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A NOTE ON WEYL'S THEOREM FOR $*$-PARANORMAL OPERATORS

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Abstract. In this note we investigate Weyl's theorem for $*$-paranormal operators on a separable infinite dimensional Hilbert space. We prove that if T is a $*$-paranormal operator satisfying Property (E) - $(T-\lambda I)H_T(\{\lambda\})$ is closed for each $\lambda \in \mathbb{C}$, where $H_T(\{\lambda\})$ is a local spectral subspace of T, then Weyl's theorem holds for T.

1. Introduction

Let H denote an infinite dimensional separable Hilbert space. Let $B(H)$ and $K(H)$ denote the algebra of bounded linear operators and the ideal of compact operators on H, respectively. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of T; $\alpha(T) := \dim N(T)$; $\beta(T) := \dim N(T^*)$; $\sigma(T)$ for the spectrum of T; $\sigma_{ap}(T)$ for the approximate point spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T.

An operator $T \in B(H)$ is called *Fredholm* if it has closed range with finite dimensional null space and its range of finite co-dimension. The *index* of a Fredholm operator $T \in B(H)$ is given by

$$\mathrm{ind}\,(T) := \alpha(T) - \beta(T).$$

An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in B(H)$ is called *Browder* if it is Fredholm "of finite ascent and descent": equivalently ([11, Theorem 7.9.3]) if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined by ([10], [11], [12])

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\};$$
$$\omega(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\};$$
$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\} :$$

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evidently,

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$.

If we write iso $K := K \setminus \operatorname{acc} K$, then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$$

denote the set of isolated eigenvalues of finite multiplicity.

To say that "Weyl's theorem holds" for an operator $T \in B(H)$ is to claim that

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T), \tag{1.1}$$

in other words, the complement in the spectrum of the Weyl spectum is precisely the isolated points of the spectrum which are eigenvalues of finite multiplicity.

H. Weyl ([16]) has shown that the equality (1.1) holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators and to Toeplitz operators ([7]), and to several classes of operators including seminormal operators ([5], [6]).

An operator $T \in B(H)$ is said to be *paranormal* if

$$||Tx||^2 \leq ||T^2x|| \quad \text{for every unit vector } x \in H,$$

and an operator $T \in B(H)$ is said to be $*$-*paranormal* if

$$||T^*x||^2 \leq ||T^2x|| \quad \text{for every unit vector } x \in H.$$

S. Prasanna ([14]) showed that Weyl's theorem holds for every paranormal operator. Evidently, every hyponormal operator T (i.e., $T^*T \geq TT^*$) is both paranormal and $*$-paranormal. The $*$-paranormality of operators has been studied in [3], [4] and others. It is known ([3]) that $T \in B(H)$ is $*$-paranormal if and only if

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0 \qquad \text{for each } \lambda > 0. \tag{1.2}$$

We emphasize that $*$-paranormality is independent of paranormality ([4, Examples 2.2 and 2.3]). We say ([2], [8], [13]) that $T \in B(H)$ has the *single valued extension property* if for every open set U of $\mathbb{C}$ the only analytic solution $f : U \longrightarrow H$ of the equation

$$(T - \lambda I)f(\lambda) = 0$$

for all $\lambda \in U$ is the zero function on U. Given an arbitrary operator $T \in B(H)$, the *local resolvent set* $\rho_T(x)$ of T at the point of $x \in H$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \longrightarrow H$ which satisfies

$$(T - \lambda I)f(\lambda) = x \quad \text{ for all } \lambda \in U.$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

For an arbitrary operator $T \in B(H)$, we define the *local spectral subspace* of T as follows:

$$H_T(F) := \{x \in H : \sigma_T(x) \subseteq F\} \quad \text{for each set } F \subseteq \mathbb{C}.$$

In this note we examine Weyl's theorem for $*$-paranormal operators. Our main result is to prove that if T is a $*$-paranormal operator satisfying Property (E) - $(T - \lambda I)H_T(\{\lambda\})$ is closed for each $\lambda \in \mathbb{C}$, then Weyl's theorem holds for T.

2. The main result

We begin with:

Lemma 1 ([3, Theorem 1.1])**.** *Every $*$-paranormal operator is normaloid, i.e., norm equals spectral radius.*

Lemma 2 ([3, Lemma 2.1])**.** *If $T \in B(H)$ is $*$-paranormal, then $N(T - \lambda I) \subseteq N(T^* - \bar{\lambda} I)$ for each $\lambda \in \mathbb{C}$. Thus $T - \lambda I$ is reduced by its eigenspaces for each $\lambda \in \mathbb{C}$.*

Definition 3. An operator $T \in B(H)$ is said to satisfy *Property* (*E*) if

$$(T - \lambda I)H_T(\{\lambda\}) \text{ is closed for each } \lambda \in \mathbb{C}.$$

