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Serendipitous Functional Relations Deducible from Certain Generalized Triple Hypergeometric Functions

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Abstract. We aim at presenting certain unexpected functional relations among various hypergeometric functions of one or several variables (for example, see the identities in Corollary 5) by making use of Carlson's method employed in his work (Some extensions of Lardner's relations between ${}_0F_3$ and Bessel functions, *SIAM J. Math. Anal.* **1**(2)(1970), 232-242).

1. Introduction

Investigation of multiple hypergeometric functions is essentially motivated by the fact that the solutions of many applied problems involving the thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, quantum mechanics and potential theory are obtainable with the help of such hypergeometric (higher and special or transcendent) functions (see [7, 11, 25, 27]). Such functions are often referred to as special functions in mathematical physics. They are mainly appeared in the solution of partial differential equations by using harmonic analysis method [9]. In view of various applications, it is important as well as interesting in itself to conduct a continuous research on multiple hypergeometric functions. In fact, in Srivastava and Karlsson's work [33], there is an extensive list of as many as 205 hypergeometric functions of three variables together with their region of convergence. It is noted that Riemann's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by using hypergeometric functions of several variables (see [2, 4, 5, 6, 12, 13, 14, 15, 16, 17, 26, 29, 36, 37, 38]). For solutions of the boundary-value problems for the involved partial differential equations, we need to investigate certain properties of hypergeometric functions of several variables (see [18, 19, 20, 21, 22, 28, 34]).

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Lardner [23] gave some connections between Bessel functions and hypergeometric ${}_0F_3$-series, for example,

$$(1.1) \qquad {}_0F_3\left(\frac{1}{2},\frac{1}{2},1;z\right)=\frac{1}{2}\left[J_0\left(4z^{\frac{1}{4}}\right)+I_0\left(4z^{\frac{1}{4}}\right)\right]$$

and

$$(1.2) \quad \operatorname{ber}(x)={}_0F_3\left(\frac{1}{2},\frac{1}{2},1;-\frac{x^4}{256}\right), \quad \text{and} \quad \operatorname{bei}(x)=\frac{x^2}{4}\,{}_0F_3\left(\frac{3}{2},\frac{3}{2},1;-\frac{x^4}{256}\right),$$

where J_ν and I_ν denote a Bessel function and a modified Bessel function of order ν (see [1]; also [35]) defined by

$$(1.3) \qquad J_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-\,;\,\nu+1\,;\,-\frac{z^2}{4}\right)$$
$$\text{and } I_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-\,;\,\nu+1\,;\,\frac{z^2}{4}\right),$$

and ber(x) and bei(x) (x real) denote the Kelvin's functions (see [10, p. 6]) defined by

$$(1.4) \qquad \operatorname{ber}(x)+i\operatorname{bei}(x)=J_0\left(x\,e^{i\,\frac{3}{4}\,\pi}\right)=I_0\left(x\,e^{i\,\frac{1}{4}\,\pi}\right).$$

Carlson [8] generalized these results for arbitrary parameters to give the following results

$$(1.5) \quad {}_0F_3\left(\frac{1}{2},c,c+\frac{1}{2};z\right)=\frac{1}{2}\,\Gamma(2c)\,\left(2z^{\frac{1}{4}}\right)^{1-2c}\left[I_{2c-1}\left(4z^{\frac{1}{4}}\right)+J_{2c-1}\left(4z^{\frac{1}{4}}\right)\right]$$

and

$$(1.6) \quad {}_0F_3\left(\frac{3}{2},c,c+\frac{1}{2};z\right)=\frac{1}{2}\,\Gamma(2c)\,\left(2z^{\frac{1}{4}}\right)^{-2c}\left[I_{2c-2}\left(4z^{\frac{1}{4}}\right)-J_{2c-2}\left(4z^{\frac{1}{4}}\right)\right].$$

Srivastava [30, 31, 33] discovered the existence of three additional complete triple hypergeometric functions H_A, H_B and H_C of the second order. One of them is presented as follows:

$$(1.7) \qquad H_A(a_1,a_2,a_3;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+p}\,(a_2)_{m+n}\,(a_3)_{n+p}}{(c_1)_m\,(c_2)_{n+p}\,m!n!p!}x^m y^n z^p$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the set of complex numbers and the set of nonpositive integers, respectively, and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [32]):

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots\lambda(\lambda+n-1) & (n\in\mathbb{N}:=\{1,2,3,...\}) \end{cases}$$
$$=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad (\lambda\in\mathbb{C}\backslash\mathbb{Z}_0^-),$$

$\Gamma(x)$ is well-known Gamma function. The three-dimensional region of convergence of (1.7) is given by Srivastava and Karlsson [33]: $(|x| := r < 1,\ |y| := s < 1,\ |z| := t < (1-r)(1-s))$, where the positive quantities r, s and t are associated with radii of convergence.

Here, by simply splitting Srivastava's hypergeometric function H_A into eight parts, we show how some useful and generalized relations between Srivastava's hypergeometric functions H_A and $F^{(3)}$ can be obtained. As a particular case, some decomposition formulas for the generalized Srivastava's hypergeometric function $F^{(3)}$ were obtained by means of Gauss's hypergeometric function and vice-versa. The other main results are shown to be specialized to yield certain relations between functions Φ_1 and Ψ_1, ${}_0F_1$, ${}_1F_1$, ${}_0F_3$, $F^{2:1;1}_{2:0;0}$. Some other interesting functional relations between the exponential function, the hyperbolic functions, and modified Bessel functions are considered as well.

2. Relationships between Srivastava's hypergeometric functions H_A and $F^{(3)}$

In this section we establish some interesting and useful identities associated with Srivastava's functions H_A and $F^{(3)}$. For this purpose we simply separate the summations in (1.7) into odd and even powers of each of x^m, y^n, and z^p. In fact, for any complex c_1, $c_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and any finite complex x, y, and z, the series H_A converges absolutely in the region of convergence and can therefore be rearranged as in the following eight summations:

$$
\begin{aligned}
(2.1)\qquad & H_A(a_1,a_2,a_3;c_1,c_2;x,y,z)\\
&= \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)}\,(a_2)_{2(i+j)}\,(a_3)_{2(j+k)}}{(c_1)_{2i}\,(c_2)_{2(j+k)}\,(2i)!\,(2j)!\,(2k)!} x^{2i}y^{2j}z^{2k}\\
&+ x \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)+1}\,(a_2)_{2(i+j)+1}\,(a_3)_{2(j+k)}}{(c_1)_{2i+1}\,(c_2)_{2(j+k)}\,(2i+1)!\,(2j)!\,(2k)!} x^{2i}y^{2j}z^{2k}\\
&+ y \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)}\,(a_2)_{2(i+j)+1}\,(a_3)_{2(j+k)+1}}{(c_1)_{2i}\,(c_2)_{2(j+k)+1}\,(2i)!\,(2j+1)!\,(2k)!} x^{2i}y^{2j}z^{2k}\\
&+ z \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)+1}\,(a_2)_{2(i+j)}\,(a_3)_{2(j+k)+1}}{(c_1)_{2i}\,(c_2)_{2(j+k)+1}\,(2i)!\,(2j)!\,(2k+1)!} x^{2i}y^{2j}z^{2k}\\
&+ xy \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)+1}\,(a_2)_{2(i+j)+2}\,(a_3)_{2(j+k)+1}}{(c_1)_{2i+1}\,(c_2)_{2(j+k)+1}\,(2i+1)!\,(2j+1)!\,(2k)!} x^{2i}y^{2j}z^{2k}\\
&+ xz \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)+2}\,(a_2)_{2(i+j)+1}\,(a_3)_{2(j+k)+1}}{(c_1)_{2i+1}\,(c_2)_{2(j+k)+1}\,(2i+1)!\,(2j)!\,(2k+1)!} x^{2i}y^{2j}z^{2k}
\end{aligned}
$$

$$+\, yz \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)+1}\,(a_2)_{2(i+j)+1}\,(a_3)_{2(j+k)+2}}{(c_1)_{2i}\,(c_2)_{2(j+k)+2}\,(2i)!\,(2j+1)!\,(2k+1)!} x^{2i} y^{2j} z^{2k}$$

$$+\, xyz \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)+2}\,(a_2)_{2(i+j)+2}\,(a_3)_{2(j+k)+2}}{(c_1)_{2i+1}\,(c_2)_{2(j+k)+2}\,(2i+1)!\,(2j+1)!\,(2k+1)!} x^{2i} y^{2j} z^{2k}.$$

