

Strongly Summable Double Sequence Spaces in n -Normed Spaces Defined by Ideal Convergence and an Orlicz Function

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ABSTRACT. In this paper we introduce some new double sequence spaces via ideal convergence and an Orlicz function in n -normed spaces and examine some properties of the resulting spaces.

1. Introduction

Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an ideal if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if $\emptyset \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called non-trivial ideal if $I \neq \emptyset$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X/A : A \in I\}$ is a filter on X . A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$. Further details on ideals of 2^X can be found in Kostyrko, et.al [5]. The notion was further investigated by Salat et.al [13], Tripathy and Hazarika [21, 23, 15, 29], Tripathy and Mahanta [27] and others.

The notion of double sequences has been investigated from different aspect by Tripathy and Dutta [25], Tripathy and Sarma [22, 24, 30] and many others in recent years.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P\text{-}\lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > n$ Pringsheim [10]. We shall write more briefly as " P -convergent".

The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l . Let l_{∞}^2 the space of all bounded double such that

$$\|x_{k,l}\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

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Recall Krasnoselski and Rutickii [6] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Nakano [9]. It was further investigated from sequence space point of view by Ruckle [12], Tripathy and Chandra [28] and many others. An Orlicz function M is said to satisfy Δ_2 -condition for all values u , if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1.1. *Let M be an Orlicz function which satisfies Δ_2 -condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Recently different classes of sequences have been introduced by using an Orlicz function and their algebraic and topological properties have been investigated by Et et.al. [1], Tripathy and Mahanta [18], Tripathy et.al. [20], Tripathy and Hazarika [15], Tripathy and Sarma [24, 30] and many others.

A double sequence space E is said to be solid or normal if $(\alpha_{k,l}x_{k,l}) \in E$, whenever $(x_{k,l}) \in E$ and for all double sequences $\alpha = (\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d , where $n \leq d$. A real-valued function $\|., \dots, .\|$ on X satisfying the following four conditions:

- (i) $\|(x_1, x_2, \dots, x_n)\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (ii) $\|(x_1, x_2, \dots, x_n)\|$ is invariant under permutation,
- (iii) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|(x_1, x_2, \dots, x_n)\|$, $\alpha \in \mathbb{R}$,
- (iv) $\|(x_1 + x_1^1, x_2, \dots, x_n)\| \leq \|(x_1, x_2, \dots, x_n)\| + \|(x_1^1, x_2, \dots, x_n)\|$

is called an n -norm on X , and the pair $(X, \|(\cdot, \dots, \cdot)\|)$ is called an n -normed space [3].

A trivial example of n -normed space is $X = \mathbb{R}$ equipped with the following Euclidean n -norm:

$$\|(x_1, x_2, \dots, x_n)\|_E = \text{abs} \left(\begin{pmatrix} x_{11} \dots x_{1n} \\ \dots \\ x_{n1} \dots x_{nn} \end{pmatrix} \right)$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$.

The notion of n -norm from different aspect by Gahler [2], Gunawan and Mashadi [4], Misiak [8], Tripathy and Dutta [26] and many others.

2. Main results

In this section we introduce the notion of different types of I -convergent double sequences. This generalizes and unifies different notions of convergence for double sequences. We shall denote the ideal of $2^{\mathbb{N} \times \mathbb{N}}$ by I_2 .

The notion of paranormed sequence space was studied at the initial stage by Simons [14]. Later on different classes of paranormed sequence spaces were intro-

duced and their different properties were investigated by Maddox [7], Rath and Tripathy [11], Tripathy [17], Tripathy and Sen [16, 19] and many others.