For example, every hyponormal operator satisfies Property (E). To see this, suppose $T \in B(H)$ is a hyponormal operator. Then we can see that

$$H_T(\{\lambda\}) = N(T - \lambda).$$

To see this we first observe that for each $\lambda \in \mathbb{C}$,

$$H_T(\{\lambda\}) = \left\{x \in H : \lim_{n\to\infty} ||(T - \lambda I)^n x||^{\frac{1}{n}} = 0\right\}. \tag{2.1}$$

Since T is hyponormal, and hence normaloid, it follows that $||(T - \lambda I)x|| \leq ||(T - \lambda I)^n x||^{\frac{1}{n}}$ for all $x \in H$ and $n \in \mathbb{N}$, which implies $H_T(\{\lambda\}) \subseteq N(T - \lambda I)$ and the converse is evident. Therefore $(T - \lambda I)H_T(\{\lambda\}) = \{0\}$, which is closed.

Our main theorem is as follows:

Theorem 4. *It $T \in B(H)$ is a $*$-paranormal operator satisfying Property (E), then Weyl's theorem holds for T.*

Proof. Suppose $T \in B(H)$ is a $*$-paranormal operator satisfying Property (E). We first claim

$$T \text{ is isoloid,} \tag{2.2}$$

in the sense that every isolated points of $\sigma(T)$ is an eigenvalue of T. To see this we suppose λ is an isolated point of $\sigma(T)$. From Lemma 2, we can see that $T - \lambda I$ has finite ascent for each $\lambda \in \mathbb{C}$. Thus by ([13, Proposition 1.8]), T has the single valued extension property. Moreover, since by our assumption

$H_T(\{\lambda\})$ and $H_T(\mathbb{C}\setminus\{\lambda\})$ are both closed, it follows from [1, Theorems 2.18, 2.20 and 3.76] that H can be decomposed as:

$$H = H_T(\{\lambda\}) \bigoplus H_T(\mathbb{C}\setminus\{\lambda\}).$$

In particular, we know (cf. [1, Theorem 2.6]) that

$$(T-\lambda I)H_T(\{\lambda\}) \subseteq H_T(\{\lambda\}) \quad \text{and} \quad (T-\lambda I)H_T(\mathbb{C}\setminus\{\lambda\}) = H_T(\mathbb{C}\setminus\{\lambda\}).$$

On the other hand, by (2.1), we have that $N(T-\lambda I) \subseteq H_T(\{\lambda\})$. Write

$$H_T(\{\lambda\}) = N(T-\lambda I) \bigoplus L \quad \text{for some closed subset } L.$$

Thus by the preceding argument, $T-\lambda I$ can be represented as follows:

$$(2.3) \qquad T-\lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix} : \begin{pmatrix} N(T-\lambda I) \\ L \\ H_T(\mathbb{C}\setminus\{\lambda\}) \end{pmatrix} \to \begin{pmatrix} N(T-\lambda I) \\ L \\ H_T(\mathbb{C}\setminus\{\lambda\}) \end{pmatrix}.$$

Assume to the contrary that $N(T-\lambda I) = \{0\}$. Since by our assumption T satisfies Property (E), A has closed range. Since A is one-one it follows that A is bounded below, i.e., there exists a constant $c > 0$ such that $||Ax|| \geq c||x||$ for each $x \in H$. Also since B is one-one and onto it follows that B is invertible. Consequently, from (2.3), we have that $T-\lambda I$ is bounded below. Hence $\lambda \notin \sigma_{ap}(T)$ and hence, by the well-known fact, $\partial\,\sigma(T) \subseteq \sigma_{ap}(T)$, where $\partial\sigma(T)$ is the topological boundary of $\sigma(T)$, we can conclude that $\lambda \notin \partial\,\sigma(T)$, which contradicts our assumption $\lambda \in \operatorname{iso}\sigma(T)$. Thus $N(T-\lambda I) \neq \{0\}$, which proves (2.2).

We now suppose $\lambda \in \sigma(T)\setminus\omega(T)$. Thus $T-\lambda I$ is Weyl. Then by the Index Product Theorem,

$$\dim N((T-\lambda I)^n) - \dim R((T-\lambda I)^n)^{\perp} = \operatorname{ind}((T-\lambda I)^n) = n \operatorname{ind}(T-\lambda I) = 0.$$

Since by Lemma 2, $T-\lambda I$ has finite ascent, $\dim N((T-\lambda I)^n)$ is a constant with respect to n, we have that $\dim R((T-\lambda I)^n)^{\perp}$ is a constant. Thus $T-\lambda I$ is Weyl of finite ascent and descent, and hence it is Browder. Therefore $\lambda \in \pi_{00}(T)$. Conversely, we suppose $\lambda \in \pi_{00}(T)$. By Lemma 2, $T-\lambda I$ is reduced by its eigenspaces. Thus we can write

$$T-\lambda I = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} : \begin{pmatrix} N(T-\lambda I) \\ N(T-\lambda I)^{\perp} \end{pmatrix} \to \begin{pmatrix} N(T-\lambda I) \\ N(T-\lambda I)^{\perp} \end{pmatrix}.$$

