FUZZY STABILITY OF THE CAUCHY ADDITIVE AND QUADRATIC TYPE FUNCTIONAL EQUATION

SunSook Jin and Yang-Hi Lee

Abstract. In this paper, we investigate a fuzzy version of stability for the functional equation

$$2f(x+y) + f(x-y) + f(y-x) - 3f(x) - f(-x) - 3f(y) - f(-y) = 0$$

in the sense of M. Mirmostafaee and M. S. Moslehian.

1. Introduction

A classical question in the theory of functional equations is "when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?". Such a problem, called *a stability problem of the functional equation*, was formulated by S. M. Ulam [22] in 1940. In the next year, D. H. Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [20] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [4], [5], [7], [9], [10], [12]-[16], [21].

In 1984, A. K. Katsaras [8] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In particular, T. Bag and S. K. Samanta [2], following Cheng and Mordeson [3], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. In 2008, A. K. Mirmostafaee and M. S. Moslehian [18] obtained a fuzzy version of stability for *the Cauchy functional equation*:

$$(1.1) \qquad f(x+y) - f(x) - f(y) = 0.$$

In the same year, they [17] proved a fuzzy version of stability for *the quadratic functional equation*:

$$(1.2) \qquad f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0.$$

A solution of (1.1) is called *an additive mapping* and a solution of (1.2) is called *a quadratic mapping.* Now we consider the functional equation:

$$(1.3) \quad 2f(x+y) + f(x-y) + f(y-x) - 3f(x) - f(-x) - 3f(y) - f(-y) = 0$$

which is called *a Cauchy additive and quadratic type functional equation.* A solution of (1.3) is called *a quadratic-additive mapping.* In 2008, C.-G. Park [19] obtained a stability of the functional equation (1.3) by taking and composing an additive mapping A and a quadratic mapping Q to prove the existence of a quadratic-additive mapping F which is close to the given mapping f. In his processing, A is approximate to the odd part $\frac{f(x)-f(-x)}{2}$ of f and Q is close to the even part $\frac{f(x)+f(-x)}{2}$ of it, respectively.

In this paper, we get a general stability result of the Cauchy additive and quadratic type functional equation (1.3) in the fuzzy normed linear space in the manner of A. K. Mirmostafaee and M. S. Moslehian [17]. To do it, we introduce a Cauchy sequence $\{J_n f(x)\}$ starting from a given mapping f, which converges to the desired mapping F in the fuzzy sense. As we mentioned before, in previous studies of stability problem of (1.3), he attempted to get stability theorems by handling the odd and even part of f, respectively. According to our proposal in this paper, we can take the desired approximate solution F at only one time. Therefore, this idea is a refinement with respect to the simplicity of the proof.

2. Main results

We use the definition of a fuzzy normed space given in [2] to exhibit a reasonable fuzzy version of stability for the quadratic-additive type functional equation in the fuzzy normed linear space.

Definition 2.1 ([2]). Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (the so-called fuzzy subset) is said to be *a fuzzy norm on X* if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x, c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
(N3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$.

The pair (X, N) is called *a fuzzy normed linear space.* Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence $\{x_n\}$* and we denote it by $N - \lim_{n \to \infty} x_n = x$. A sequence $\{x_n\}$ in X is called *Cauchy* if for each

$\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called *a fuzzy Banach space.*

Let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space. For a given mapping $f : X \to Y$, we use the abbreviation

$$Df(x, y) := 2f(x + y) + f(x - y) + f(y - x) - 3f(x) - f(-x) - 3f(y) - f(-y)$$

for all $x, y \in X$. For given $q > 0$, the mapping f is called *a fuzzy q-almost quadratic-additive mapping,* if

$$(2.1) \qquad N'(Df(x, y), t + s) \geq \min\{N(x, s^q), N(y, t^q)\}$$

for all $x, y \in X$ and all $s, t \in (0, \infty)$. Now we get the general stability result in the fuzzy normed linear space.