Now making use of the following well-known (or easily derivable) identity for the Pochhammer symbol (see [23, 24]):

$$(\alpha)_{2m} = 2^{2m} \left(\frac{\alpha}{2}\right)_m \left(\frac{\alpha}{2}+\frac{1}{2}\right)_m \qquad \left(m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right),$$

after some simplification, we obtain

Theorem 1. *The following relationship between H_A and $F^{(3)}$ holds true.*

$$\text{(2.2)} \qquad H_A\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right)$$

$$= F^{(3)}\left[\begin{matrix} - :: & \frac{a_2}{2}, \frac{a_2+1}{2}; & \frac{a_3}{2}, \frac{a_3+1}{2}; & \frac{a_1}{2}, \frac{a_1+1}{2}; & & -; & -; & -; \\ - :: & -; & \frac{c_2}{2}, \frac{c_2+1}{2}; & -; & \frac{c_1}{2}, \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\; x^2, y^2, z^2\right]$$

$$+ \frac{a_1\, a_2}{c_1}\, x\, F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & \frac{a_3}{2}, \frac{a_3+1}{2}; & \frac{a_1}{2}, \frac{a_1+1}{2}; & & -; & -; & -; \\ - :: & -; & \frac{c_2}{2}, \frac{c_2+1}{2}; & -; & \frac{c_1+1}{2}, \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\; x^2, y^2, z^2\right]$$

$$+ \frac{a_2\, a_3}{c_2}\, y\, F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & \frac{a_3+1}{2}, \frac{a_3+2}{2}; & \frac{a_1}{2}, \frac{a_1+1}{2}; & & -; & -; & -; \\ - :: & -; & \frac{c_2+1}{2}, \frac{c_2+2}{2}; & -; & \frac{c_1}{2}, \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\; x^2, y^2, z^2\right]$$

$$+ \frac{a_1\, a_3}{c_2}\, z\, F^{(3)}\left[\begin{matrix} - :: & \frac{a_2}{2}, \frac{a_2+1}{2}; & \frac{a_3+1}{2}, \frac{a_3+2}{2}; & \frac{a_1+1}{2}, \frac{a_1+2}{2}; & & -; & -; & -; \\ - :: & -; & \frac{c_2+1}{2}, \frac{c_2+2}{2}; & -; & \frac{c_1}{2}, \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\; x^2, y^2, z^2\right]$$

$$+ \frac{a_1\, a_2\, (a_2+1)\, a_3}{c_1\, c_2}\, x\, y\, F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+2}{2}, \frac{a_2+3}{2}; & \frac{a_3+1}{2}, \frac{a_3+2}{2}; & \frac{a_1+1}{2}, \frac{a_1+2}{2}; & & -; & -; & -; \\ - :: & -; & \frac{c_2+1}{2}, \frac{c_2+2}{2}; & -; & \frac{c_1+1}{2}, \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\; x^2, y^2, z^2\right]$$

$$+\frac{a_1\,(a_1+1)\,a_2\,a_3}{c_1\,c_2}\,x\,z\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2+1}{2},\,\frac{a_2+2}{2};\ \frac{a_3+1}{2},\,\frac{a_3+2}{2};\ \frac{a_1+2}{2},\,\frac{a_1+3}{2};\\ -:: \qquad -;\ \frac{c_2+1}{2},\,\frac{c_2+2}{2};\qquad -;\end{array}\right.$$
$$\left.\begin{array}{l} \qquad\qquad -;\ -;\ -;\\ \frac{c_1+1}{2},\,\frac{c_1+2}{2},\,\frac{3}{2};\ \frac{1}{2};\ \frac{3}{2};\end{array} x^2,\,y^2,\,z^2\right]$$

$$+\frac{a_1\,a_2\,a_3\,(a_3+1)}{c_2\,(c_2+1)}\,y\,z\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2+1}{2},\,\frac{a_2+2}{2};\ \frac{a_3+2}{2},\,\frac{a_3+3}{2};\ \frac{a_1+1}{2},\,\frac{a_1+2}{2};\\ -:: \qquad -;\ \frac{c_2+2}{2},\,\frac{c_2+3}{2};\qquad -;\end{array}\right.$$
$$\left.\begin{array}{l} \qquad\qquad -;\ -;\ -;\\ \frac{c_1}{2},\,\frac{c_1+1}{2},\,\frac{1}{2};\ \frac{3}{2};\ \frac{3}{2};\end{array} x^2,\,y^2,\,z^2\right]$$

$$+\frac{a_1\,(a_1+1)\,a_2\,(a_2+1)\,a_3\,(a_3+1)}{c_1\,c_2\,(c_2+1)}\,x\,y\,z$$
$$\cdot F^{(3)}\left[\begin{array}{l} -:: \frac{a_2+2}{2},\,\frac{a_2+3}{2};\ \frac{a_3+2}{2},\,\frac{a_3+3}{2};\ \frac{a_1+2}{2},\,\frac{a_1+3}{2};\\ -:: \qquad -;\ \frac{c_2+2}{2},\,\frac{c_2+3}{2};\qquad -;\end{array}\right.$$
$$\left.\begin{array}{l} \qquad\qquad -;\ -;\ -;\\ \frac{c_1+1}{2},\,\frac{c_1+2}{2},\,\frac{3}{2};\ \frac{3}{2};\ \frac{3}{2};\end{array} x^2,\,y^2,\,z^2\right],$$

where $F^{(3)}$ *is Srivastava's generalized hypergeometric function* (*see* [33]):

$$F^{(3)}\left[\begin{array}{l} -:: b_1,b_2;\ b_1',b_2';\ b_1'',b_2'';\qquad -;\ -;\ -;\\ -:: \quad -;\ g_1,g_2;\quad -;\ h_1,h_2,h_3;\ h_1';\ h_1'';\end{array} x,y,z\right]$$
(2.3)
$$=\sum_{i,j,k=0}^{\infty}\frac{(b_1)_{i+j}\,(b_2)_{i+j}\left(b_1'\right)_{j+k}\left(b_2'\right)_{j+k}\left(b_1''\right)_{i+k}\left(b_2''\right)_{i+k}}{(g_1)_{j+k}\,(g_2)_{j+k}\,(h_1)_i\,(h_2)_i\,(h_3)_i\left(h_1'\right)_j\left(h_1''\right)_k\,i!j!k!}x^i y^j z^k.$$

Conversely, combining the signs of x, y and z in the definition of H_A, from (2.2) we readily express $F^{(3)}$ in terms of H_A's.

Theorem 2. *The following eight relationships between* $F^{(3)}$ *and* H_A *hold true.*

(2.4)
$$8\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2}{2},\,\frac{a_2+1}{2};\ \frac{a_3}{2},\,\frac{a_3+1}{2};\ \frac{a_1}{2},\,\frac{a_1+1}{2};\\ -:: \qquad -;\ \frac{c_2}{2},\,\frac{c_2+1}{2};\qquad -;\end{array}\right.$$
$$\left.\begin{array}{l} \qquad\qquad -;\ -;\ -;\\ \frac{c_1}{2},\,\frac{c_1+1}{2},\,\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2};\end{array} x^2,\,y^2,\,z^2\right]$$
$$=H_A\,(x,y,z)+H_A\,(-x,y,z)+H_A\,(x,y,-z)+H_A\,(x,-y,z)$$
$$+H_A\,(-x,-y,z)+H_A\,(-x,y,-z)+H_A\,(x,-y,-z)+H_A\,(-x,-y,-z)\,;$$

$$
(2.5)\quad \frac{8\,a_1\,a_2\,x}{c_1}\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2+1}{2},\,\frac{a_2+2}{2};\ \frac{a_3}{2},\,\frac{a_3+1}{2};\ \frac{a_1}{2},\,\frac{a_1+1}{2};\ -;\ -;\ -; \\ -:: -;\ \frac{c_2}{2},\,\frac{c_2+1}{2};\ -;\ \frac{c_1+1}{2},\,\frac{c_1+2}{2},\,\frac{3}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array} x^2,\,y^2,\,z^2\right]
$$
$$
\begin{aligned}
&= H_A(x,y,z) - H_A(-x,y,z) + H_A(x,-y,z) + H_A(x,y,-z) \\
&\quad - H_A(-x,-y,z) - H_A(-x,y,-z) + H_A(x,-y,-z) - H_A(-x,-y,-z)\,;
\end{aligned}
$$

$$
(2.6)\quad \frac{8\,a_2\,a_3\,y}{c_2}\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2+1}{2},\,\frac{a_2+2}{2};\ \frac{a_3+1}{2},\,\frac{a_3+2}{2};\ \frac{a_1}{2},\,\frac{a_1+1}{2};\ -;\ -;\ -; \\ -:: -;\ \frac{c_2+1}{2},\,\frac{c_2+2}{2};\ -;\ \frac{c_1}{2},\,\frac{c_1+1}{2},\,\frac{1}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array} x^2,\,y^2,\,z^2\right]
$$
$$
\begin{aligned}
&= H_A(x,y,z) + H_A(-x,y,z) - H_A(x,-y,z) + H_A(x,y,-z) \\
&\quad - H_A(-x,-y,z) + H_A(-x,y,-z) - H_A(x,-y,-z) - H_A(-x,-y,-z)\,;
\end{aligned}
$$

$$
(2.7)\quad \frac{8\,a_1\,a_3\,z}{c_2}\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2}{2},\,\frac{a_2+1}{2};\ \frac{a_3+1}{2},\,\frac{a_3+2}{2};\ \frac{a_1+1}{2},\,\frac{a_1+2}{2};\ -;\ -;\ -; \\ -:: -;\ \frac{c_2+1}{2},\,\frac{c_2+2}{2};\ -;\ \frac{c_1}{2},\,\frac{c_1+1}{2},\,\frac{1}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array} x^2,\,y^2,\,z^2\right]
$$
$$
\begin{aligned}
&= H_A(x,y,z) + H_A(-x,y,z) + H_A(x,-y,z) - H_A(x,y,-z) \\
&\quad + H_A(-x,-y,z) - H_A(-x,y,-z) - H_A(x,-y,-z) - H_A(-x,-y,-z)\,;
\end{aligned}
$$

