LIMIT WEAK SHADOWABLE TRANSITIVE SETS OF C^1-GENERIC DIFFEOMORPHISMS

Manseob Lee and Gang Lu

Abstract. In this paper, we prove that locally maximal transitive set of a C^1-generic diffeomorphism is hyperbolic if and only if it is limit weak shadowable.

1. Introduction

The theory of shadowing was developed intensively in recent years and became a significant part of the qualitative theory of dynamical systems containing a lot of interesting and deep results. The weak shadowing property is investigated in [7, 8, 10, 11], and a remarkable example having the property is treated in [9] at first. In fact, every homeomorphism having the shadowing property has the weak shadowing property but its converse is not true. An irrational rotation map ρ on the unit circle has the weak shadowing property, but ρ does not have the shadowing property. Many recent papers explored their "hyperbolic-like" properties (for more details, see [1, 2, 4, 5]). In this paper, we study the hyperbolicity of limit weak shadowable transitive sets of C^1-generic diffeomorphism.

Let M be a C^∞ closed Riemannian manifold, and d be the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let Diff(M) denote the set of C^1 diffeomorphism on M endowed with the C^1 topology.

For $f \in \text{Diff(M)}$, denote by $\mathcal{O}_f(x)$ the f-orbit $\{f^i(x)\}_{i\in\mathbb{Z}}$ of $x \in M$. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in M is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{Z}$. Given $\varepsilon > 0$, $\{x_i\}_{i\in\mathbb{Z}}$ is said to be ε-*shadowed* by $y \in M$ if $d(f^i(y), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. For a closed f-invariant set $\Lambda \subset M$, we say that f has the *shadowing property* on Λ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit of f in Λ can be ε-shadowed by some points. Notice

that only δ-pseudo orbits of f contain in Λ are allowed to be ε-shadowed, but the shadowing point $y \in M$ is not necessarily contained in Λ.

Given $\varepsilon > 0$, $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ is said to be *weakly ε-shadowed* by $y \in M$ if $d(\mathcal{O}_f(y), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. We say that f has the (usual) *weak shadowing property* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit of f can be weakly ε-shadowed by some point, that is $\{x_i\}_{i \in \mathbb{Z}} \subset B(\mathcal{O}_f(y), \varepsilon)$. From now, we introduce the notion of the limit weak shadowing property which was studied in [12]. At first, we introduce the following notions which is called the limit shadowing property. Eirola *et al.* in [3] has defined that f has the *limit shadowing property* if for any sequence $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ such that $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$, there exists a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. We say f has the *limit weak shadowing property* on Λ if for any sequence $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ with $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$, then there exists a point $y \in M$ such that $d(\mathcal{O}_f(y), x_i) \to 0$ as $i \to \pm\infty$.

In [12], the author showed that there is a diffeomorphism f on 2 dimensional torus belonging to the C^1-interior of the set of diffeomorphisms possessing the limit weak shadowing property such that f does not satisfy the strong transversality condition. For the usual weak shadowing property, the existence of such the map has already known. Especially, in this paper we study that dimension of whole space is any dimension which is more general for the results of Sakai ([12]).

A closed f-invariant set $\Lambda \subset M$ is said to be *transitive* if there is a point $x \in \Lambda$ such that the ω-limit set $\omega(x)$ of x coincides with Λ. We say that Λ is *locally maximal* if there is an open neighborhood U of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$.

We say that a closed f-invariant set $\Lambda \subset M$ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s(x)}\| \leq C\lambda^n$$

and

$$\|Df^{-n}|_{E^u(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. Moreover, we say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for any $x \in \Lambda$ and $n \geq 0$.

We say that a subset $\mathcal{R} \subset \mathrm{Diff(M)}$ is *residual* if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff(M)}$; in this case $\mathcal{R}$ is dense in $\mathrm{Diff(M)}$. A property (P) is said to be (C^1)-*generic* if (P) holds for all diffeomorphisms which belong to some residual subset of $\mathrm{Diff(M)}$.

Recently, in [2] Abdenur and Díaz proved that for a locally maximal transitive set Λ of a generic diffeomorphisms f then either Λ is hyperbolic, or there

are a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood V of Λ such that every $g \in \mathcal{U}(f)$ does not have the shadowing property on the neighborhood V (for more details, [2, Theorem 3]). In this paper, the following result is obtained.