Theorem 2.2. *Let q be a positive real number with $q \neq \frac{1}{2}, 1$. And let f be a fuzzy q-almost quadratic-additive mapping from a fuzzy normed space (X, N) into a fuzzy Banach space (Y, N'). Then there is a unique quadratic-additive mapping $F : X \to Y$ such that*
(2.2)

$$N'(F(x) - f(x), t) \geq \begin{cases} \sup_{t' < t} N\left(x, (2 - 2^p)^q t'^q\right) & \text{if } q > 1, \\ \sup_{t' < t} N\left(x, \left(\frac{(4 - 2^p)(2^p - 2)}{2}\right)^q t'^q\right) & \text{if } \frac{1}{2} < q < 1 \\ \sup_{t' < t} N\left(x, (2^p - 4)^q t'^q\right) & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

for each $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. It follows from (2.1) and (N4) that

$$N'(f(0), t) = N'(Df(0, 0), 4t) \geq N(0, (2t)^q) = 1$$

for all $t \in (0, \infty)$. By (N2), we have $f(0) = 0$. We will prove the theorem in three cases, $q > 1$, $\frac{1}{2} < q < 1$, and $0 < q < \frac{1}{2}$.

Case 1. Let $q > 1$ and let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x)\right) + 2^{-n}\left(f(2^n x) - f(-2^n x)\right)\right)$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$(2.3) \quad J_j f(x) - J_{j+1} f(x) = \frac{2^{j+1} - 1}{4^{j+2}} Df(-2^j x, -2^j x) - \frac{2^{j+1} + 1}{4^{j+2}} Df(2^j x, 2^j x)$$

for all $x \in X$ and $j \geq 0$. Together with (N3), (N4) and (2.1), this equation implies that if $n + m > m \geq 0$, then

$$(2.4) \quad N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{1}{2}\left(\frac{2^p}{2}\right)^j t^p\right)$$

$$\geq N'\left(\sum_{j=m}^{n+m-1}(J_jf(x)-J_{j+1}f(x)),\ \sum_{j=m}^{n+m-1}\frac{1}{2}\left(\frac{2^p}{2}\right)^j t^p\right)$$

$$\geq \min_{j=m}^{n+m-1}\bigcup\left\{N'\left(J_jf(x)-J_{j+1}f(x),\frac{1}{2}\left(\frac{2^p}{2}\right)^j\right)t^p\right\}$$

$$\geq \min_{j=m}^{n+m-1}\bigcup\left\{\min\left\{N'\left(-\frac{(2^{j+1}+1)Df(2^jx,2^jx)}{4^{j+2}},\frac{(2^{j+1}+1)2^{jp}t^p}{8\cdot4^j}\right),\right.\right.$$

$$\left.\left.N'\left(\frac{(2^{j+1}-1)Df(-2^jx,-2^jx)}{4^{j+2}},\frac{(2^{j+1}-1)2^{jp}t^p}{8\cdot4^j}\right)\right\}\right\}$$

$$\geq \min_{j=m}^{n+m-1}\bigcup\left\{N(2^jx,2^jt)\right\}=N(x,t)$$

for all $x\in X$ and $t>0$. Let $\varepsilon>0$ be given. Since $\lim_{t\to\infty}N(x,t)=1$, there is $t_0>0$ such that

$$N(x,t_0)\geq1-\varepsilon.$$

Observe that for some $\tilde{t}>t_0$, the series $\sum_{j=0}^{\infty}\frac{1}{2}\left(\frac{2^p}{2}\right)^j\tilde{t}^p$ converges for $p=\frac{1}{q}<1$. It guarantees that, for an arbitrary given $c>0$, there exists $n_0\geq0$ such that

$$\sum_{j=m}^{n+m-1}\frac{1}{2}\left(\frac{2^p}{2}\right)^j\tilde{t}^p<c$$

for each $m\geq n_0$ and $n>0$. Together with (N5) and (2.4), this implies that

$$N'(J_mf(x)-J_{n+m}f(x),c)$$

$$\geq N'\left(J_mf(x)-J_{n+m}f(x),\ \sum_{j=m}^{n+m-1}\frac{1}{2}\left(\frac{2^p}{2}\right)^j\tilde{t}^p\right)$$

$$\geq N(x,\tilde{t})\geq N(x,t_0)\geq1-\varepsilon$$

for all $x\in X$. Hence $\{J_nf(x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y,N'), and so we can define a mapping $F:X\to Y$ by

$$F(x):=N'-\lim_{n\to\infty}J_nf(x).$$

Moreover, if we put $m=0$ in (2.4), we have

$$(2.5)\qquad N'(f(x)-J_nf(x),t)\geq N\left(x,\frac{t^q}{\left(\sum_{j=0}^{n-1}\frac{1}{2}\left(\frac{2^p}{2}\right)^j\right)^q}\right)$$

for all $x \in X$. Next we will show that F is the desired quadratic additive function. Using (N4), we have