$$
(2.8)\quad \frac{8\,a_1\,a_2\,(a_2+1)\,a_3\,x\,y}{c_1\,c_2}\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2+2}{2},\,\frac{a_2+3}{2};\ \frac{a_3+1}{2},\,\frac{a_3+2}{2};\ \frac{a_1+1}{2},\,\frac{a_1+2}{2};\ -;\ -;\ -; \\ -:: -;\ \frac{c_2+1}{2},\,\frac{c_2+2}{2};\ -;\ \frac{c_1+1}{2},\,\frac{c_1+2}{2},\,\frac{3}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array} x^2,\,y^2,\,z^2\right]
$$
$$
\begin{aligned}
&= H_A(x,y,z) - H_A(-x,y,z) - H_A(x,-y,z) + H_A(x,y,-z) \\
&\quad + H_A(-x,-y,z) - H_A(-x,y,-z) - H_A(x,-y,-z) + H_A(-x,-y,-z)\,;
\end{aligned}
$$

$$
(2.9)\quad \frac{8\,a_1\,(a_1+1)\,a_2\,a_3\,x\,z}{c_1\,c_2}\,F^{(3)}\left[\begin{array}{l} -:: \frac{a_2+1}{2},\,\frac{a_2+2}{2};\ \frac{a_3+1}{2},\,\frac{a_3+2}{2};\ \frac{a_1+2}{2},\,\frac{a_1+3}{2};\ -;\ -;\ -; \\ -:: -;\ \frac{c_2+1}{2},\,\frac{c_2+2}{2};\ -;\ \frac{c_1+1}{2},\,\frac{c_1+2}{2},\,\frac{3}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array} x^2,\,y^2,\,z^2\right]
$$
$$
\begin{aligned}
&= H_A(x,y,z) - H_A(-x,y,z) + H_A(x,-y,z) - H_A(x,y,-z) \\
&\quad - H_A(-x,-y,z) + H_A(-x,y,-z) - H_A(x,-y,-z) + H_A(-x,-y,-z)\,;
\end{aligned}
$$

$$(2.10)\quad \frac{8\,a_1a_2a_3(a_3+1)\;y\,z}{c_2\,(c_2+1)}\,F^{(3)}\left[\begin{matrix} -:: & \frac{a_2+1}{2},\ \frac{a_2+2}{2}; & \frac{a_3+2}{2},\ \frac{a_3+3}{2}; & \frac{a_1+1}{2},\ \frac{a_1+2}{2}; \\ -:: & -; & \frac{c_2+2}{2},\ \frac{c_2+3}{2}; & -; \end{matrix}\right.$$

$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1}{2},\ \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ x^2,\,y^2,\,z^2\right]$$

$$= H_A\,(x,y,z) + H_A\,(-x,y,z) - H_A\,(x,-y,z) - H_A\,(x,y,-z)$$
$$- H_A\,(-x,-y,z) - H_A\,(-x,y,-z) + H_A\,(x,-y,-z) + H_A\,(-x,-y,-z)\,;$$

$$(2.11)\quad \frac{8\,a_1\ (a_1+1)\ a_2\ (a_2+1)\ a_3\ (a_3+1)\ x\,y\,z}{c_1\,c_2\ (c_2+1)}$$

$$\cdot\,F^{(3)}\left[\begin{matrix} -:: & \frac{a_2+2}{2},\ \frac{a_2+3}{2}; & \frac{a_3+2}{2},\ \frac{a_3+3}{2}; & \frac{a_1+2}{2},\ \frac{a_1+3}{2}; \\ -:: & -; & \frac{c_2+2}{2},\ \frac{c_2+3}{2}; & -; \end{matrix}\right.$$

$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1+1}{2},\ \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ x^2,\,y^2,\,z^2\right]$$

$$= H_A\,(x,y,z) - H_A\,(-x,y,z) - H_A\,(x,-y,z) - H_A\,(x,y,-z)$$
$$+ H_A\,(-x,-y,z) + H_A\,(-x,y,-z) + H_A\,(x,-y,-z) - H_A\,(-x,-y,-z)\,,$$

where, for simplicity, $H_A(x,y,z) := H_A\,(a_1,a_2,a_3;c_1,c_2;x,y,z)\,.$

3. Limiting Cases

Here we want to express the triple hypergeometric functions in terms of simpler hypergeometric functions. For this purpose we begin by providing functional relationships between a little simpler function of H_A and $F^{(3)}$ as in Corollary 1. Indeed, in order to use the method suggested in [8], employing the following transformations $a_1 \sim 1/\varepsilon,\ x \sim \varepsilon x,\ \ z \sim \varepsilon z$ in identities (2.1) and (2.4) to (2.11), and taking the limit of the resulting identities as $\varepsilon \to 0$, we obtain

Corollary 1. *Each of the following relationships holds true.*

$$(3.1)\quad {}_1H_A\,(a_2,a_3;c_1,c_2;x,y,z)$$

$$= F^{(3)}\left[\begin{matrix} -:: & \frac{a_2}{2},\ \frac{a_2+1}{2}; & \frac{a_3}{2},\ \frac{a_3+1}{2}; & -; & & -; & -; & -; \\ -:: & -; & \frac{c_2}{2},\ \frac{c_2+1}{2}; & -; & \frac{c_1}{2},\ \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]$$

$$+\,\frac{a_2\,x}{c_1}\,F^{(3)}\left[\begin{matrix} -:: & \frac{a_2+1}{2},\ \frac{a_2+2}{2}; & \frac{a_3}{2},\ \frac{a_3+1}{2}; & -; \\ -:: & -; & \frac{c_2}{2},\ \frac{c_2+1}{2}; & -; \end{matrix}\right.$$

$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1+1}{2},\ \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]$$

$$
+\frac{a_2\,a_3\,y}{c_2}\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+1}{2},\frac{a_2+2}{2};\ \frac{a_3+1}{2},\frac{a_3+2}{2};\ -;\ -;\ -;\ -; \\ -:: \ -;\ \frac{c_2+1}{2},\frac{c_2+2}{2};\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]
$$

$$
+\frac{a_3\,z}{c_2}\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2}{2},\frac{a_2+1}{2};\ \frac{a_3+1}{2},\frac{a_3+2}{2};\ -;\ -;\ -;\ -; \\ -:: \ -;\ \frac{c_2+1}{2},\frac{c_2+2}{2};\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]
$$

$$
+\frac{a_2\,(a_2+1)\,a_3\,x\,y}{c_1\,c_2}\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+2}{2},\frac{a_2+3}{2};\ \frac{a_3+1}{2},\frac{a_3+2}{2};\ -;\ -;\ -;\ -; \\ -:: \ -;\ \frac{c_2+1}{2},\frac{c_2+2}{2};\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]
$$

$$
+\frac{a_2\,a_3\,x\,z}{c_1\,c_2}\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+1}{2},\frac{a_2+2}{2};\ \frac{a_3+1}{2},\frac{a_3+2}{2};\ -;\ -;\ -;\ -; \\ -:: \ -;\ \frac{c_2+1}{2},\frac{c_2+2}{2};\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]
$$

$$
+\frac{a_2\,a_3\,(a_3+1)\,y\,z}{c_2\,(c_2+1)}\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+1}{2},\frac{a_2+2}{2};\ \frac{a_3+2}{2},\frac{a_3+3}{2};\ -;\ -;\ -;\ -; \\ -:: \ -;\ \frac{c_2+2}{2},\frac{c_2+3}{2};\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{3}{2};\ \frac{3}{2}; \end{array}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]
$$

$$
+\frac{a_2\,(a_2+1)\,a_3\,(a_3+1)\,x\,y\,z}{c_1\,c_2\,(c_2+1)}
$$

$$
\cdot\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+2}{2},\frac{a_2+3}{2};\ \frac{a_3+2}{2},\frac{a_3+3}{2};\ -;\ -;\ -;\ -; \\ -:: \ -;\ \frac{c_2+2}{2},\frac{c_2+3}{2};\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{3}{2};\ \frac{3}{2}; \end{array}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right];
$$

(3.2)

$$
8\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2}{2},\frac{a_2+1}{2};\ \frac{a_3}{2},\frac{a_3+1}{2};\ -;\ -;\ -;\ -; \\ -:: \ -;\ \frac{c_2}{2},\frac{c_2+1}{2};\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array}\ \frac{x^2}{4},\,y^2,\,\frac{z^2}{4}\right]
$$

$$
\begin{aligned}
&= {}_1H_A\,(x,y,z) + {}_1H_A\,(-x,y,z) + {}_1H_A\,(x,y,-z) + {}_1H_A\,(x,-y,z) \\
&\quad + {}_1H_A\,(-x,-y,z) + {}_1H_A\,(-x,y,-z) + {}_1H_A\,(x,-y,-z) + {}_1H_A\,(-x,-y,-z)\,;
\end{aligned}
$$

$$(3.3)\quad \frac{8\,a_2\,x}{c_1}\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+1}{2},\ \frac{a_2+2}{2};\ \ \frac{a_3}{2},\ \frac{a_3+1}{2};\ \ -;\\ -::\ \ -;\ \ \frac{c_2}{2},\ \frac{c_2+1}{2};\ \ -;\end{array}\right.$$

$$\left.\begin{array}{l} -;\ \ -;\ \ -;\\ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \ \frac{1}{2};\ \ \frac{1}{2};\end{array}\ \frac{x^2}{4},\ y^2,\ \frac{z^2}{4}\right];$$

$$\begin{aligned}&= {}_1H_A\,(x,y,z) - {}_1H_A\,(-x,y,z) + {}_1H_A\,(x,-y,z)\\ &\quad + {}_1H_A\,(x,y,-z) - {}_1H_A\,(-x,-y,z) - {}_1H_A\,(-x,y,-z)\\ &\quad + {}_1H_A\,(x,-y,-z) - {}_1H_A\,(-x,-y,-z)\,;\end{aligned}$$