Let I_2 be an ideal of $2^{\mathbb{N} \times \mathbb{N}}$, M be an Orlicz function, $p = (p_{k,l})$ be a bounded double sequence of strictly positive real numbers and $(X, \|(\cdot, \dots, \cdot)\|)$ be an n -normed space. Further $w(n - X)$ denotes X -valued sequence space. Now, we define the following double sequence spaces:

$$w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o = \left\{ \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \begin{array}{l} x = (x_{k,l}) \in w(n - X) : \forall \varepsilon > 0, \\ \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \varepsilon \end{array} \right\} \in I_2 \right\},$$

for some $\rho > 0$ and for every $z_1, \dots, z_{n-1} \in X$

$$w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|] = \left\{ \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \begin{array}{l} x = (x_{k,l}) \in w(n - X) : \forall \varepsilon > 0, \\ \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}-L}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \varepsilon \end{array} \right\} \in I_2 \right\},$$

for some $\rho > 0, L \in X$ and for every $z_1, \dots, z_{n-1} \in X$

$$w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_\infty = \left\{ \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \begin{array}{l} x = (x_{k,l}) \in w(n - X) : \exists K > 0, \\ \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq K \end{array} \right\} \in I_2 \right\}$$

for some $\rho > 0$ and for every $z_1, \dots, z_{n-1} \in X$

and

$$w [M, p, \|(\cdot, \dots, \cdot)\|]_\infty = \left\{ \begin{array}{l} x = (x_{k,l}) \in w(n - X) : \\ \exists K > 0, \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq K \end{array} \right\}.$$

for some $\rho > 0$ and for every $z_1, \dots, z_{n-1} \in X$

If $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$ we obtain $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|] = w^{I_2} [M, \|(\cdot, \dots, \cdot)\|]$, $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o = w^{I_2} [M, \|(\cdot, \dots, \cdot)\|]_o$, $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_\infty = w^{I_2} [M, \|(\cdot, \dots, \cdot)\|]_\infty$ and $w [M, p, \|(\cdot, \dots, \cdot)\|]_\infty = w [M, \|(\cdot, \dots, \cdot)\|]_\infty$.

The following well-known inequality will be used in this study: If $0 \leq \inf_{k,l} p_{k,l} = H_o \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$, $D = \max(1, 2^{H-1})$, then

$$|x_{k,l} + y_{k,l}|^{p_{k,l}} \leq D \{ |x_{k,l}|^{p_{k,l}} + |y_{k,l}|^{p_{k,l}} \}$$

for all $k, l \in \mathbb{N}$ and $x_{k,l}, y_{k,l} \in \mathbb{C}$. Also $|x_{k,l}|^{p_{k,l}} \leq \max(1, |x_{k,l}|^H)$ for all $x_{k,l} \in \mathbb{C}$.

Theorem 2.1. *The sets $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o, w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]$ and $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ are linear spaces over the complex field \mathbb{C} .*

Proof. We will prove only for $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|_o]$ and the others can be proved similarly. Let $x, y \in w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|_o]$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \frac{\varepsilon}{2} \right\} \in I_2, \text{ for some } \rho_1 > 0$$

and

$$\left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \frac{\varepsilon}{2} \right\} \in I_2, \text{ for some } \rho_2 > 0.$$

Since $\| \cdot, \dots, \cdot \|$ is a n -norm and M is an Orlicz function, the following inequality holds:

$$\begin{aligned} & \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{\alpha x_{k,l} + \beta y_{k,l}}{|\alpha| \rho_1 + |\beta| \rho_2}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \\ & \leq \frac{D}{nm} \sum_{k,l=1,1}^{m,n} \left[\frac{|\alpha|}{|\alpha| \rho_1 + |\beta| \rho_2} M \left(\left\| \left(\frac{x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \\ & \quad + \frac{D}{nm} \sum_{k,l=1,1}^{m,n} \left[\frac{|\beta|}{|\alpha| \rho_1 + |\beta| \rho_2} M \left(\left\| \left(\frac{y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \\ & \leq \frac{D}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \\ & \quad + \frac{D}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \end{aligned}$$

From the above inequality we get

$$\begin{aligned} & \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{\alpha x_{k,l} + \beta y_{k,l}}{|\alpha| \rho_1 + |\beta| \rho_2}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \\ & \subset \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{D}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \frac{\varepsilon}{2} \right\} \\ & \cup \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{D}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \frac{\varepsilon}{2} \right\}. \end{aligned}$$

Two sets on the right hand side belong to I_2 and this completes the proof. \square

It is also easy verify that the space $w[M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ is also a linear space.