(2.6)

$$N'(DF(x,y),t)$$

$$\geq \min \left\{ N'\left(2F(x+y) - 2J_n f(x+y), \frac{t}{14}\right), N'\left(F(x-y) - J_n f(x-y), \frac{t}{14}\right), \right.$$

$$N'\left(F(y-x) - J_n f(y-x), \frac{t}{14}\right), N'\left(3J_n f(x) - 3F(x), \frac{t}{14}\right),$$

$$N'\left(J_n f(-x) - F(-x), \frac{t}{14}\right), N'\left(3J_n f(y) - 3F(y), \frac{t}{14}\right),$$

$$\left. N'\left(J_n f(-y) - F(-y), \frac{t}{14}\right), N'\left(DJ_n f(x,y), \frac{t}{2}\right) \right\}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. The first seven terms on the right hand side of (2.6) tend to 1 as $n \to \infty$ by the definition of F and (N2), and the last term holds

$$N'\left(DJ_n f(x,y), \frac{t}{2}\right)$$

$$\geq \min \left\{ N'\left(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right), \right.$$

$$\left. N'\left(\frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}, \frac{t}{8}\right) \right\}$$

for all $x, y \in X$. By (N3) and (2.1), we obtain

$$N'\left(\frac{Df(\pm 2^n x, \pm 2^n y))}{2 \cdot 4^n}, \frac{t}{8}\right) = N'\left(Df(\pm 2^n x, \pm 2^n y), \frac{4^n t}{4}\right)$$

$$\geq \min \left\{ N\left(2^n x, \left(\frac{4^n t}{8}\right)^q\right), N\left(2^n y, \left(\frac{4^n t}{8}\right)^q\right) \right\}$$

$$\geq \min \left\{ N\left(x, \frac{2^{(2q-1)n}}{2^{3q}} t^q\right), N\left(y, \frac{2^{(2q-1)n}}{2^{3q}} t^q\right) \right\}$$

and

$$N'\left(\frac{Df(\pm 2^n x, \pm 2^n y))}{2 \cdot 2^n}, \frac{t}{8}\right) \geq \min \left\{ N\left(x, \frac{2^{(q-1)n}}{2^{3q}} t^q\right), N\left(y, \frac{2^{(q-1)n}}{2^{3q}} t^q\right) \right\}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $q > 1$, together with (N5), we can deduce that the last term of (2.6) also tends to 1 as $n \to \infty$. It follows from (2.6) that

$$N'(DF(x,y),t) = 1$$

for each $x, y \in X$ and $t > 0$. By (N2), this means that $DF(x,y) = 0$ for all $x, y \in X$.

Next we approximate the difference between f and F in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since F is the limit of $\{J_n f(x)\}$, there is $n \in \mathbb{N}$ such that

$$N'\left(F(x) - J_n f(x), t - t'\right) \geq 1 - \varepsilon.$$

By (2.5), we have

$$N'(F(x) - f(x), t) \geq \min\left\{N'\left(F(x) - J_n f(x), t - t'\right), N'\left(J_n f(x) - f(x), t'\right)\right\}$$

$$\geq \min\left\{1 - \varepsilon, N\left(x, \frac{t'^q}{\left(\frac{1}{2}\sum_{j=0}^{n-1}\left(\frac{2^p}{2}\right)^j\right)^q}\right)\right\}$$

$$\geq \min\left\{1 - \varepsilon, N\left(x, (2 - 2^p)^q t'^q\right)\right\}.$$

Because $0 < \varepsilon < 1$ is arbitrary, we get the inequality (2.2) in this case. Finally, to prove the uniqueness of F, let $F' : X \to Y$ be another quadratic-additive mapping satisfying (2.2). Then by (2.3), we get