$$(3.4)\quad \frac{8\,a_2\,a_3\,y}{c_2}\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+1}{2},\ \frac{a_2+2}{2};\ \ \frac{a_3+1}{2},\ \frac{a_3+2}{2};\ \ -;\\ -::\ \ -;\ \ \frac{c_2+1}{2},\ \frac{c_2+2}{2};\ \ -;\end{array}\right.$$

$$\left.\begin{array}{l} -;\ \ -;\ \ -;\\ \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2};\ \ \frac{3}{2};\ \ \frac{1}{2};\end{array}\ \frac{x^2}{4},\ y^2,\ \frac{z^2}{4}\right];$$

$$\begin{aligned}&= {}_1H_A\,(x,y,z) + {}_1H_A\,(-x,y,z) - {}_1H_A\,(x,-y,z)\\ &\quad + {}_1H_A\,(x,y,-z) - {}_1H_A\,(-x,-y,z) + {}_1H_A\,(-x,y,-z)\\ &\quad - {}_1H_A\,(x,-y,-z) - {}_1H_A\,(-x,-y,-z)\,;\end{aligned}$$

$$(3.5)\quad \frac{8\,a_3\,z}{c_2}\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2}{2},\ \frac{a_2+1}{2};\ \ \frac{a_3+1}{2},\ \frac{a_3+2}{2};\ \ -;\\ -::\ \ -;\ \ \frac{c_2+1}{2},\ \frac{c_2+2}{2};\ \ -;\end{array}\right.$$

$$\left.\begin{array}{l} -;\ \ -;\ \ -;\\ \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2};\ \ \frac{1}{2};\ \ \frac{3}{2};\end{array}\ \frac{x^2}{4},\ y^2,\ \frac{z^2}{4}\right];$$

$$\begin{aligned}&= {}_1H_A\,(x,y,z) + {}_1H_A\,(-x,y,z) - {}_1H_A\,(x,y,-z)\\ &\quad + {}_1H_A\,(x,-y,z) + {}_1H_A\,(-x,-y,z) - {}_1H_A\,(-x,y,-z)\\ &\quad - {}_1H_A\,(x,-y,-z) - {}_1H_A\,(-x,-y,-z)\,;\end{aligned}$$

$$(3.6)\quad \frac{8\,a_2\ (a_2+1)\ a_3\,x\,y}{c_1\,c_2}\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+2}{2},\ \frac{a_2+3}{2};\ \ \frac{a_3+1}{2},\ \frac{a_3+2}{2};\ \ -;\\ -::\ \ -;\ \ \frac{c_2+1}{2},\ \frac{c_2+2}{2};\ \ -;\end{array}\right.$$

$$\left.\begin{array}{l} -;\ \ -;\ \ -;\\ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \ \frac{3}{2};\ \ \frac{1}{2};\end{array}\ \frac{x^2}{4},\ y^2,\ \frac{z^2}{4}\right];$$

$$\begin{aligned}&= {}_1H_A\,(x,y,z) - {}_1H_A\,(-x,y,z) - {}_1H_A\,(x,-y,z)\\ &\quad + {}_1H_A\,(x,y,-z) + {}_1H_A\,(-x,-y,z) - {}_1H_A\,(-x,y,-z)\\ &\quad - {}_1H_A\,(x,-y,-z) + {}_1H_A\,(-x,-y,-z)\,;\end{aligned}$$

$$(3.7)\quad \frac{8\,a_2\,a_3\,x\,z}{c_1\,c_2}\,F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+1}{2},\ \frac{a_2+2}{2}; & \frac{a_3+1}{2},\ \frac{a_3+2}{2}; & -; \\ - :: & -; & \frac{c_2+1}{2},\ \frac{c_2+2}{2}; & -; \end{matrix}\right.$$
$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1+1}{2},\ \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\, y^2,\, \frac{z^2}{4}\right];$$
$$\begin{aligned} &= {}_1H_A\,(x,y,z) - {}_1H_A\,(-x,y,z) + {}_1H_A\,(x,-y,z) \\ &\quad - {}_1H_A\,(x,y,-z) - {}_1H_A\,(-x,-y,z) + {}_1H_A\,(-x,y,-z) \\ &\quad - {}_1H_A\,(x,-y,-z) + {}_1H_A\,(-x,-y,-z)\,; \end{aligned}$$

$$(3.8)\quad \frac{8\,a_2\,a_3\,(a_3+1)\,y\,z}{c_2\,(c_2+1)}\,F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+1}{2},\ \frac{a_2+2}{2}; & \frac{a_3+2}{2},\ \frac{a_3+3}{2}; & -; \\ - :: & -; & \frac{c_2+2}{2},\ \frac{c_2+3}{2}; & -; \end{matrix}\right.$$
$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1}{2},\ \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\, y^2,\, \frac{z^2}{4}\right];$$
$$\begin{aligned} &= {}_1H_A\,(x,y,z) + {}_1H_A\,(-x,y,z) - {}_1H_A\,(x,-y,z) \\ &\quad - {}_1H_A\,(x,y,-z) - {}_1H_A\,(-x,-y,z) - {}_1H_A\,(-x,y,-z) \\ &\quad + {}_1H_A\,(x,-y,-z) + {}_1H_A\,(-x,-y,-z)\,; \end{aligned}$$

$$(3.9)\quad \frac{8\,a_2\,(a_2+1)\,a_3\,(a_3+1)\,x\,y\,z}{c_1\,c_2\,(c_2+1)}\,F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+2}{2},\ \frac{a_2+3}{2}; & \frac{a_3+2}{2},\ \frac{a_3+3}{2}; & -; \\ - :: & -; & \frac{c_2+2}{2},\ \frac{c_2+3}{2}; & -; \end{matrix}\right.$$
$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1+1}{2},\ \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\, y^2,\, \frac{z^2}{4}\right];$$
$$\begin{aligned} &= {}_1H_A\,(x,y,z) - {}_1H_A\,(-x,y,z) - {}_1H_A\,(x,-y,z) \\ &\quad - {}_1H_A\,(x,y,-z) + {}_1H_A\,(-x,-y,z) + {}_1H_A\,(-x,y,-z) \\ &\quad + {}_1H_A\,(x,-y,-z) - {}_1H_A\,(-x,-y,-z)\,, \end{aligned}$$

where ${}_1H_A(x,y,z) := {}_1H_A\,(a_2,a_3;c_1,c_2;x,y,z)$ *and*

$$(3.10)\quad \begin{aligned} {}_1H_A\,(a_2,a_3;c_1,c_2;x,y,z) &= \lim_{\varepsilon\to 0} H_A\left(\frac{1}{\varepsilon},a_2,a_3;c_1,c_2;\varepsilon x,y,\varepsilon z\right) \\ &= \sum_{m,n,p=0}^{\infty} \frac{(a_2)_{m+n}\,(a_3)_{n+p}}{(c_1)_m\,(c_2)_{n+p}\,m!n!p!}\,x^m y^n z^p. \end{aligned}$$

For further specializations we start with observing the following limits:

Lemma 1. *Each of the following relationships holds true.*

$$\lim_{\varepsilon\to 0} F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2}{2},\frac{a_2+1}{2};\ \frac{a_3}{2},\frac{a_3+1}{2};\ \frac{1}{2\varepsilon},\frac{1+\varepsilon}{2\varepsilon};\\ -::\ -;\ \frac{c_2}{2},\frac{c_2+1}{2};\ -;\\ \quad -;\ -;\ -;\\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array} (\varepsilon x)^2, y^2, (\varepsilon z)^2\right] \tag{3.11}$$

$$= F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2}{2},\frac{a_2+1}{2};\ \frac{a_3}{2},\frac{a_3+1}{2};\ -;\\ -::\ -;\ \frac{c_2}{2},\frac{c_2+1}{2};\ -;\\ \quad -;\ -;\ -;\\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array} \frac{x^2}{4}, y^2, \frac{z^2}{4}\right];$$

$$\lim_{\varepsilon\to 0} {}_1H_A\left(\frac{1}{\varepsilon}, a_3; c_1, c_2; \varepsilon x, \varepsilon y, z\right) = {}_0F_1\left(c_1; x\right) {}_1F_1\left(a_3; c_2; y+z\right); \tag{3.12}$$

$$\lim_{\varepsilon\to 0} F^{(3)}\left[\begin{array}{l} -::\ \frac{1}{2\varepsilon},\frac{1+\varepsilon}{2\varepsilon};\ \frac{a_3}{2},\frac{a_3+1}{2};\ -;\\ -::\ -;\ \frac{c_2}{2},\frac{c_2+1}{2};\ -;\\ \quad -;\ -;\ -;\\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array} \frac{(\varepsilon x)^2}{4}, (\varepsilon y)^2, \frac{z^2}{4}\right] \tag{3.13}$$

$$= {}_0F_3\left(\frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \frac{x^2}{16}\right) F^{2:0;0}_{2:1;1}\left[\begin{array}{l} \frac{a_3}{2},\frac{a_3+1}{2}:\ -;\ -;\\ \frac{c_2}{2},\frac{c_2+1}{2}:\ \frac{1}{2}\ \frac{1}{2}; \end{array} \frac{y^2}{4}, \frac{z^2}{4}\right],$$

where $F^{2:0;0}_{2:1;1}$ *is a hypergeometric Kampé de Fériet function* (*see* [3, 33]) *of two variables defined by*

$$F^{2:0;0}_{2:1;1}\left[\begin{array}{l} a_1, a_2:\ -;\ -;\\ c_1, c_2:\ d;\ e; \end{array} y, z\right] = \sum_{n,p=0}^{\infty} \frac{(a_1)_{n+p}\,(a_2)_{n+p}}{(c_1)_{n+p}\,(c_2)_{n+p}\,(d)_n\,(e)_p\, n!p!} y^n z^p, \tag{3.14}$$

and ${}_pF_q$ *denotes the generalized hypergeometric function* (*see* [33]).