Thus

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & S+\lambda I \end{pmatrix}.$$

We now claim that S is invertible. Assume to the contrary that S is not invertible. Then $0 \in \operatorname{iso}\sigma(S)$ since $\lambda \in \operatorname{iso}\sigma(T)$. Thus $\lambda \in \operatorname{iso}\sigma(S+\lambda I)$. But since $S+\lambda I$ is also a $*$-paranormal operator satisfying Property (E), it follows from (2.2) that λ is an eigenvalue of $S+\lambda I$. Thus $0 \in \pi_0(S)$. But this contradicts to the fact that S is one-one. Therefore S should be invertible.

Note that $N(T-\lambda I)$ is finite-dimensional. Thus evidently, $T-\lambda I$ is Weyl, so that $\lambda \in \sigma(T)\setminus\omega(T)$. This completes the proof. □

In general, the spectral mapping theorem for the Weyl spectrum may fail. However, it was known ([9]) that for ∗-paranormal operators, Weyl spectrum obeys the spectral mapping theorem, i.e., if $T \in B(H)$ is ∗-paranormal, then

$$\omega(f(T)) = f(\omega(T)) \quad \text{for every } f \in A(\sigma(T)), \tag{2.4}$$

where $A(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

We thus have:

Corollary 5. *If $T \in B(H)$ is a ∗-paranormal operator satisfying Property (E), then for every $f \in A(\sigma(T))$, Weyl's theorem holds for $f(T)$.*

Proof. By Theorem 4, Weyl's theorem holds for every ∗-paranormal operator satisfying Property (E). Remembering ([12]) that if T is isoloid, then

$$f\big(\sigma(T)\setminus\pi_{00}(T)\big) = \sigma(f(T))\setminus\pi_{00}(f(T)) \quad \text{for every } f \in A(\sigma(T)),$$

it follows from (2.2) and (2.4) that

$$\sigma(f(T))\setminus\pi_{00}(f(T)) = f\big(\sigma(T)\setminus\pi_{00}(T)\big) = f(\omega(T)) = \omega(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. □

We conclude with an interesting structure theorem for ∗-paranormal operators.

Theorem 6. *If $T \in B(H)$ is ∗-paranormal and Riesz (i.e., $\sigma_e(T)=\{0\}$), then T is compact and normal.*

Proof. Suppose T is ∗-paranormal. Then by (1.2), $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0$ for each $\lambda > 0$. Write π for the Calkin homomorphism from $B(H)$ to the Calkin algebra $B(H)/K(H)$. Then $0 \leq \pi\left(T^{*2}T^2 - 2\lambda TT^* + \lambda^2\right) = \pi(T)^{*2}\pi(T)^2 - 2\lambda\pi(T)\pi(T)^* + \lambda^2$, which shows that $\pi(T)$ is ∗-paranormal and hence by Lemma 1, it is normaloid. If T is Riesz, then by the West Decomposition Theorem ([15]), we can write

$$T = K + Q, \quad \text{where } K \text{ is compact and } Q \text{ is quasinilpotent.}$$

Since $\pi(T)=\pi(Q)$, and hence $\sigma(\pi(T)) = \sigma(\pi(Q)) = \sigma_e(Q) = \{0\}$, we have that $\pi(T)$ is quasinilpotent. Therefore $||\pi(T)|| = r(\pi(T)) = 0$, and hence $\pi(T)=0$. Therefore T is compact. For the normality of T, observe that by Lemma 2,

$$\mathfrak{M} := \bigoplus_{\lambda\in\sigma(T)} N(T-\lambda I)$$

reduces T. Thus we can write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : \mathfrak{M}\oplus\mathfrak{M}^\perp \to \mathfrak{M}\oplus\mathfrak{M}^\perp.$$

Note that T_1 is normal. If $\mathfrak{M}^\perp = \{0\}$, then evidently T is normal. Thus we assume that $\mathfrak{M}^\perp \neq \{0\}$. We now claim that $T_2 = 0$. Assume to the contrary that $T_2 \neq 0$. Since T_2 is $*$-paranormal, and hence normaloid, we can find $\gamma \in \sigma(T_2)$ such that $||T_2|| = |\gamma|$. But since T is compact and $\gamma \neq 0$, we have that $\gamma \in \pi_0(T)$, which contradicts to the construction of $\mathfrak{M}$. Therefore $T_2 = 0$, and therefore we can conclude that T is normal. □

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