$$(2.7) \qquad \begin{cases} F(x) - J_n F(x) = \sum_{j=0}^{n-1}\left(J_j F(x) - J_{j+1} F(x)\right) = 0 \\ F'(x) - J_n F'(x) = \sum_{j=0}^{n-1}\left(J_j F'(x) - J_{j+1} F'(x)\right) = 0 \end{cases}$$

for all $x \in X$ and $n \in \mathbb{N}$. Together with (N4) and (2.2), this implies that

$$N'(F(x) - F'(x), t)$$
$$= N'(J_n F(x) - J_n F'(x), t)$$
$$\geq \min\left\{N'\left(J_n F(x) - J_n f(x), \frac{t}{2}\right), N'\left(J_n f(x) - J_n F'(x), \frac{t}{2}\right)\right\}$$
$$\geq \min\left\{N'\left(\frac{(F-f)(2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right),\right.$$
$$N'\left(\frac{(F-f)(-2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right),$$
$$N'\left(\frac{(F-f)(2^n x)}{2 \cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^n x)}{2 \cdot 2^n}, \frac{t}{8}\right),$$
$$\left. N'\left(\frac{(F-f)(-2^n x)}{2 \cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^n x)}{2 \cdot 2^n}, \frac{t}{8}\right)\right\}$$
$$\geq \sup_{t'<t} N\left(x, 2^{(q-1)n-2q}(2 - 2^p)^q t'^q\right)$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that, for $q = \frac{1}{p} > 1$, the last term of the above inequality tends to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and so we get

$$F(x) = F'(x)$$

for all $x \in X$ by (N2).

Case 2. Let $\frac{1}{2} < q < 1$ and let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x)\right) + 2^n \left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)\right)$$

for all $x \in X$. Then we have $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = -\frac{1}{4^{j+2}} Df(2^j x, 2^j x) - \frac{1}{4^{j+2}} Df(-2^j x, -2^j x)$$
$$+ 2^{j-2} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) - 2^{j-2} Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right)$$

for all $x \in X$ and $j \geq 0$. If $n + m > m \geq 0$, then we have

$$N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1}\left(\frac{1}{4}\left(\frac{2^p}{4}\right)^j + \frac{1}{2^p}\left(\frac{2}{2^p}\right)^j\right)t^p\right)$$

$$\geq \min_{j=m}^{n+m-1}\bigcup\left\{\min\left\{N'\left(\frac{Df(2^j x, 0)}{2 \cdot 4^{j+1}}, \frac{2^{jp}t^p}{2 \cdot 4^{j+1}}\right),\right.\right.$$

$$N'\left(\frac{Df(-2^j x, 0)}{2 \cdot 4^{j+1}}, \frac{2^{jp}t^p}{2 \cdot 4^{j+1}}\right), N'\left(-2^{j-1}Df\left(\frac{x}{2^{j+1}}, 0\right), \frac{2^{j-1}t^p}{2^{(j+1)p}}\right),$$

$$\left.\left.N'\left(2^{j-1}Df\left(-\frac{x}{2^{j+1}}, 0\right), \frac{2^{j-1}t^p}{2^{(j+1)p}}\right)\right\}\right\}$$

$$\geq \min_{j=m}^{n+m-1}\bigcup\left\{N(2^j x, 2^j t), N\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right)\right\}$$

$$= N(x, t)$$

for all $x \in X$ and $t > 0$. In the similar argument following (2.4) of the previous case, we can define the limit $F(x) := N' - \lim_{n \to \infty} J_n f(x)$ of the Cauchy sequence $\{J_n f(x)\}$ in the Banach fuzzy space Y. Moreover, putting $m = 0$ in the above inequality, we have

$$(2.8) \qquad N'(f(x) - J_n f(x), t) \geq N\left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1}\left(\frac{1}{4}\left(\frac{2^p}{4}\right)^j + \frac{1}{2^p}\left(\frac{2}{2^p}\right)^j\right)\right)^q}\right)$$

for each $x \in X$ and $t > 0$. To prove that F is a quadratic additive function, we have enough to show that the last term of (2.6) in Case 1 tends to 1 as $n \to \infty$. By (N3) and (2.1), we get

$$N'\left(DJ_n f(x, y), \frac{t}{2}\right)$$

$$\geq \min\left\{N'\left(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right),\right.$$

