 0$ and from (3.4), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-b}{a+c+d}\,G(x_{n-1}, x_n, x_n) \tag{3.5}$$

where $0 < \frac{1-b}{a+c+d} < 1$.

Let $q = \frac{1-b}{a+c+d}$. Then $0 < q < 1$ and by repeated application of (3.5), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n\, G(x_0, x_1, x_1). \tag{3.6}$$

Then for all $n, m \in N,\ n < m$, we have by repeated use of the rectangle inequality and (3.6) that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq \left(q^n + q^{n+1} + \cdots + q^{m-1}\right) G(x_0, x_1, x_1) \\ &\leq \frac{q^n}{1-q}\, G(x_0, x_1, x_1). \end{aligned}$$

Then, $lim\, G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, since $lim \frac{q^n}{1-q}\, G(x_0, x_1, x_1) = 0$, as $n, m \to \infty$. For $n, m, l \in N$, (G_5) implies that

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_l, x_m, x_m),$$

taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$. So (x_n) is a G-Cauchy sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-convergent to u.

Let $y \in T^{-1}(u)$ and so, $u = T(y)$. Then

$$\begin{aligned} G(x_n, u, u) &= G(T(x_{n+1}), T(y), T(y)) \\ &\geq a\, G(x_{n+1}, y, y) + b\, G(x_{n+1}, x_{n+1}, x_n) \\ &\quad + c\, G(y, y, u) + d\, G(y, y, u). \end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous on its variables, we have

$$0 \geq a\, G(u, y, y) + b\, G(u, u, u) + c\, G(y, y, u) + d\, G(y, y, u).$$

So,

$$(a + c + d)\, G(u, y, y) \leq 0,$$

which implies $G(u, y, y) = 0$, since $a + c + d \neq 0$. Therefore, by Proposition 2.3, we have $u = y$ and hence, $u = T(u)$. □

Remark 3.5. The above theorem states that a fixed point of T is not unique. The identity mapping I satisfies the condition of Theorem 3.4. But a fixed point of I is not unique.

Remark 3.6. Theorem 2.15 is a special case of Theorem 3.4.

Theorem 3.7. *Let (X,G) be a complete G-metric space, and let $T : X \longrightarrow X$ be an onto G-continuous mapping satisfying*

$$G(T(x), T^2(x), T^2(x)) \geq a\, G(x, T(x), T(x)) \tag{3.7}$$

for all $x \in X$, where $a > 1$. Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence (x_n) by $x_n = T^{-1}(x_{n-1})$ for $n = 1, 2, 3, \cdots$, since T is onto. We may assume that $x_n \neq x_{n-1}$ for all $n \in N$. Then by (3.7), we have

$$G(T(x_{n+1}), T^2(x_{n+1}), T^2(x_{n+1})) \geq a\, G(x_{n+1}, T(x_{n+1}), T(x_{n+1})).$$

So,

$$G(x_n, x_{n-1}, x_{n-1}) \geq a\, G(x_{n+1}, x_n, x_n)$$

which implies

$$G(x_{n+1}, x_n, x_n) \leq \frac{1}{a}\, G(x_n, x_{n-1}, x_{n-1}) \tag{3.8}$$

where $0 < \frac{1}{a} < 1$.

Let $q = \frac{1}{a}$, then $0 < q < 1$ since $a > 1$ and by repeated application of (3.8), we have

$$G(x_{n+1}, x_n, x_n) \leq q^n\, G(x_1, x_0, x_0). \tag{3.9}$$

Then for all $n, m \in N,\ n < m$, we have by repeated use of the rectangle inequality and (3.9) that

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\
&\leq 2\, G(x_{n+1}, x_n, x_n) + 2\, G(x_{n+2}, x_{n+1}, x_{n+1}) \\
&\quad + 2\, G(x_{n+3}, x_{n+2}, x_{n+2}) + \cdots + 2\, G(x_m, x_{m-1}, x_{m-1}) \\
&\leq 2\, \left(q^n + q^{n+1} + \cdots + q^{m-1}\right) G(x_1, x_0, x_0) \\
&\leq \frac{2\, q^n}{1-q}\, G(x_1, x_0, x_0).
\end{aligned}$$

Then, $lim\, G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, since $lim \frac{2\, q^n}{1-q}\, G(x_1, x_0, x_0) = 0$, as $n, m \to \infty$. For $n, m, l \in N$, (G_5) implies that

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_l, x_m, x_m),$$

taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$. So (x_n) is a G-Cauchy sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-convergent to u. By G-continuity of T, we have

$$T(x_n) = x_{n-1} \xrightarrow{(G)} T(u)$$

which implies $u = T(u)$. $\square$

As an application of Theorem 3.7, we have the following results.

Corollary 3.8. *Let (X, G) be a complete G-metric space, and let $T : X \longrightarrow X$ be an onto G-continuous mapping satisfying*

$$G(T(x), T(y), T(z)) \geq a \min \left\{ \begin{array}{l} G(x, T(x), T(x)),\ G(y, T(y), T(y)), \\ \\ G(z, T(z), T(z)),\ G(x, y, z) \end{array} \right\} \tag{3.10}$$

for all $x, y, z \in X$, where $a > 1$. Then T has a fixed point in X.