Theorem 1.1. *A locally maximal transitive set of a C^1-generic diffeomorphism is hyperbolic if and only if it is limit weak shadowable.*

2. Proof of Theorem 1.1

In dynamical systems, the periodic orbit plays an important role. Many dynamical invariants are associated to them. In fact, they also can be followed after perturbation of the dynamics.

Next, we present some results of this theory which will be used in the proof of Theorem 1.1. The following is well known result which is so-called Kupka-Samle Theorem.

Lemma 2.1. *There is a residual set $\mathcal{R}_1 \subset \mathrm{Diff(M)}$ such that every $f \in \mathcal{R}_1$ satisfies the following property:*

  (I) *every periodic point of f is hyperbolic,*
  (II) *if p and q are periodic points of f, then $W^s(p)$ is transversal to $W^u(q)$.*

From now, we may assume that Λ is a nontrivial transitive set which means not one orbit. Then we can get the following result.

Lemma 2.2 ([13, Corollary 2.7.1])**.** *Let Λ be a nontrivial transitive set of f. There are a sequence of diffeomorphisms $\{f_n\}$ and a sequence of points $\{p_n\}$ such that p_n is a periodic point of f_n and $\lim f_n = f$ and $\lim \mathcal{O}(p_n) = \Lambda$.*

Lemma 2.3 ([5, Lemma 2.2])**.** *There is a residual set $\mathcal{R}_2 \subset \mathrm{Diff(M)}$ such that every $f \in \mathcal{R}_2$ satisfies the following property: For any closed f-invariant set $\Lambda \subset M$, if there are a sequence of diffeomorphisms f_n converging to f and a sequence of hyperbolic periodic orbits $\mathcal{O}(p_n)$ of f_n with index k verifying $\lim_{n \to \infty} \mathcal{O}(p_n) = \Lambda$, then there is a sequence of hyperbolic periodic orbits $\mathcal{O}(q_n)$ of f with index k such that Λ is the Hausdorff limit of $\mathcal{O}(q_n)$.*

We say that a point x in M is *well closable* for f if for any $\varepsilon > 0$, there are $g \in \mathrm{Diff(M)}$ with $d_{C^1}(g, f) < \varepsilon$ and a periodic point p of g such that $d(f^n(x), g^n(p)) < \varepsilon$ for all $0 \le n \le \pi(p)$, where d_{C^1} is the usual C^1-metric, and $\pi(p)$ is the period of p. Let Σ_f denote the set of well closable points of f. Mañé's ergodic closing lemma ([6]) says that $\mu(\Sigma_f) = 1$ for any f-invariant Borel probability measure μ on M.

Let $\mathcal{M}$ be the space of all Borel measures μ on M with the weak* topology. It is easy to check that, for any ergodic measure $\mu \in \mathcal{M}$ of f, μ is supported on a periodic orbit $\mathcal{O}(p) = \{p, f(p), \dots, f^{\pi(p)-1}(p)\}$ of f if and only if

$$\mu = \frac{1}{\pi(p)} \sum_{i=0}^{\pi(p)-1} \delta_{f^i(p)},$$

where δ_x is the atomic measure respecting x.

The following lemma comes from the Mañé's ergodic closing lemma which gives the measure theoretical viewpoint on the approximation by periodic orbits.

Lemma 2.4 ([5, Lemma 2.3]). *There is a residual set $\mathcal{R}_3 \subset \mathrm{Diff}(M)$ such that every $f \in \mathcal{R}_3$ satisfies the following property: Any ergodic invariant measure μ of f is the limit of sequence of ergodic invariant measures supported by periodic orbits $\mathcal{O}(p_n)$ of f in the weak* topology. Moreover, the orbits $\mathcal{O}(p_n)$ converges to the support of μ in the Hausdorff topology.*

The *stable manifold* $W^s(p)$ and the *unstable manifold* $W^u(p)$ of p with respect to f are defined as usual. Let $p, q \in P(f)$ be a saddle. We say that p and q are *homoclinically related*, and write $p \sim q$ if $W^s(p)$ (resp. $W^u(p)$) and $W^u(q)$ (resp. $W^s(q)$) have non-empty transverse intersections. It is clear that if $p \sim q$, then $\mathrm{index}(p) = \mathrm{index}(q)$. Here $\mathrm{index}(p)$ is the *index* of p, namely, the dimension of the stable eigenspace E^s_p of p. The following lemma, we can know that every periodic point of a limit weak shadowable transitive set Λ of $f \in \mathcal{R}_1$ has the same index.