$$\left.N'\left(2^{n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(2^{n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right)\right\}$$

$$\geq \min \left\{ N(x, 2^{(2q-1)n-3q}t^q), N(y, 2^{(2q-1)n-3q}t^q), \right.$$

$$\left. N(x, 2^{(1-q)n-3q}t^q), N(y, 2^{(1-q)n-3q}t^q) \right\}$$

for each $x, y \in X$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, since $\frac{1}{2} < q < 1$. Hence, together with the similar argument after (2.6), we can say that $DF(x, y) = 0$ for all $x, y \in X$. Recall, in Case 1, the inequality (2.2) follows from (2.5). By the same reasoning, we get (2.2) from (2.8) in this case. Now to prove the uniqueness of F, let F' be another quadratic additive mapping satisfying (2.2). Then, together with (N4), (2.2), and (2.7), we have

$$N'(F(x) - F'(x), t)$$
$$= N'(J_n F(x) - J_n F'(x), t)$$
$$\geq \min \left\{ N'\left(J_n F(x) - J_n f(x), \frac{t}{2} \right), N'\left(J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\}$$
$$\geq \min \left\{ N'\left(\frac{(F-f)(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), \left(\frac{(f-F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), \right.$$
$$N'\left(\frac{(F-f)(-2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right), N'\left(\frac{(f-F')(-2^n x)}{2 \cdot 4^n}, \frac{t}{8} \right),$$
$$N'\left(2^{n-1}\left((F-f)\left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N'\left(2^{n-1}\left((f-F')\left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right),$$
$$N'\left(2^{n-1}\left((F-f)\left(\frac{-x}{2^n} \right) \right), \frac{t}{8} \right), N'\left(2^{n-1}\left((f-F')\left(\frac{-x}{2^n} \right) \right), \frac{t}{8} \right) \right\}$$
$$\geq \min \left\{ \sup_{t' < t} N\left(x, 2^{(2q-1)n-2q}\left(\frac{(4-2^p)(2^p-2)}{2} \right)^q t'^q \right), \right.$$
$$\left. \sup_{t' < t} N\left(x, 2^{(1-q)n-2q}\left(\frac{(4-2^p)(2^p-2)}{2} \right)^q t'^q \right) \right\}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $\lim_{n \to \infty} 2^{(2q-1)n-2q} = \lim_{n \to \infty} 2^{(1-q)n-2q} = \infty$ in this case, both terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and so $F(x) = F'(x)$ for all $x \in X$ by (N2).

Case 3. Finally, we take $0 < q < \frac{1}{2}$ and define $J_n f : X \to Y$ by

$$J_n f(x) = \frac{1}{2}\left(4^n \left(f(2^{-n}x) + f(-2^{-n}x) \right) + 2^n \left(f\left(\frac{x}{2^n} \right) - f\left(-\frac{x}{2^n} \right) \right) \right)$$

for all $x \in X$. Then we have $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = (4^{j-1} + 2^{j-2}) Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right)$$

$$+ (4^{j-1} - 2^{j-2})Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right)$$

which implies that if $n + m > m \geq 0$, then

$$N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1}\left(\frac{4}{2^p}\right)^j \frac{t^p}{2^p}\right)$$

$$\geq \min_{j=m}^{n+m-1}\bigcup\left\{\min\left\{N'\left((4^{j-1}+2^{j-2})Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \frac{(2^{2j-1}+2^{j-1})t^p}{2^{(j+1)p}}\right),\right.\right.$$

$$\left.\left. N'\left((4^{j-1} - 2^{j-2})Df\left(-\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right), \frac{(2^{2j-1}-2^{j-1})t^p}{2^{(j+1)p}}\right)\right\}\right\}$$

$$\geq \min_{j=m}^{n+m-1}\bigcup\left\{N\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right)\right\}$$

$$= N(x, t)$$

for all $x \in X$ and $t > 0$. Similar to the previous cases, it leads us to define the mapping $F : X \to Y$ by $F(x) := N' - \lim_{n\to\infty} J_n f(x)$. Putting $m = 0$ in the above inequality, we have