Proof. Replacing y and z by $T(x)$ in (3.10), we obtain

$$\begin{aligned} &G(T(x), T^2(x), T^2(x)) \\ &\geq a \min \left\{ \begin{array}{l} G(x, T(x), T(x)),\ G(T(x), T^2(x), T^2(x)), \\ \\ G(T(x), T^2(x), T^2(x)),\ G(x, T(x), T(x)) \end{array} \right\} \\ &= a \min\{G(x, T(x), T(x)),\ G(T(x), T^2(x), T^2(x))\} \end{aligned} \tag{3.11}$$

Without loss of generality, we may assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point. Then $T(x) \neq T^2(x)$ and condition (3.11) imply that

$$G(T(x), T^2(x), T^2(x)) \geq a\, G(x, T(x), T(x))$$

which is Condition (3.7). Hence the result follows from Theorem 3.7. $\square$

Corollary 3.9. *Let (X, G) be a complete non-symmetric G-metric space, and $T : X \longrightarrow X$ be onto G-continuous. If there exist non-negative reals a, b, c, d with $a + b + c + d > 1$ and $b + c < 1$ such that*

$$G(T(x), T(y), T(z)) \geq \left\{ \begin{array}{l} a\, G(x, T(x), T(x)) + b\, G(y, T(y), T(y)) \\ \\ +c\, G(z, T(z), T(z)) + d\, G(x, y, z) \end{array} \right\} \tag{3.12}$$

for all $x, y, z \in X$, then T has a fixed point in X.

Proof. Replacing y and z by $T(x)$ in (3.12), we obtain

$$G(T(x), T^2(x), T^2(x)) \geq \left\{ \begin{array}{l} a\, G(x, T(x), T(x)) + (b + c)\, G(T(x), T^2(x), T^2(x)) \\ \\ +d\, G(x, T(x), T(x)) \end{array} \right\}.$$

So, it must be the case that

$$G(T(x), T^2(x), T^2(x)) \geq \frac{a + d}{1 - b - c}\, G(x, T(x), T(x)), \text{ since } b + c < 1. \tag{3.13}$$

Let $k = \frac{a+d}{1-b-c}$, then $k > 1$ since $a+b+c+d > 1$ and (3.13) becomes

$$G(T(x), T^2(x), T^2(x)) \geq k\, G(x, T(x), T(x)),$$

which is Condition (3.7). Hence the result follows from Theorem 3.7. □

Theorem 3.10. *Let (X, G) be a complete G-metric space, and $S, T : X \longrightarrow X$ be onto G-continuous. If there exists a with $1 < 2a < 2$ such that*

$$\min\{G(S(x), T(y), T(y)), G(T(y), S(x), S(x))\} \geq a\{G(S(x), x, x) + G(T(y), y, y)\}$$

for all $x, y \in X$, then S and T have a common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since S is onto, there is an element x_1, satisfying $x_1 \in S^{-1}(x_0)$. Since T is also onto, there is an element x_2, satisfying $x_2 \in T^{-1}(x_1)$. Proceeding in the same way, we can find $x_{2n+1} \in S^{-1}(x_{2n})$ and $x_{2n+2} \in T^{-1}(x_{2n+1})$ for $n = 1, 2, 3, \cdots$. Therefore, $x_{2n} = S(x_{2n+1})$ and $x_{2n+1} = T(x_{2n+2})$, for $n = 0, 1, 2, \cdots$.

Now, if $n = 2m$, then

$$\begin{aligned}
G(x_{n-1}, x_n, x_n) &= G(x_{2m-1}, x_{2m}, x_{2m}) = G(T(x_{2m}), S(x_{2m+1}), S(x_{2m+1})) \\
&\geq \min\left\{\begin{array}{l} G(S(x_{2m+1}), T(x_{2m}), T(x_{2m})), \\ \\ G(T(x_{2m}), S(x_{2m+1}), S(x_{2m+1})) \end{array}\right\} \\
&\geq a\{G(S(x_{2m+1}), x_{2m+1}, x_{2m+1}) + G(T(x_{2m}), x_{2m}, x_{2m})\} \\
&= a\{G(x_{2m}, x_{2m+1}, x_{2m+1}) + G(x_{2m-1}, x_{2m}, x_{2m})\} \\
&= a\{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_n, x_n)\}.
\end{aligned}$$

Therefore, $G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-a}{a} G(x_{n-1}, x_n, x_n)$.

If $n = 2m+1$, then by the same argument used in above, we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-a}{a} G(x_{n-1}, x_n, x_n).$$

Thus for any positive integer n,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-a}{a} G(x_{n-1}, x_n, x_n). \tag{3.14}$$

Let $q = \frac{1-a}{a}$. Then $0 < q < 1$ since $1 < 2a < 2$ and by repeated application of (3.14), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n\, G(x_0, x_1, x_1). \tag{3.15}$$

Then for all $n, m \in N,\ n < m$, we have by repeated use of the rectangle inequality and (3.15) that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq \left(q^n + q^{n+1} + \cdots + q^{m-1}\right) G(x_0, x_1, x_1) \\ &\leq \frac{q^n}{1-q} G(x_0, x_1, x_1). \end{aligned}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$. So, by Proposition 2.7, the sequence (x_n) is G-Cauchy sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-convergent to u. By G-continuity of S and T, we have

$$S(x_{2m+1}) = x_{2m} \stackrel{(G)}{\longrightarrow} S(u) \ as\ m \longrightarrow \infty$$

and

$$T(x_{2m+2}) = x_{2m+1} \stackrel{(G)}{\longrightarrow} T(u) \ as\ m \longrightarrow \infty.$$

Hence $S(u) = u$ and $T(u) = u$, which means that u is a common fixed point of S and T. □

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