Setting $a_2 \sim 1/\varepsilon,\ x \sim \varepsilon x,\ \ y \sim \varepsilon y,$ in (3.1) to (3.9), and taking the limit of the resulting identities as $\varepsilon \to 0$, and using the identities in Lemma1, we get

Corollary 2. *Each of the following relationships holds true.*

$$(3.15)\quad {}_0F_1\left(c_1;x\right){}_1F_1\left(a_3;c_2;y+z\right)$$
$$=\left[{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)+\frac{x}{c_1}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)\right]$$
$$\cdot\left\{F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3}{2},\frac{a_3+1}{2}:\ -;\ -;\\ \frac{c_2}{2},\frac{c_2+1}{2}:\ \frac{1}{2};\ \frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]\right.$$
$$+\frac{a_3\,y}{c_2}F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}:\ -;\ -;\\ \frac{c_2+1}{2},\frac{c_2+2}{2}:\ \frac{3}{2};\ \frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$+\frac{a_3\,z}{c_2}F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}:\ -;\ -;\\ \frac{c_2+1}{2},\frac{c_2+2}{2}:\ \frac{1}{2};\ \frac{3}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$\left.+\frac{a_3\,(a_3+1)\,y\,z}{c_2\,(c_2+1)}F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3+2}{2},\frac{a_3+3}{2}:\ -;\ -;\\ \frac{c_2+2}{2},\frac{c_2+3}{2}:\ \frac{3}{2};\ \frac{3}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]\right\};$$

$$(3.16)\quad 8\,{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3}{2},\frac{a_3+1}{2}:\ -;\ -;\\ \frac{c_2}{2},\frac{c_2+1}{2}:\ \frac{1}{2};\ \frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$=\left[{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)\right]$$
$$\cdot\left[{}_1F_1\left(a_3;c_2;y+z\right)+{}_1F_1\left(a_3;c_2;y-z\right)+{}_1F_1\left(a_3;c_2;-y+z\right)+{}_1F_1\left(a_3;c_2;-y-z\right)\right];$$

$$(3.17)\quad \frac{8x}{c_1}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3}{2},\frac{a_3+1}{2}:\ -;\ -;\\ \frac{c_2}{2},\frac{c_2+1}{2}:\ \frac{1}{2};\ \frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$=\left[{}_0F_1\left(c_1;x\right)-{}_0F_1\left(c_1;-x\right)\right]$$
$$\cdot\left[{}_1F_1\left(a_3;c_2;y+z\right)+{}_1F_1\left(a_3;c_2;y-z\right)+{}_1F_1\left(a_3;c_2;-y+z\right)+{}_1F_1\left(a_3;c_2;-y-z\right)\right];$$

$$(3.18)\quad \frac{8a_3y}{c_2}{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}:\ -;\ -;\\ \frac{c_2+1}{2},\frac{c_2+2}{2}:\ \frac{3}{2};\ \frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$=\left[{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)\right]$$
$$\cdot\left[{}_1F_1\left(a_3;c_2;y+z\right)-{}_1F_1\left(a_3;c_2;-y+z\right)+{}_1F_1\left(a_3;c_2;y-z\right)-{}_1F_1\left(a_3;c_2;-y-z\right)\right];$$

$$(3.19)\quad 8\frac{a_3z}{c_2}{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)F^{2:0;0}_{2:1;1}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}:\ -;\ -;\\ \frac{c_2+1}{2},\frac{c_2+2}{2}:\ \frac{1}{2};\ \frac{3}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$= \left[{}_0F_1(c_1;x) + {}_0F_1(c_1;-x)\right]$$
$$\cdot \left[{}_1F_1(a_3;c_2;y+z) + {}_1F_1(a_3;c_2;-y+z) - {}_1F_1(a_3;c_2;y-z) - {}_1F_1(a_3;c_2;-y-z)\right];$$

(3.20)
$$\frac{8a_3xy}{c_1c_2}\,{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right) \cdot F^{2:0;0}_{2:1;1}\left[\begin{array}{l} \frac{a_3+1}{2},\frac{a_3+2}{2}: \;\; -; \;\; -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: \;\; \frac{3}{2}; \;\; \frac{1}{2}; \end{array} \frac{y^2}{4},\frac{z^2}{4}\right]$$

$$= \left[{}_0F_1(c_1;x) - {}_0F_1(c_1;-x)\right]$$
$$\cdot \left[{}_1F_1(a_3;c_2;y+z) - {}_1F_1(a_3;c_2;-y+z) + {}_1F_1(a_3;c_2;y-z) - {}_1F_1(a_3;c_2;-y-z)\right];$$

(3.21)
$$\frac{8a_3xz}{c_1c_2}\,{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right) \cdot F^{2:0;0}_{2:1;1}\left[\begin{array}{l} \frac{a_3+1}{2},\frac{a_3+2}{2}: \;\; -; \;\; -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: \;\; \frac{1}{2}; \;\; \frac{3}{2}; \end{array} \frac{y^2}{4},\frac{z^2}{4}\right]$$

$$= \left[{}_0F_1(c_1;x) - {}_0F_1(c_1;-x)\right]$$
$$\cdot \left[{}_1F_1(a_3;c_2;y+z) + {}_1F_1(a_3;c_2;-y+z) - {}_1F_1(a_3;c_2;y-z) - {}_1F_1(a_3;c_2;-y-z)\right];$$

(3.22)
$$\frac{8a_3(a_3+1)yz}{c_2(c_2+1)}\,{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right) \cdot F^{2:0;0}_{2:1;1}\left[\begin{array}{l} \frac{a_3+2}{2},\frac{a_3+3}{2}: \;\; -; \;\; -; \\ \frac{c_2+2}{2},\frac{c_2+3}{2}: \;\; \frac{3}{2}; \;\; \frac{3}{2}; \end{array} \frac{y^2}{4},\frac{z^2}{4}\right]$$

$$= \left[{}_0F_1(c_1;x) + {}_0F_1(c_1;-x)\right]$$
$$\cdot \left[{}_1F_1(a_3;c_2;y+z) - {}_1F_1(a_3;c_2;-y+z) - {}_1F_1(a_3;c_2;y-z) + {}_1F_1(a_3;c_2;-y-z)\right];$$

(3.23)
$$\frac{8a_3(a_3+1)xyz}{c_1c_2(c_2+1)}\,{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right) \cdot F^{2:0;0}_{2:1;1}\left[\begin{array}{l} \frac{a_3+2}{2},\frac{a_3+3}{2}: \;\; -; \;\; -; \\ \frac{c_2+2}{2},\frac{c_2+3}{2}: \;\; \frac{3}{2}; \;\; \frac{3}{2}; \end{array} \frac{y^2}{4},\frac{z^2}{4}\right]$$

$$= \left[{}_0F_1(c_1;x) - {}_0F_1(c_1;-x)\right]$$
$$\cdot \left[{}_1F_1(a_3;c_2;y+z) - {}_1F_1(a_3;c_2;-y+z) - {}_1F_1(a_3;c_2;y-z) + {}_1F_1(a_3;c_2;-y-z)\right].$$

Setting $a_3 \sim 1/\varepsilon,\ y \sim \varepsilon y,\ \ z \sim \varepsilon z$ in (3.15) to (3.23) and taking the limit of the resulting identities as $\varepsilon \to 0$, we find

Corollary 3. *Each of the following relationships holds true.*

$$(3.24)\qquad {}_0F_1(c_1;x)\,{}_0F_1(c_2;y+z) = \left[{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)+\frac{1}{c_1}xF_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)\right]$$
$$\cdot\left\{F^{2:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2}{2},\frac{c_2+1}{2}: & \frac{1}{2} & \frac{1}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right] + \frac{y}{c_2}F^{2:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: & \frac{3}{2} & \frac{1}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right]\right.$$
$$+\frac{z}{c_2}F^{2:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: & \frac{1}{2} & \frac{3}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right]$$
$$\left.+\frac{y\,z}{c_2\,(c_2+1)}F^{2:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2+2}{2},\frac{c_2+3}{2}: & \frac{3}{2} & \frac{3}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right]\right\};$$

$$(3.25)\qquad \frac{8x}{c_1}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)F^{0:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2}{2},\frac{c_2+1}{2}: & \frac{1}{2}; & \frac{1}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right]$$
$$= \left[{}_0F_1(c_1;x)-{}_0F_1(c_1;-x)\right]$$
$$\cdot\left[{}_0F_1(c_2;y+z)+{}_0F_1(c_2;y-z)+{}_0F_1(c_2;-y+z)+{}_0F_1(c_2;-y-z)\right];$$

$$(3.26)\qquad \frac{8y}{c_2}{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)F^{0:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: & \frac{3}{2}; & \frac{1}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right]$$
$$= \left[{}_0F_1(c_1;x)+{}_0F_1(c_1;-x)\right]$$
$$\cdot\left[{}_0F_1(c_2;y+z)-{}_0F_1(c_2;-y+z)+{}_0F_1(c_2;y-z)-{}_0F_1(c_2;-y-z)\right];$$

$$(3.27)\qquad \frac{8z}{c_2}{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)F^{0:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: & \frac{1}{2}; & \frac{3}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right]$$
$$= \left[{}_0F_1(c_1;x)+{}_0F_1(c_1;-x)\right]$$
$$\cdot\left[{}_0F_1(c_2;y+z)+{}_0F_1(c_2;-y+z)-{}_0F_1(c_2;y-z)-{}_0F_1(c_2;-y-z)\right];$$

$$(3.28)\qquad \frac{8xy}{c_1c_2}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)$$
$$\cdot F^{0:0;0}_{2:1;1}\left[\begin{matrix} -: & -; & -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: & \frac{3}{2}; & \frac{1}{2};\end{matrix}\ \frac{y^2}{16},\frac{z^2}{16}\right]$$

$$= \left[{}_0F_1(c_1;x) - {}_0F_1(c_1;-x)\right] \cdot \left[{}_0F_1(c_2;y+z) - {}_0F_1(c_2;-y+z) + {}_0F_1(c_2;y-z) - {}_0F_1(c_2;-y-z)\right];$$