$$(2.9) \qquad N'(f(x) - J_n f(x), t) \geq N\left(x, \frac{t^q}{\left(\frac{1}{2^p}\sum_{j=0}^{n-1}\left(\frac{4}{2^p}\right)^j\right)^q}\right)$$

for all $x \in X$ and $t > 0$. Notice that

$$N'\left(DJ_n f(x, y), \frac{t}{2}\right)$$

$$\geq \min\left\{N'\left(\frac{4^n}{2}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(\frac{4^n}{2}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right),\right.$$

$$\left. N'\left(2^{n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(2^{n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right)\right\}$$

$$\geq \min\left\{N\left(x, 2^{(1-2q)n-3q}t^q\right), N\left(y, 2^{(1-2q)n-3q}t^q\right),\right.$$

$$\left. N\left(x, 2^{(1-q)n-3q}t^q\right), N\left(y, 2^{(1-q)n-3q}t^q\right)\right\}$$

for each $x, y \in X$ and $t > 0$. Since $0 < q < \frac{1}{2}$, all terms on the right hand side tend to 1 as $n \to \infty$, which implies that the last term of (2.6) tends to 1 as $n \to \infty$. Therefore, we can say that $DF \equiv 0$. Moreover, using the similar argument after (2.6) in Case 1, we get the inequality (2.2) from (2.9) in this case. To prove the uniqueness of F, let $F' : X \to Y$ be another quadratic additive function satisfying (2.2). Then by (2.7), we get

$$N'(F(x) - F'(x), t)$$

$$\geq \min \left\{ N' \left(J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left(J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\}$$

$$\geq \min \left\{ N' \left(\frac{4^n}{2} \left((F - f) \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(\frac{4^n}{2} \left((f - F') \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \right.$$

$$N' \left(\frac{4^n}{2} \left((F - f) \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(\frac{4^n}{2} \left((f - F') \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right),$$

$$N' \left(2^{n-1} \left((F - f) \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right),$$

$$\left. N' \left(2^{n-1} \left((F - f) \left(\frac{-x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(\frac{-x}{2^n} \right) \right), \frac{t}{8} \right) \right\}$$

$$\geq \sup_{t' < t} N \left(x, 2^{(1-2q)n-2q} \left(2^p - 4 \right)^q t'^q \right)$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that, for $0 < q < \frac{1}{2}$, the last term tends to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and $F(x) = F'(x)$ for all $x \in X$ by (N2).   □

Remark 2.3. Consider a mapping $f : X \to Y$ satisfying (2.1) for all $x, y \in X$ and a real number $q < 0$. Take any $t > 0$. If we choose a real number s with $0 < 2s < t$, then we have

$$N'(Df(x, y), t) \geq N'(Df(x, y), 2s) \geq \min\{N(x, s^q), N(y, s^q)\}$$

for all $x, y \in X$. Since $q < 0$, we have $\lim_{s \to 0^+} s^q = \infty$. This implies that

$$\lim_{s \to 0^+} N(x, s^q) = \lim_{s \to 0^+} N(y, s^q) = 1$$

and so

$$N'(Df(x, y), t) = 1$$

for all $x, y \in X$ and $t > 0$. By (N2), it allows us to get $Df(x, y) = 0$ for all $x, y \in X$. In other words, f is itself a quadratic additive mapping if f is a fuzzy q-almost quadratic-additive mapping for the case $q < 0$.

Corollary 2.4. *Let f be an even mapping satisfying all of the conditions of Theorem 2.2. Then there is a unique quadratic mapping $F : X \to Y$ such that*

$$(2.10) \qquad N'(F(x) - f(x), t) \geq \sup_{t' < t} N \left(x, (|4 - 2^p| t')^q \right)$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. Let $J_n f$ be defined as in Theorem 2.2. Since f is an even mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} & \text{if } 0 < q < \frac{1}{2}, \\ \frac{1}{2} \left(4^n \left(f(2^{-n} x) + f(-2^{-n} x) \right) \right) & \text{if } q > \frac{1}{2} \end{cases}$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{-1}{4^{j+2}}(Df(2^j x, 2^j x) + Df(-2^j x, -2^j x)) & \text{if } 0 < q < \frac{1}{2}, \\ 4^{j-1}\left(Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) + Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right)\right) & \text{if } q > \frac{1}{2} \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.2, we obtain the quadratic-additive mapping F satisfying (2.10). Notice that F is also even, $F(x) := N' - \lim_{n \to \infty} J_n f(x)$ for all $x \in X$, and $DF(x, y) = 0$ for all $x, y \in X$. Hence, we get

