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The Prime Avoidance Lemma Revisited

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Abstract. We show that the above lemma and its well-known refinement are valid, in a general setting, in non-commutative rings. Some interesting consequences are also observed.

The above lemma is in every single text book on commutative algebra, but not even a single book on non-commutative algebra, as yet (except [12, Proposition 2.12.7], where the validity of a special variant of this lemma is shown), proves it for non-commutative rings. We recall that, in the proof of [7, Proposition 2], which is in fact the proof of the converse of the Generalized Principal Ideal Theorem of Krull for non-commutative rings, see also [5, Theorem 153], we invoked the lemma for the non-commutative case, even without mentioning it, and gave no proof for it (note, it seems (at least to us) that the lemma for non-commutative rings was overlooked in the literature, at that time, and although I knew of a proof for a generalization of this lemma then, but did not present it in [7], on purpose, for the reason that we will see, shortly). Let R be a ring with identity, if S is a subring of R without containing the identity of R, we say that S is a subring-1 (see [4, Chapter 16]). The above lemma states that, if $P_1, P_2, \dots, P_n$ are ideals of a commutative ring R with identity such that at most two of the P_i's are not prime and S is a subring-1 of R contained in $P_1 \cup P_2 \cup \dots \cup P_n$, then $S \subseteq P_k$ for some P_k (note, in most cases S is an ideal). Without any doubt, as it is rightly mentioned in [13], the Prime Avoidance Lemma is one of the fundamental cornerstones of commutative algebra. Its numerous applications in the field, makes one surely claim that no one working in the commutative algebra can do without it. The lemma for commutative rings goes back to a 1957 paper by McCoy, and Kaplansky generalized it in his 1974 book on commutative rings, see [4], [5], respectively. The reader might consult ([4, Chapter 16]), [9] and [11]) for a short history and some variations of this lemma. There is also a very useful refinement of the lemma, which says that whenever $P_1, P_2, \dots, P_n$ are prime ideals of the commutative ring R and I is an ideal of R, $a \in R$ with $aR + I \nsubseteq \bigcup\limits_{i=1}^{n} P_i$, then $a + c \notin \bigcup\limits_{i=1}^{n} P_i$ for some $c \in I$ (equivalently, $a + I \subseteq \bigcup\limits_{i=1}^{n} P_i$

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implies that $I \cup \{a\} \subseteq P_i$ for some $i \geq 1$), see ([11], Theorem 3.64). There are also the Sharp-Vamos prime avoidance theorems for noetherian complete local rings for countable prime ideals, stated as 16.7 A and 16.7 B in [4], see also [13], [10]. We have extended the latter result to complete noetherian semi-local rings in [8], for some purpose. Our aim, in this short note, is to generalize the Prime Avoidance Lemma and its refinement to non-commutative rings and obtain some new and interesting consequences. Let us, without further ado, make some definitions. If $S, P_1, P_2, \ldots, P_n$ are subsets of a ring R such that $S \subseteq \bigcup\limits_{i=1}^{n} P_i$ implies that S is contained in the union of a smaller number of P_i's, we shall say that $S \subseteq \bigcup\limits_{i=1}^{n} P_i$ is reducible. Let S be a subring-1 of a ring R, then S is called prime preserving with respect to $\bigcup\limits_{i=1}^{n} P_i$, where each P_i is an ideal in R, if $S \cap P_i$ is a prime ideal in S whenever P_i is prime in R (e.g., any subring-1 of a commutative ring, or even of a right (left) duo ring (i.e., each right (left) ideal is two sided)). Finally, if S is a subring-1 of a ring R, then S is said to be a right (left) ideal with respect to a subset T of R if $ST \subseteq S$ ($TS \subseteq S$) (e.g., any right (left) ideal of R). Before stating our results, we should remind the reader that there have also been some generalizations of this lemma to prime submodules, see [3], [1] and [2]. Noting that the prime submodules are more general than the prime ideals and, in particular, the latter article deals with the prime submodules over non-commutative rings, however, non of our results can be deduced from these prime submodule versions of the lemma, in any way.

We should remind the reader that the following proof, which is given for a general form of the above lemma, is nothing but the trivial proof of the fact that no union of two noncomparable subgroups in any group can be a subgroup.

Theorem A(Prime Avoidance Lemma)**.** *Suppose that $P_1, P_2, \ldots, P_n$ are ideals of a ring R and S is a subring-1 of R with $S \subseteq \bigcup\limits_{i=1}^{n} P_i$. If at most two of the P_i's are not prime and S is either prime preserving or a right (left) ideal with respect to $\bigcup\limits_{i=1}^{n} P_i$, then $S \subseteq \bigcup\limits_{i=1}^{n} P_i$ is reducible. In particular, $S \subseteq P_i$ for some P_i.*

Proof. Let $S \subseteq \bigcup\limits_{i=1}^{n} P_i$ be irreducible and obtain a contradiction. Clearly, $S = \bigcup\limits_{i=1}^{n} Q_i$, where $Q_i = S \cap P_i$, is irreducible. Hence, we can infer that there exist $x \in Q_1 \setminus \bigcup\limits_{i=2}^{n} Q_i$ and $y \in Q_2 Q_3 \ldots Q_n \setminus Q_1$ (note, if $n = 2$, x, y exist even if P_1, P_2 are not prime and if $n \geq 3$, then we may assume that P_1 is prime and hence, in this case, x, y exist too). But $x + y \in S \setminus Q_i$ for all $i \geq 1$, which is absurd. The final part is now evident. $\square$

Next, we should emphasize that, our statement of the refinement of the lemma, which follows, is stronger than the one in the commutative case. We note that our proof of this refinement, in contrast to the usual proof in the commutative case, makes no use of the Prime Avoidance Lemma (cf. [11, Theorem 3.64], or [9]). Finally, we should also admit that we found this proof a very long time ago (see [6, p. 87]), but did not publish it (because we could not get rid of of this constant nagging feeling that this cannot be new). However my searches, after all these years, turned up nothing.