(3.29)
$$\frac{8xz}{c_1c_2}\,{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right) \cdot F^{0:0;0}_{2:1;1}\left[\begin{array}{c} -: \quad -; \quad -; \\ \frac{c_2+1}{2},\frac{c_2+2}{2}: \frac{1}{2}; \frac{3}{2}; \end{array} \frac{y^2}{16},\frac{z^2}{16}\right]$$
$$= \left[{}_0F_1(c_1;x) - {}_0F_1(c_1;-x)\right] \cdot \left[{}_0F_1(c_2;y+z) + {}_0F_1(c_2;-y+z) - {}_0F_1(c_2;y-z) - {}_0F_1(c_2;-y-z)\right];$$

(3.30)
$$\frac{8yz}{c_2(c_2+1)}\,{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right) \cdot F^{0:0;0}_{2:1;1}\left[\begin{array}{c} -: \quad -; \quad -; \\ \frac{c_2+2}{2},\frac{c_2+3}{2}: \frac{3}{2}; \frac{3}{2}; \end{array} \frac{y^2}{16},\frac{z^2}{16}\right]$$
$$= \left[{}_0F_1(c_1;x) + {}_0F_1(c_1;-x)\right] \cdot \left[{}_0F_1(c_2;y+z) - {}_0F_1(c_2;-y+z) - {}_0F_1(c_2;y-z) + {}_0F_1(c_2;-y-z)\right];$$

(3.31)
$$\frac{8xyz}{c_1c_2(c_2+1)}\,{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right) \cdot F^{0:0;0}_{2:1;1}\left[\begin{array}{c} -: \quad -; \quad -; \\ \frac{c_2+2}{2},\frac{c_2+3}{2}: \frac{3}{2}; \frac{3}{2}; \end{array} \frac{y^2}{16},\frac{z^2}{16}\right]$$
$$= \left[{}_0F_1(c_1;x) - {}_0F_1(c_1;-x)\right] \cdot \left[{}_0F_1(c_2;y+z) - {}_0F_1(c_2;-y+z) - {}_0F_1(c_2;y-z) + {}_0F_1(c_2;-y-z)\right].$$

4. Special cases

For certain special cases of some identities in the previous sections, we introduce the case $a_3 = c_2$ of H_A as in the following lemma.

Lemma 2. *The function H_A when $a_3 = c_2$ is seen to reduce to a Gauss hypergeometric function ${}_2F_1 = F$:*

(4.1)
$$H_A(a_1,a_2,a_3;c_1,a_3;x,y,z) = (1-y)^{-a_2}(1-z)^{-a_1} F\left(a_1,a_2;c_1;\frac{x}{(1-y)(1-z)}\right).$$

Setting $a_3 = c_2$ in (2.2) and (2.4) to (2.11) and considering (4.1), we obtain

Corollary 4. *Each of the following relationships holds true.*

$$(4.2)\qquad (1-y)^{-a_2}\,(1-z)^{-a_1}\,F\left(a_1,a_2;c_1;\frac{x}{(1-y)\,(1-z)}\right)$$

$$= F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2}{2},\frac{a_2+1}{2};\ -;\ \frac{a_1}{2},\frac{a_1+1}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array} x^2,\,y^2,\,z^2\right]$$

$$+\frac{a_1\,a_2}{c_1}\,x\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+1}{2},\frac{a_2+2}{2};\ -;\ \frac{a_1}{2},\frac{a_1+1}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array} x^2,\,y^2,\,z^2\right]$$

$$+\,a_2\,y\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+1}{2},\frac{a_2+2}{2};\ -;\ \frac{a_1}{2},\frac{a_1+1}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array} x^2,\,y^2,\,z^2\right]$$

$$+\,a_1\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2}{2},\frac{a_2+1}{2};\ -;\ \frac{a_1+1}{2},\frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array} x^2,\,y^2,\,z^2\right]$$

$$+\frac{a_1\,a_2\,(a_2+1)}{c_1}\,x\,y\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+2}{2},\frac{a_2+3}{2};\ -;\ \frac{a_1+1}{2},\frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array} x^2,\,y^2,\,z^2\right]$$

$$+\frac{a_1\,(a_1+1)\,a_2}{c_1}\,x\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+1}{2},\frac{a_2+2}{2};\ -;\ \frac{a_1+2}{2},\frac{a_1+3}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array} x^2,\,y^2,\,z^2\right]$$

$$+\,a_1\,a_2\,y\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_2+1}{2},\frac{a_2+2}{2};\ -;\ \frac{a_1+1}{2},\frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{3}{2};\ \frac{3}{2}; \end{array} x^2,\,y^2,\,z^2\right]$$

$$+\frac{a_1\ (a_1+1)\ a_2\ (a_2+1)}{c_1}\, x\ y\, z\, F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+2}{2},\ \frac{a_2+3}{2};\ \ -;\ \ \frac{a_1+2}{2},\ \frac{a_1+3}{2}; \\ -:: \ \ -;\ \ -;\ \ -; \end{array}\right.$$

$$\left.\begin{array}{l} \qquad\qquad\qquad\qquad -;\ \ -;\ \ -; \\ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \ \frac{3}{2};\ \ \frac{3}{2}; \end{array} x^2,\ y^2,\ z^2\right];$$

(4.3)
$$8\, F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2}{2},\ \frac{a_2+1}{2};\ \ -;\ \ \frac{a_1}{2},\ \frac{a_1+1}{2};\ \ \qquad\qquad -;\ \ -;\ \ -; \\ -:: \ \ -;\ \ -;\ \ -;\ \ \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2};\ \ \frac{1}{2};\ \ \frac{1}{2}; \end{array} x^2,\ y^2,\ z^2\right]$$

$$\begin{aligned}
&= (1-y)^{-a_2}\,(1-z)^{-a_1} \\
&\quad \cdot\left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1-z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1-z)}\right)\right] \\
&+ (1-y)^{-a_2}\,(1+z)^{-a_1} \\
&\quad \cdot\left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1+z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1+y)\,(1-z)}\right)\right] \\
&+ (1+y)^{-a_2}\,(1-z)^{-a_1} \\
&\quad \cdot\left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)\,(1-z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1+z)}\right)\right] \\
&+ (1+y)^{-a_2}\,(1+z)^{-a_1} \\
&\quad \cdot\left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)\,(1+z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1+y)\,(1+z)}\right)\right];
\end{aligned}$$

(4.4)
$$\frac{8\, a_1\, a_2}{c_1}\, x\, F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+1}{2},\ \frac{a_2+2}{2};\ \ -;\ \ \frac{a_1}{2},\ \frac{a_1+1}{2}; \\ -:: \ \ -;\ \ -;\ \ -; \end{array}\right.$$

$$\left.\begin{array}{l} \qquad\qquad\qquad\qquad -;\ \ -;\ \ -; \\ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \ \frac{1}{2};\ \ \frac{1}{2}; \end{array} x^2,\ y^2,\ z^2\right]$$

$$\begin{aligned}
&= (1-y)^{-a_2}\,(1-z)^{-a_1} \\
&\quad \cdot\left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1-z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1-z)}\right)\right] \\
&- (1+y)^{-a_2}\,(1-z)^{-a_1} \\
&\quad \cdot\left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)\,(1-z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1+y)\,(1-z)}\right)\right] \\
&+ (1-y)^{-a_2}\,(1+z)^{-a_1} \\
&\quad \cdot\left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1+z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1+z)}\right)\right]
\end{aligned}$$

$$- (1+y)^{-a_2}(1+z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1+y)(1+z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1+y)(1+z)}\right)\right];$$

(4.5) $$8\,a_2\,y\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2+1}{2},\frac{a_2+2}{2};\ -;\ \frac{a_1}{2},\frac{a_1+1}{2};\ -;\ -;\ -;\\ -:: \ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array} x^2,y^2,z^2\right]$$

$$= (1-y)^{-a_2}(1-z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1-y)(1-z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1-y)(1-z)}\right)\right]$$

$$- (1+y)^{-a_2}(1-z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1+y)(1-z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1+y)(1-z)}\right)\right]$$

$$+ (1-y)^{-a_2}(1+z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1-y)(1+z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1-y)(1+z)}\right)\right]$$

$$- (1+y)^{-a_2}(1+z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1+y)(1+z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1+y)(1+z)}\right)\right];$$