Lemma 2.5. *Let $f \in \mathcal{R}_1$, and Λ be a limit weak shadowable transitive set of f. Then all periodic points in Λ have the same index.*

Proof. Let p and q be two periodic points of f in Λ, and let $\varepsilon > 0$ be a small constant such that the local stable manifold $W^s_\varepsilon(p) = \{x \in M : d(f^n(x), f^n(p)) \leq \varepsilon, n \geq 0\}$ and the local unstable manifold $W^u_\varepsilon(q) = \{x \in M : d(f^{-n}(x), f^{-n}(p)) \leq \varepsilon, n \geq 0\}$ are well defined. To simplify notation in this proof, we may assume that $f(p) = p$ and $f(q) = q$. Take $\varepsilon = \{\varepsilon(p), \varepsilon(q)\}$. Since Λ is transitive, there exists a point $x \in \Lambda$, $\omega(x) = \Lambda$. Then for this ε, there exist $l > 0$ and $k > 0$, such that

$$d(f^l(x), p) < \varepsilon \quad \text{and} \quad d(f^k(x), q) < \varepsilon.$$

Without loss of generality, we may assume that $l < k$. Then one can construct a sequence ξ_1 in Λ as follows:

$$\xi = \{p, f^l(x), f^{l+1}(x), \ldots, f^{k-1}(x), q\}.$$

We extend ξ_1 as follows:

- $x_i = q$ for $i \leq 0$;
- $x_i = f^{l+i-1}(x)$ for $1 \leq i \leq k - l$;
- $x_i = p$ for $i \geq k - l + 1$.

Then

$$\xi_1 = \{\ldots, p, x_1, x_2, \ldots, x_{k-l-1}, q, q \ldots\},$$

and ξ_1 is a limit sequence of f. That is,

$$d(f(x_i), x_{i+1}) \to \infty \quad \text{as} \quad i \to \pm\infty.$$

Clearly, $\xi_1 \subset \Lambda$. By the limit weak shadowing property, there is a point $y \in M$ such that $d(\mathcal{O}_f(y), x_i) \to 0$ as $i \to \pm\infty$. Thus $\mathcal{O}_f(y) \cap W_\varepsilon^s(q) \neq \emptyset$ and $\mathcal{O}_f(y) \cap W_\varepsilon^u(p) \neq \emptyset$. Therefore,

$$y \in W^s(q) \cap W^u(p).$$

By Lemma 2.1, we know that the index of p and index of q should be same. Otherwise it will contradicts the fact that the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ are transverse, and so completes the proof. $\qquad \square$

Now we construct the residual subset $\mathcal{R}$ of Diff(M) required in the statement of Theorem 1.1 as follow: $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$. Then we have the following proposition which is important to prove Theorem 1.1.

Proposition 2.6. *Let $f \in \mathcal{R}$, and let Λ be a limit weak shadowable transitive set of f which is locally maximal. Then there exist constants $K > 0$, $m > 0$ and $0 < \lambda < 1$ such that for any periodic point $p \in \Lambda$,*

$$\prod_{i=0}^{k-1} \|Df^m|_{E^s(f^{im}(p))}\| < K\lambda^k,$$

$$\prod_{i=0}^{k-1} \|Df^{-m}|_{E^u(f^{-im}(p))}\| < K\lambda^k$$

and

$$\|Df^m|_{E^s(p)}\| \cdot \|Df^{-m}|_{E^u(f^m(p))}\| < \lambda^2,$$

where $k = [\pi(p)/m]$.

Proof. See [5, Proposition 2.1] $\qquad \square$

End of the proof of Theorem 1.1. Let $f \in \mathcal{R}$, and let U be a neighborhood of Λ such that $\bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda$. By Lemma 2.1 and Proposition 2.6, we can that Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ which satisfies $E(p) = E^s(p)$ and $F(p) = E^u(p)$ for every periodic point $p \in \Lambda$. To complete the proof of Theorem 1.1, it is enough to show that Df^m is contracting on E, and Df^m is expanding on F if Λ is limit weak shadowable for f. To derive a contradiction, we may assume that Df^m is not contracting on E. If Df^m is not contracting on E, then there are a subsequence $\{j_n\}_{n \in \mathbb{N}}$ and an f^m-invariant probability measure μ on Λ sun that

$$\int_\Lambda \log \|D_x f^m|_{E_x}\| d\mu = \lim_{n \to \infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log \|D_{f^{mi}(x)} f^m|_{f^{mi}}\| \geq 0.$$