Theorem 2.2. For fixed $(n, m) \in \mathbb{N} \times \mathbb{N}$, $w[M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ paranormed space with respect to the paranorm defined by

$$g_{(n,m)}(x) = \inf \left\{ \rho^{\frac{pn,m}{H}} : \left(\sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \right)^{\frac{1}{H}} \leq 1, \right\}.$$

Proof. $g_{(n,m)}(\theta) = 0$ and $g_{(n,m)}(-x) = g_{(n,m)}(x)$ are easy to prove, so we omit them. Let us take $x, y \in w[M, p, \|(\cdot, \dots, \cdot)\|]_\infty$. Let

$$A(x) = \left\{ \rho > 0 : \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq 1, \forall z \in X \right\}$$

and

$$A(y) = \left\{ \rho > 0 : \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{y_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq 1, \forall z \in X \right\}.$$

Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$. If $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned} & \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l} + y_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right] \\ & \leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1} \right) \right\| \right) \right] \\ & \quad + \frac{\rho_2}{\rho_1 + \rho_2} \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{y_{k,l}}{\rho_1}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]. \end{aligned}$$

Thus

$$\sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l} + y_{k,l}}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq 1$$

and

$$\begin{aligned} g_{(n,m)}(x+y) &= \inf \left\{ (\rho_1 + \rho_2)^{\frac{pn,m}{H}} : \rho_1 \in A(x) \text{ and } \rho_2 \in A(y) \right\} \\ &\leq \inf \left\{ (\rho_1)^{\frac{pn,m}{H}} : \rho_1 \in A(x) \right\} + \inf \left\{ (\rho_2)^{\frac{pn,m}{H}} : \rho_2 \in A(y) \right\} \\ &= g_{(n,m)}(x) + g_{(n,m)}(y). \end{aligned}$$

Now, let $\lambda_{k,l}^u \rightarrow \lambda$, where $\lambda_{k,l}^u, \lambda \in \mathbb{C}$ and $g_{(n,m)}(x_{k,l}^u - x_{k,l}) \rightarrow 0$ as $u \rightarrow \infty$. We have to show that $g_{(n,m)}(\lambda_{k,l} x_{k,l}^u - \lambda x_{k,l}) \rightarrow 0$ as $u \rightarrow \infty$. Let

$$A(x^u) = \left\{ \rho_u > 0 : \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}^u}{\rho_u}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq 1, \forall z \in X \right\}$$

and

$$A(x^u - x) = \left\{ \rho_u^i > 0 : \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}^u - x_{k,l}}{\rho_u^i}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq 1, \forall z \in X \right\}.$$

If $\rho_u \in A(x^u)$ and $\rho_u^i \in A(x^u - x)$ then we observe that

$$\begin{aligned} & M \left(\left\| \left(\frac{\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}}{\rho_u |\lambda_{k,l}^u - \lambda| + \rho_u^i |\lambda|}, z_1, \dots, z_{n-1} \right) \right\| \right) \\ & \leq M \left(\left\| \left(\frac{\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}^u}{\rho_u |\lambda_{k,l}^u - \lambda| + \rho_u^i |\lambda|}, z_1, \dots, z_{n-1} \right) \right\| \right) \\ & \quad + \left\| \left(\frac{\lambda x_{k,l}^u - \lambda x_{k,l}}{\rho_u |\lambda_{k,l}^u - \lambda| + \rho_u^i |\lambda|}, z_1, \dots, z_{n-1} \right) \right\| \\ & \leq \frac{\rho_u |\lambda_{k,l}^u - \lambda|}{\rho_u |\lambda_{k,l}^u - \lambda| + \rho_u^i |\lambda|} M \left(\left\| \left(\frac{x_{k,l}^u}{\rho_u}, z_1, \dots, z_{n-1} \right) \right\| \right) \\ & \quad + \frac{\rho_u^i |\lambda|}{\rho_u |\lambda_{k,l}^u - \lambda| + \rho_u^i |\lambda|} M \left(\left\| \left(\frac{x_{k,l}^u - x_{k,l}}{\rho_u^i}, z_1, \dots, z_{n-1} \right) \right\| \right). \end{aligned}$$