(4.6) $$8\,a_1\,z\,F^{(3)}\left[\begin{array}{l} -:: \ \frac{a_2}{2},\frac{a_2+1}{2};\ -;\ \frac{a_1+1}{2},\frac{a_1+2}{2};\ -;\ -;\ -;\\ -:: \ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array} x^2,y^2,z^2\right]$$

$$= (1-y)^{-a_2}(1-z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1-y)(1-z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1-y)(1-z)}\right)\right]$$

$$+ (1+y)^{-a_2}(1-z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1+y)(1-z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1+y)(1-z)}\right)\right]$$

$$- (1-y)^{-a_2}(1+z)^{-a_1} \cdot \left[F\left(a_1,a_2;c_1;\frac{x}{(1-y)(1+z)}\right)+F\left(a_1,a_2;c_1;\frac{-x}{(1-y)(1+z)}\right)\right]$$

$$- (1+y)^{-a_2}\,(1+z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)\,(1+z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1+y)\,(1+z)}\right)\right];$$

(4.7)
$$\frac{8\,a_1\,a_2\,(a_2+1)}{c_1}\,x\,y\,F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+2}{2}, \frac{a_2+3}{2}; & -; & \frac{a_1+1}{2}, \frac{a_1+2}{2}; \\ - :: & -; & -; & -; \end{matrix}\right.$$
$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1+1}{2}, \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\; x^2,\, y^2,\, z^2\right]$$
$$= (1-y)^{-a_2}\,(1-z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1-z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1-z)}\right)\right]$$
$$- (1+y)^{-a_2}\,(1-z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)\,(1-z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1+y)\,(1-z)}\right)\right]$$
$$+ (1-y)^{-a_2}\,(1+z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1+z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1+z)}\right)\right]$$
$$- (1+y)^{-a_2}\,(1+z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)\,(1+z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1+y)\,(1+z)}\right)\right];$$

(4.8)
$$\frac{8\,a_1\,(a_1+1)\,a_2}{c_1}\,x\,z\,F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & -; & \frac{a_1+2}{2}, \frac{a_1+3}{2}; \\ - :: & -; & -; & -; \end{matrix}\right.$$
$$\left.\begin{matrix} & -; & -; & -; \\ \frac{c_1+1}{2}, \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\; x^2,\, y^2,\, z^2\right]$$
$$= (1-y)^{-a_2}\,(1-z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1-z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1-z)}\right)\right]$$
$$+ (1+y)^{-a_2}\,(1-z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)\,(1-z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1+y)\,(1-z)}\right)\right]$$
$$- (1-y)^{-a_2}\,(1+z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)\,(1+z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1-y)\,(1+z)}\right)\right]$$

$$- (1+y)^{-a_2} (1+z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)(1+z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1+y)(1+z)}\right)\right];$$

(4.9)
$$8\, a_1\, a_2\, y\, z\, F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & -; & \frac{a_1+1}{2}, \frac{a_1+2}{2}; \\ - :: & -; & -; & -; \\ \frac{c_1}{2}, \frac{c_1+1}{2}, & \begin{matrix} -; \\ \frac{1}{2}; \end{matrix} & \begin{matrix} -; \\ \frac{3}{2}; \end{matrix} & \begin{matrix} -; \\ \frac{3}{2}; \end{matrix} \end{matrix} x^2, y^2, z^2 \right]$$

$$= (1-y)^{-a_2} (1-z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)(1-z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1-y)(1-z)}\right)\right]$$
$$- (1+y)^{-a_2} (1-z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)(1-z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1+y)(1-z)}\right)\right]$$
$$- (1-y)^{-a_2} (1+z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)(1+z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1-y)(1+z)}\right)\right]$$
$$+ (1+y)^{-a_2} (1+z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)(1+z)}\right) + F\left(a_1, a_2; c_1; \frac{-x}{(1+y)(1+z)}\right)\right];$$

(4.10)
$$\frac{8\, a_1\, (a_1+1)\, a_2\, (a_2+1)}{c_1}\, x\, y\, z\, F^{(3)}\left[\begin{matrix} - :: & \frac{a_2+2}{2}, \frac{a_2+3}{2}; & -; & \frac{a_1+2}{2}, \frac{a_1+3}{2}; \\ - :: & -; & -; & -; \\ \frac{c_1+1}{2}, \frac{c_1+2}{2}, & \begin{matrix} -; \\ \frac{3}{2}; \end{matrix} & \begin{matrix} -; \\ \frac{3}{2}; \end{matrix} & \begin{matrix} -; \\ \frac{3}{2}; \end{matrix} \end{matrix} x^2, y^2, z^2 \right]$$

$$= (1-y)^{-a_2} (1-z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)(1-z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1-y)(1-z)}\right)\right]$$
$$- (1-y)^{-a_2} (1+z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1-y)(1+z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1-y)(1+z)}\right)\right]$$
$$- (1+y)^{-a_2} (1-z)^{-a_1} \cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)(1-z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1+y)(1-z)}\right)\right]$$

$$+ (1+y)^{-a_2} (1+z)^{-a_1}$$
$$\cdot \left[F\left(a_1, a_2; c_1; \frac{x}{(1+y)(1+z)}\right) - F\left(a_1, a_2; c_1; \frac{-x}{(1+y)(1+z)}\right)\right].$$

Similarly, setting $a_3 = c_2$ in formulas (3.15) and (3.16) to (3.23), we find

Corollary 5. *Each of the following relationships holds true.*

$$\tag{4.11} {}_0F_1(c_1; x)\, e^{y+z} = \left[{}_0F_3\left(\frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \frac{x^2}{16}\right) + \frac{x}{c_1}{}_0F_3\left(\frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2}; \frac{x^2}{16}\right)\right] \cdot \left[{}_0F_1\left(\frac{1}{2}; \frac{y^2}{4}\right) + y\,{}_0F_1\left(\frac{3}{2}; \frac{y^2}{4}\right)\right]\left[{}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right) + z\,{}_0F_1\left(\frac{3}{2}; \frac{z^2}{4}\right)\right];$$

$$\tag{4.12} 8\,{}_0F_3\left(\frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \frac{x^2}{16}\right){}_0F_1\left(\frac{1}{2}; \frac{y^2}{4}\right){}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right) = \left[{}_0F_1(c_1; x) + {}_0F_1(c_1; -x)\right]\left(e^y + e^{-y}\right)\left(e^z + e^{-z}\right);$$

$$\tag{4.13} \frac{8x}{c_1}{}_0F_3\left(\frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2}; \frac{x^2}{16}\right){}_0F_1\left(\frac{1}{2}; \frac{y^2}{4}\right){}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right) = \left[{}_0F_1(c_1; x) - {}_0F_1(c_1; -x)\right]\left(e^y + e^{-y}\right)\left(e^z + e^{-z}\right);$$

$$\tag{4.14} 8y\,{}_0F_3\left(\frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \frac{x^2}{16}\right){}_0F_1\left(\frac{3}{2}; \frac{y^2}{4}\right){}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right) = \left[{}_0F_1(c_1; x) + {}_0F_1(c_1; -x)\right]\left(e^y - e^{-y}\right)\left(e^z + e^{-z}\right);$$

$$\tag{4.15} 8z\,{}_0F_3\left(\frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \frac{x^2}{16}\right){}_0F_1\left(\frac{1}{2}; \frac{y^2}{4}\right){}_0F_1\left(\frac{3}{2}; \frac{z^2}{4}\right) = \left[{}_0F_1(c_1; x) + {}_0F_1(c_1; -x)\right]\left(e^y + e^{-y}\right)\left(e^z - e^{-z}\right);$$

$$\tag{4.16} \frac{8xy}{c_1}{}_0F_3\left(\frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2}; \frac{x^2}{16}\right){}_0F_1\left(\frac{3}{2}; \frac{y^2}{4}\right){}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right) = \left[{}_0F_1(c_1; x) - {}_0F_1(c_1; -x)\right]\left(e^y - e^{-y}\right)\left(e^z + e^{-z}\right);$$

$$\tag{4.17} \frac{8xz}{c_1}{}_0F_3\left(\frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2}; \frac{x^2}{16}\right){}_0F_1\left(\frac{1}{2}; \frac{y^2}{4}\right){}_0F_1\left(\frac{3}{2}; \frac{z^2}{4}\right) = \left[{}_0F_1(c_1; x) - {}_0F_1(c_1; -x)\right]\left(e^y + e^{-y}\right)\left(e^z - e^{-z}\right);$$

$$8yz\,{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right){}_0F_1\left(\frac{3}{2};\frac{y^2}{4}\right){}_0F_1\left(\frac{3}{2};\frac{z^2}{4}\right) = \left[{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)\right]\left(e^y-e^{-y}\right)\left(e^z-e^{-z}\right); \tag{4.18}$$

$$\frac{8xyz}{c_1}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right){}_0F_1\left(\frac{3}{2};\frac{y^2}{4}\right){}_0F_1\left(\frac{3}{2};\frac{z^2}{4}\right) = \left[{}_0F_1\left(c_1;x\right)-{}_0F_1\left(c_1;-x\right)\right]\left(e^y-e^{-y}\right)\left(e^z-e^{-z}\right); \tag{4.19}$$

$$\frac{x}{c_1}\frac{{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)}{{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)} = \frac{{}_0F_1\left(c_1;x\right)-{}_0F_1\left(c_1;-x\right)}{{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)}; \tag{4.20}$$

$$y\frac{{}_0F_1\left(\frac{3}{2};\frac{y^2}{4}\right)}{{}_0F_1\left(\frac{1}{2};\frac{y^2}{4}\right)} = \frac{e^y-e^{-y}}{e^y+e^{-y}}. \tag{4.21}$$

Setting $a_2 = c_1$ in formulas (4.2) to (4.10), we can also express elementary power functions in terms of the Srivastava's function $F^{(3)}$ and vice-versa.