By Birkhoff's theorem, and Mañé's ergodic closing lemma, we can find $y \in \Lambda \cap \Sigma_f$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{f^{mi}(y)} f^m|_{E_{f^{im}(y)}}\| \geq 0$$

(see [6, pp. 521–522]). Here Σ_f is the set of Mañé's ergodic closing lemma. By Proposition 2.6, one may see $y \notin P(f)$. Take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{f^{im}(y)} f^m|_{E_{f^{im}(y)}}\| \geq \log \lambda_0$$

when $n > n_0$. Then, by Mañé's ergodic closing lemma we can find $g \in \mathcal{U}(f)$ ($g = f$ on $M \setminus U_p$) and $\tilde{y} \in \bigcap_{n \in \mathbb{N}} g^n(U_p) \cap P(g)$ nearby y, where U_p is the neighborhood of the orbit of p and $U_p \subset U$. Moreover $\text{index}(\tilde{y}) = \text{index}(p)$ by Lemma 2.5. By applying Franks' lemma, one can construct $\tilde{g} \in \mathcal{U}(f)$ C^1-close to g such that

$$\lambda_0^k \leq \prod_{i=0}^{k-1} |D_{\tilde{g}^{im}(\tilde{y})} \tilde{g}^m|_{E_{\tilde{g}^{im}(\tilde{y})}}\|$$

(see, [6, p. 523]). On the other hand, by Proposition 2.6, one can get

$$\prod_{i=0}^{k-1} \|D_{\tilde{g}^{im}(\tilde{y})} \tilde{g}^m|_{E_{\tilde{g}^{mi}(\tilde{y})}}\| < K\lambda^k.$$

One can choose the period $\pi(\tilde{y})(> n_0)$ of $\tilde{y}$ as large as $\lambda_0^k \geq K\lambda^k$. Here $k = [\pi(\tilde{y})/m]$. This is a contradiction. One proves that Df^m is contracting on E. Similarly one can show that Df^m is expanding on F. $\square$

References

[1] F. Abdenur, *Generic robustness of spectral decompositions*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 2, 213–224.

[2] F. Abdenur and L. J. Díaz, *Pseudo-orbit shadowing in the C^1 topology*, Discrete Contin. Dyn. Syst. **17** (2007), no. 2, 223–245.

[3] T. Eirola, O. Nevanlinna, and S. Pilyugin, *Limit shadowing property*, Numer. Funct. Anal. Optim. **18** (1997), no. 1-2, 75–92.

[4] K. Lee and M. Lee, *Hyperbolic of C^1-stably expansive homoclinic classes*, Discrete Contin. Dyn. Syst. **27** (2010), no. 3, 1133–1145.

[5] K. Lee and X. Wen, *Shadowable chain transitive sets of C^1-generic diffeomorphisms*, to apear in Bull. Korean. Math. Soc.

[6] R. Mañé, *An ergodic closing lemma*, Ann. of Math. (2) **116** (1982), no. 3, 503–540.

[7] S. Pilyugin, *Shadowing in Dynamical Systems*, Lecture Notes in Math., 1706, Springer 1999.

[8] S. Pilyugin, K. Sakai, and O. Tarakanov, *Transversality properties and C^1-open sets of diffeomorphisms with weak shadowing*, Discrete Contin. Dyn. Syst. **16** (2006), no. 4, 871–882.

[9] O. B. Plamenevskaya, *Weak shadowing in two-dimensional diffeomorphisms*, Vestnik St. Petersburg Univ. Math. **31** (1999), no. 3, 49–56.

[10] K. Sakai, *Diffeomorphisms with weak shadowing*, Fund. Math. **168** (2001), no. 1, 57–75.

[11] ———, *A note on weak shadowing*, Far East J. Dyn. Syst. **3** (2001), no. 1, 45–49.

[12] ———, *Diffeomorphisms with limit weak shadowing*, Trends in Mathematics **7** (2004), 19–27.

[13] D. Yang, *Stably weakly shadowing transitive sets and dominated splittings*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2747–2751.

MANSEOB LEE
DEPARTMENT OF MATHEMATICS
MOKWON UNIVERSITY
DAEJEON 302-729, KOREA
E-mail address: lmsds@mokwon.ac.kr

GANG LU
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE
SHENYANG UNIVERSITY OF TECHNOLOGY
SHENYANG 110178, P. R. CHINA
E-mail address: lvgang1234@hanmail.net