We should also emphasis that in the next result, which is the refinement of the lemma, usually in the commutative case, all P_i's are taken to be prime.

Theorem B. *Let S be a subring–1 of a ring R which is a right (left) ideal with respect to $\bigcup_{i=1}^{n} P_i$, where each P_i is an ideal of R such that at most one of P_i's is not prime. If T is a subset of R with $S+T \subseteq \bigcup_{i=1}^{n} P_i$, then there exists $t \in T$ such that $S \cup \{t\} \subseteq P_i$ for some P_i.*

Proof. For $n = 1$, we note that $S \cup \{t\} \subseteq P_1$ for all $t \in T$, even if P_1 is not prime. Hence, let $n \geq 2$ and we may assume that P_1 is prime and $P_i \nsubseteq P_j$ for $i \neq j$. Now by induction we may suppose that $S + T \nsubseteq \bigcup_{i=2}^{n} P_i$. Hence there are $x \in S$, $t \in T$ with $x+t \in P_1 \setminus \bigcup_{i=2}^{n} P_i$. We claim that $S \cup \{t\} \subseteq P_1$ and we are done. To see this, it suffices to show that $S \subseteq P_1$. Let us put $J = \bigcap_{i=2}^{n} P_i$ and note that for each $y \in S \cap J$ we have $x + t + y \notin P_i$ for all $i \geq 2$, hence $x + t + y \in P_1$ which means $y \in P_1$ and therefore $S \cap J \subseteq P_1$. Since P_1 is a prime ideal and $J \nsubseteq P_1$, we infer that $S \subseteq P_1$ (note, if S is a right ideal with respect to $\bigcup_{i=1}^{n} P_i$, then $SJ \subseteq S \cap J \subseteq P_1$, otherwise $JS \subseteq P_1$). □

The following interesting corollary is new, even in the commutative case.

Corollary. *Let S be a subring–1 of a ring R which is a right (left) ideal with respect to $\bigcup_{i=1}^{n} P_i$, where each P_i is an ideal such that at most one of P_i's is not prime. If $S+T \subseteq \bigcup_{i=1}^{n} P_i$ with $S+T \subseteq \bigcup_{i=1}^{n} P_i$ (resp. $T \subseteq \bigcup_{i=1}^{n} P_i$) irreducible, then $S \subseteq \bigcap_{i=1}^{n} P_i$.*

Proof. For $n = 1$, it is evident. Hence let $n \geq 2$. We also note that whenever $T \subseteq \bigcup_{i=1}^{n} P_i$ is irreducible, so too is $S + T \subseteq \bigcup_{i=1}^{n} P_i$. Hence we may only assume that $S+T \subseteq \bigcup_{i=1}^{n} P_i$ is irreducible. By Theorem B, there exists $t_1 \in T$ with $S \cup \{t_1\} \subseteq P_i$

for some $i \geq 1$. Since $S + T \subseteq \bigcup_{i=1}^{n} P_i$ is irreducible, there exists $t_2 \in T$ with $S + \{t_2\} \nsubseteq P_i$. But $S + \{t_2\} \subseteq \bigcup_{i=1}^{n} P_i$, hence by Theorem B, there exists $P_j \neq P_i$ with $S \cup \{t_2\} \subseteq P_j$. Now if $n \geq 3$, then $S \subseteq P_i \cap P_j$ and $S + T \nsubseteq P_i \cup P_j$ imply that there exists $t_3 \in T$ such that $S + \{t_3\} \nsubseteq P_i$, $S + \{t_3\} \nsubseteq P_j$. But $S + \{t_3\} \subseteq \bigcup_{i=1}^{n} P_i$ implies that $S \cup \{t_3\} \subseteq P_k$ for some $k \neq i, j$, hence $S \subseteq P_i \cap P_j \cap P_k$. If we repeat this process n times, we are done. $\square$

Remark. We observe that in the previous corollary, if $S + T \subseteq \bigcup_{i=1}^{n} P_i$, then $S + T \subseteq \bigcup_{i=1}^{n} P_i$ is irreducible if and only if $T \subseteq \bigcup_{i=1}^{n} P_i$ is irreducible.

Finally, we apply the previous corollary to give a solution to the following number theory exercise and it would be interesting if one could find a solution to this exercise without using the above corollary or its proof.

Exercise. Let $p_1, p_2, \dots p_{n+1}$ be distinct positive integers such that all p_i's, except possibly p_{n+1}, are prime and $(p_i, p_{n+1}) = 1$ for all $i \leq n$. Show that there exists the largest abelian subgroup G of the integer numbers such that for each $g \in G$ and each p_i there exists some p_j such that p_j divides $g + p_i$.

Solution. If there exists such a subgroup G we must have $G + T \subseteq \bigcup_{i=1}^{n+1} (p_i)$, where $T = \{p_1, p_2, \dots p_{n+1}\}$. Clearly, $T \subseteq \bigcup_{i=1}^{n+1} (p_i)$ is irreducible. Consequently, in view of the above corollary, we must have $G \subseteq \bigcap_{i=1}^{n+1} (p_i) = (p_1 p_2 \dots p_{n+1})$. But clearly $G = (p_1 p_2 \dots p_{n+1})$ satisfies the property mentioned in the statement of the exercise and therefore this G is the largest subgroup with this property.

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