Corollary 6. *Each of the following relationships holds true.*

$$(1-y)^{a_1-c_1}\left[(1-y)(1-z)-x\right]^{-a_1} \tag{4.22}$$

$$= F^{(3)}\left[\begin{matrix} - :: & \frac{c_1}{2},\ \frac{c_1+1}{2}; & -; & \frac{a_1}{2},\ \frac{a_1+1}{2}; & -; & -; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2}; & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ x^2,\ y^2,\ z^2\right]$$

$$+ a_1\,x\,F^{(3)}\left[\begin{matrix} - :: & \frac{c_1+1}{2},\ \frac{c_1+2}{2}; & -; & \frac{a_1}{2},\ \frac{a_1+1}{2}; & -; & -; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2}; & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ x^2,\ y^2,\ z^2\right]$$

$$+ c_1\,y\,F^{(3)}\left[\begin{matrix} - :: & \frac{c_1+1}{2},\ \frac{c_1+2}{2}; & -; & \frac{a_1}{2},\ \frac{a_1+1}{2}; & -; & -; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2}; & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\ x^2,\ y^2,\ z^2\right]$$

$$+\,a_1\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1}{2},\frac{c_1+1}{2};\ -;\ \frac{a_1+1}{2},\frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array}\ x^2,\,y^2,\,z^2\right]$$

$$+\,a_1\,(c_1+1)\,x\,y\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1+2}{2},\frac{c_1+3}{2};\ -;\ \frac{a_1+1}{2},\frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array}\ x^2,\,y^2,\,z^2\right]$$

$$+\,a_1\,(a_1+1)\,x\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1+1}{2},\frac{c_1+2}{2};\ -;\ \frac{a_1+2}{2},\frac{a_1+3}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array}\ x^2,\,y^2,\,z^2\right]$$

$$+\,a_1\,c_1\,y\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1+1}{2},\frac{c_1+2}{2};\ -;\ \frac{a_1+1}{2},\frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{3}{2};\ \frac{3}{2}; \end{array}\ x^2,\,y^2,\,z^2\right]$$

$$+\,a_1\,(a_1+1)\,(c_1+1)\,x\,y\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1+2}{2},\frac{c_1+3}{2};\ -;\ \frac{a_1+2}{2},\frac{a_1+3}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\ \frac{3}{2};\ \frac{3}{2}; \end{array}\ x^2,\,y^2,\,z^2\right];$$

(4.23)
$$8\,c_1\,y\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1}{2},\frac{c_1+1}{2};\ -;\ \frac{a_1}{2},\frac{a_1+1}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \end{array}\ x^2,\,y^2,\,z^2\right]$$
$$= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1}+[(1-y)(1-z)+x]^{-a_1}\right\}$$
$$+\,(1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1}+[(1-y)(1+z)+x]^{-a_1}\right\}$$
$$+\,(1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1}+[(1+y)(1-z)+x]^{-a_1}\right\}$$
$$+\,(1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1}+[(1+y)(1+z)+x]^{-a_1}\right\};$$

$$(4.24)\qquad 8\,a_1\,x\,F^{(3)}\left[\begin{matrix} - :: & \frac{c_1+1}{2},\ \frac{c_1+2}{2}; & -; & \frac{a_1}{2},\ \frac{a_1+1}{2}; & & -; & -; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2},\ \frac{c_1+2}{2}, & \frac{3}{2}; & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ x^2,\ y^2,\ z^2\right]$$

$$\begin{aligned} &= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1} - [(1-y)(1-z)+x]^{-a_1}\right\} \\ &\quad + (1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1} - [(1+y)(1-z)+x]^{-a_1}\right\} \\ &\quad + (1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1} - [(1-y)(1+z)+x]^{-a_1}\right\} \\ &\quad + (1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1} - [(1+y)(1+z)+x]^{-a_1}\right\}; \end{aligned}$$

$$(4.25)\qquad 8\,c_1\,y\,F^{(3)}\left[\begin{matrix} - :: & \frac{c_1+1}{2},\ \frac{c_1+2}{2}; & -; & \frac{a_1}{2},\ \frac{a_1+1}{2}; & & -; & -; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2},\ \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\ x^2,\ y^2,\ z^2\right]$$

$$\begin{aligned} &= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1} + [(1-y)(1-z)+x]^{-a_1}\right\} \\ &\quad - (1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1} + [(1+y)(1-z)+x]^{-a_1}\right\} \\ &\quad + (1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1} + [(1-y)(1+z)+x]^{-a_1}\right\} \\ &\quad - (1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1} + [(1+y)(1+z)+x]^{-a_1}\right\}; \end{aligned}$$

$$(4.26)\qquad 8\,a_1\,z\,F^{(3)}\left[\begin{matrix} - :: & \frac{c_1}{2},\ \frac{c_1+1}{2}; & -; & \frac{a_1+1}{2},\ \frac{a_1+2}{2}; & & -; & -; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2},\ \frac{c_1+1}{2}, & \frac{1}{2}; & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\ x^2,\ y^2,\ z^2\right]$$

$$\begin{aligned} &= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1} + [(1-y)(1-z)+x]^{-a_1}\right\} \\ &\quad + (1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1} + [(1+y)(1-z)+x]^{-a_1}\right\} \\ &\quad - (1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1} + [(1-y)(1+z)+x]^{-a_1}\right\} \\ &\quad - (1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1} + [(1+y)(1+z)+x]^{-a_1}\right\}; \end{aligned}$$

$$
(4.27)\qquad 8\,a_1\,(c_1+1)\,x\,y\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1+2}{2},\ \frac{c_1+3}{2};\ -;\ \frac{a_1+1}{2},\ \frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \frac{3}{2};\ \frac{1}{2}; \end{array} x^2,\ y^2,\ z^2\right]
$$

$$
\begin{aligned}
&= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1} - [(1-y)(1-z)+x]^{-a_1}\right\} \\
&\quad - (1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1} - [(1+y)(1-z)+x]^{-a_1}\right\} \\
&\quad + (1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1} - [(1-y)(1+z)+x]^{-a_1}\right\} \\
&\quad - (1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1} - [(1+y)(1+z)+x]^{-a_1}\right\};
\end{aligned}
$$

$$
(4.28)\qquad 8\,a_1\,(a_1+1)\,x\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1+1}{2},\ \frac{c_1+2}{2};\ -;\ \frac{a_1+2}{2},\ \frac{a_1+3}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \frac{1}{2};\ \frac{3}{2}; \end{array} x^2,\ y^2,\ z^2\right]
$$

$$
\begin{aligned}
&= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1} - [(1-y)(1-z)+x]^{-a_1}\right\} \\
&\quad + (1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1} - [(1+y)(1-z)+x]^{-a_1}\right\} \\
&\quad - (1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1} - [(1-y)(1+z)+x]^{-a_1}\right\} \\
&\quad - (1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1} - [(1+y)(1+z)+x]^{-a_1}\right\};
\end{aligned}
$$

$$
(4.29)\qquad 8\,a_1\,c_1\,y\,z\,F^{(3)}\left[\begin{array}{l} -::\ \frac{c_1+1}{2},\ \frac{c_1+2}{2};\ -;\ \frac{a_1+1}{2},\ \frac{a_1+2}{2};\ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2};\ \frac{3}{2};\ \frac{3}{2}; \end{array} x^2,\ y^2,\ z^2\right]
$$

$$
\begin{aligned}
&= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1} + [(1-y)(1-z)+x]^{-a_1}\right\} \\
&\quad - (1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1} + [(1+y)(1-z)+x]^{-a_1}\right\} \\
&\quad - (1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1} + [(1-y)(1+z)+x]^{-a_1}\right\} \\
&\quad + (1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1} + [(1+y)(1+z)+x]^{-a_1}\right\};
\end{aligned}
$$

$$(4.30) \quad 8\,a_1\,(a_1+1)\,(c_1+1)\,x\,y\,z\,F^{(3)}\left[\begin{array}{l} -:: \frac{c_1+2}{2}, \frac{c_1+3}{2};\; -;\; \frac{a_1+2}{2}, \frac{a_1+3}{2};\\ -:: \quad -;\; -;\; -;\\ \frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2};\; \begin{array}{c}-;\\ \frac{3}{2};\end{array}\; \begin{array}{c}-;\\ \frac{3}{2};\end{array}\; \begin{array}{c}-;\\ \frac{3}{2};\end{array}\; x^2, y^2, z^2 \end{array}\right]$$
$$= (1-y)^{a_1-c_1}\left\{[(1-y)(1-z)-x]^{-a_1} - [(1-y)(1-z)+x]^{-a_1}\right\}$$
$$- (1-y)^{a_1-c_1}\left\{[(1-y)(1+z)-x]^{-a_1} - [(1-y)(1+z)+x]^{-a_1}\right\}$$
$$- (1+y)^{a_1-c_1}\left\{[(1+y)(1-z)-x]^{-a_1} - [(1+y)(1-z)+x]^{-a_1}\right\}$$
$$+ (1+y)^{a_1-c_1}\left\{[(1+y)(1+z)-x]^{-a_1} - [(1+y)(1+z)+x]^{-a_1}\right\}.$$

5. Concluding remarks

We note that in a specialized parameters we can easily obtain many interesting functional relations from the identities established here. For instance, at $x = 0$ and $z = 0$ from (2.2) and (2.4) to (2.11) we can get decompositions for Appell's functions F_1 and F_2 in terms of Srivastava's function $F^{(3)}$.
Applying this method to some other special functions, instead of Srivastava's functions H_A and $F^{(3)}$, defined by power series, interested readers can find certain other unexpected functional relations.

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