$$F(x + y) + F(x - y) - 2F(x) - 2F(y) = \frac{1}{2}DF(x, y) = 0$$

for all $x, y \in X$. This means that F is a quadratic mapping. $\qquad \square$

Corollary 2.5. *Let f be an odd mapping satisfying all of the conditions of Theorem 2.2. Then there is a unique additive mapping $F : X \to Y$ such that*

$$(2.11) \qquad N'(F(x) - f(x), t) \geq \sup_{t' < t} N\left(x, (|2 - 2^p|t')^q\right)$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. Let $J_n f$ be defined as in Theorem 2.2. Since f is an odd mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(2^n x) + f(-2^n x)}{2^{n+1}} & \text{if } 0 < q < 1, \\ 2^{n-1}\left(f(2^{-n} x) + f(-2^{-n} x)\right) & \text{if } q > 1 \end{cases}$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{1}{2^{j+3}}(Df(-2^j x, -2^j x) - Df(2^j x, 2^j x)) & \text{if } 0 < q < 1 \\ 2^{j-2}\left(Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) - Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right)\right) & \text{if } q > 1 \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.2, we obtain the quadratic-additive mapping F satisfying (2.11). Notice that F is also odd, $F(x) := N' - \lim_{n \to \infty} J_n f(x)$ for all $x \in X$, and $DF(x, y) = 0$ for all $x, y \in X$. Hence, we get

$$F(x + y) - F(x) - F(y) = \frac{1}{2}DF(x, y) = 0$$

for all $x, y \in X$. This means that F is an additive mapping. $\qquad \square$

We can use Theorem 2.2 to get a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed linear space. Then we can define a fuzzy norm N_X on X by following

$$N_X(x, t) = \begin{cases} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$, see [17]. Suppose that $f : X \to Y$ is a mapping into a Banach space $(Y, \|\|\cdot\|\|)$ such that

$$\|\|Df(x, y)\|\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$. Let N_Y be a fuzzy norm on Y. Then we get

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & s+t \leq |||Df(x,y)||| \\ 1, & s+t > |||Df(x,y)||| \end{cases}$$

for all $x, y \in X$ and $s, t \in \mathbb{R}$. Consider the case $N_Y(Df(x,y), s+t) = 0$. This implies that

$$\|x\|^p + \|y\|^p \geq |||Df(x,y)||| \geq s+t$$

and so either $\|x\|^p \geq s$ or $\|y\|^p \geq t$ in this case. Hence, for $q = \frac{1}{p}$, we have

$$\min\{N_X(x, s^q), N_X(y, t^q)\} = 0$$

for all $x, y \in X$ and $s, t > 0$. Therefore, in every case, the inequality

$$N_Y(Df(x,y), s+t) \geq \min\{N_X(x, s^q), N_X(y, t^q)\}$$

holds. It means that f is a fuzzy q-almost quadratic additive mapping, and by Theorem 2.2, we get the following stability result.

Corollary 2.6. *Let $(X, \|\cdot\|)$ be a normed linear space and let $(Y, |||\cdot|||)$ be a Banach space. If*

$$|||Df(x,y)||| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$, then there is a unique quadratic-additive mapping $F : X \to Y$ such that

$$|||F(x) - f(x)||| \leq \begin{cases} \frac{\|x\|^p}{2 - 2^p} & \text{if } 0 < p < 1, \\ \frac{2\|x\|^p}{(2 - 2^p)(2^p - 4)} & \text{if } 1 < p < 2, \\ \frac{\|x\|^p}{2^p - 4} & \text{if } 2 < p \end{cases}$$

for all $x \in X$.

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SunSook Jin
Department of Mathematics Education
Gongju National University of Education
Gongju 314-711, Korea
E-mail address: ssjin@gjue.ac.kr

Yang-Hi Lee
Department of Mathematics Education
Gongju National University of Education
Gongju 314-711, Korea
E-mail address: lyhmzi@gjue.ac.kr