From this inequality, it follows that

$$\left[M \left(\left\| \left(\frac{\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}}{\rho_u |\lambda_{k,l}^u - \lambda| + \rho_u^i |\lambda|}, z \right) \right\| \right) \right]^{p_{k,l}} \leq 1$$

and consequently

$$\begin{aligned}
& g_{(n,m)} (\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}) \\
&= \inf \left\{ (\rho_u |\lambda_{k,l}^u - \lambda| + \rho_u^i |\lambda|)^{\frac{p_{n,m}}{H}} : \rho_u \in A(x^u) \text{ and } \rho_u^i \in A(x^u - x) \right\} \\
&\leq (|\lambda_{k,l}^u - \lambda|)^{\frac{p_{n,m}}{H}} \inf \left\{ (\rho_u)^{\frac{p_{n,m}}{H}} : \rho_u \in A(x^u) \right\} \\
&\quad + (|\lambda|)^{\frac{p_{n,m}}{H}} \inf \left\{ (\rho_u^i)^{\frac{p_{n,m}}{H}} : \rho_u^i \in A(x^u - x) \right\} \\
&\leq \max \left\{ |\lambda|, (|\lambda|)^{\frac{p_{n,m}}{H}} \right\} g_{(n,m)} (x_{k,l}^u - x_{k,l}).
\end{aligned}$$

Hence by our assumption the right hand side tends to zero as $u \rightarrow \infty$. This completes the proof. \square

Corollary 2.3. *It can be noted that $g = \inf_{n,m \in \mathbb{N}} g_{(n,m)}$ also gives a paranorm on the above sequence spaces. However if one consider the sequence space $w[M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ which is larger space than the space $w^{I_2}[M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ the construction of the paranorm is not clear and we leave it as an open problem. However it should be noted that for a fixed $G \in I_2$, the space*

$$\begin{aligned}
& w_G [M, p, \|(\cdot, \dots, \cdot)\|]_\infty \\
&= \left\{ \left\{ \begin{array}{l} x = (x_{k,l}) \in w(n-X) : \exists K > 0, \\ (n, m) \in \mathbb{N} \times \mathbb{N} : \\ \sup_{(n,m) \in \mathbb{N} \times \mathbb{N}/F} \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq K \end{array} \right\} \in I_2 \right\} \\
&\quad \text{for some } \rho > 0 \text{ and for every } z_1, \dots, z_{n-1} \in X
\end{aligned}$$

which is subspace of the space $w^{I_2}[M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ is a paranormed space with the paranorms $g_{(n,m)}$ for $(n, m) \notin G$ and $g_G = \inf_{(n,m) \in \mathbb{N} \times \mathbb{N}/G} g_{(n,m)}$.

Theorem 2.4. *Let M, M_1 and M_2 be Orlicz functions. Then we have*

- (i) $w^{I_2}[M_1, p, \|(\cdot, \dots, \cdot)\|]_o \subset w^{I_2}[M \circ M_1, p, \|(\cdot, \dots, \cdot)\|]_o$ provided that $p = (p_{k,l})$ is such that $H_o > 0$.
- (ii) $w^{I_2}[M_1, p, \|(\cdot, \dots, \cdot)\|]_o \cap w^{I_2}[M_2, p, \|(\cdot, \dots, \cdot)\|]_o \subset w^{I_2}[M_1 + M_2, p, \|(\cdot, \dots, \cdot)\|]_o$.

Proof. (i). Let $\inf_{k,l} p_{k,l} = H_o$. For given $\varepsilon > 0$, we first choose $\varepsilon_o > 0$ such that $\max \{ \varepsilon_o^H, \varepsilon_o^{H_o} \} < \varepsilon$. Now using the continuity of M , choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $M(t) < \varepsilon_o$. Let $x \in w^{I_2}[M_1, p, \|(\cdot, \dots, \cdot)\|]_o$. Now from the definition of the space $w^{I_2}[M_1, p, \|(\cdot, \dots, \cdot)\|]_o$, for some $\rho > 0$

$$A(\delta) = \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \geq \delta^H \right\} \in I_2.$$

Thus if $(n, m) \notin A(\delta)$ then

$$\begin{aligned} & \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} < \delta^H \\ \Rightarrow & \sum_{k,l=1,1}^{m,n} \left[M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} < nm\delta^H, \\ \Rightarrow & \left[M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} < \delta^H \text{ for all } k, l = 1, 2, \dots, \\ \Rightarrow & M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) < \delta \text{ for all } k, l = 1, 2, \dots \end{aligned}$$

Hence from above inequality and using continuity of M , we must have

$$M \left(M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right) < \varepsilon_o \text{ for all } k, l = 1, 2, \dots$$

which consequently implies that

$$\begin{aligned} & \sum_{k,l=1,1}^{m,n} \left[M \left(M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right) \right]^{p_{k,l}} < nm \max \{ \varepsilon_o^H, \varepsilon_o^{H_o} \} < nm\varepsilon, \\ \Rightarrow & \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right) \right]^{p_{k,l}} < \varepsilon. \end{aligned}$$

This shows that

$$\left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \subset A(\delta)$$

and so belongs to I_2 . This completes the proof.

(ii) Let $x \in w^{I_2} [M_1, p, \|(\cdot, \dots, \cdot)\|]_o \cap w^{I_2} [M_2, p, \|(\cdot, \dots, \cdot)\|]_o$. Then the fact that

$$\begin{aligned} & \frac{1}{nm} \left[(M_1 + M_2) \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \\ \leq & \frac{D}{nm} \left[M_1 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} + \frac{D}{nm} \left[M_2 \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \end{aligned}$$

gives us the result. \square

Theorem 2.5. (i) If $0 < \inf_{k,l} p_{k,l} = H_o \leq p_{k,l} < 1$, then $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o \subset w^{I_2} [M, \|(\cdot, \dots, \cdot)\|]_o$.

(ii) If $1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$, then $w^{I_2} [M, \|(\cdot, \dots, \cdot)\|]_o \subset w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o$.

(iii) If $0 < p_{k,l} < q_{k,l} < \infty$ and $\frac{q_{k,l}}{p_{k,l}}$ is bounded, then $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o \subset w^{I_2} [M, q, \|(\cdot, \dots, \cdot)\|]_o$.

Proof. The proof is standard, so we omit it. \square

Theorem 2.6. *The sequence spaces $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o$, $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]$, $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ and $w [M, p, \|(\cdot, \dots, \cdot)\|]_\infty$ are solid.*

Proof. We give the proof for only $w^{I_2} [M, p, \|(\cdot, \dots, \cdot)\|]_o$. The others can be proved similarly. Let $x \in w^{I_2} [M_1, p, \|(\cdot, \dots, \cdot)\|]_o$ and $\alpha = (\alpha_{k,l})$ be a double sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$. Then we have

$$\left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{\alpha_{k,l} x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq \varepsilon \right\} \\ \subset \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} : \frac{T}{nm} \sum_{k,l=1,1}^{m,n} \left[M \left(\left\| \left(\frac{x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right) \right\| \right) \right]^{p_{k,l}} \leq \varepsilon \right\} \in I_2,$$

where $T = \max_{k,l} \{1, |\alpha_{k,l}|^H\}$. Hence $\alpha x \in w^{I_2} [M_1, p, \|(\cdot, \dots, \cdot)\|]_o$ for all double sequences $\alpha = (\alpha_{k,l})$ with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$ whenever $x \in w^{I_2} [M_1, p, \|(\cdot, \dots, \cdot)\|]_o$. \square

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