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WEAK CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS AND NONEXPANSIVE MAPPINGS AND NONSPREADING MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mappings and nonspreading mappings and the set of solution of an equilibrium problem on the setting of real Hilbert spaces.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Then a mapping $T : C \to C$ is said to be nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{z \in C : Tz = z\}$.

A mapping F is said to be firmly nonexpansive if

$$\| Fx - Fy \|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$. On the other hand, a mapping $Q : C \to C$ is said to be quasi-nonexpansive if $F(Q) \neq \emptyset$ and

$$\| Qx - q \| \leq \| x - q \|$$

for all $x \in C$ and $q \in F(Q)$, where $F(Q)$ is the set of fixed points of Q. Obviously if $T : C \to C$ is nonexpansive and the set $F(T)$ of fixed points of T is nonempty, then T is quasi-nonexpansive.

The nonspreading mapping was introduced by Kohsaka and Takahashi [1]. Let E be a smooth, strictly convex and reflexive Banach space. Let J be the duality mapping of E and let C be a nonempty closed convex subset of E. Then a mapping $S : C \to C$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x)$$

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for all $x, y \in C$, where $\phi(x,y) = \| x \|^2 - 2\langle x, Jy\rangle + \| y \|^2$ for all $x, y \in E$. They considered such a mapping to study the resolvents of a maximal monotone operator in a Banach space. In the case when E is a Hilbert space, we know that $\phi(x,y) = \| x - y \|^2$ for all $x, y \in E$, then a nonspreading mapping $S : C \to C$ in a Hilbert space H is defined as follows:

$$2 \| Sx - Sy \|^2 \leq \| Sx - y \|^2 + \| x - Sy \|^2$$

for all $x, y \in C$.

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let f be a bifunction form $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that $f(x,y) \geq 0$ for all $y \in C$. The set of such solutions is denoted by $EP(f)$. Numerous problems in physics, optimization, and economics reduce to finding a solution of the equilibrium problem.

In this paper, we will establish the convergence theorems for finding common solutions of equilibrium problem and fixed problems of nonexpansive mappings and nonspreading mappings.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. In a Hilbert space, it is known that

$$\| \alpha x + (1-\alpha) y \|^2 = \alpha \| x \|^2 + (1-\alpha) \| y \|^2 - \alpha(1-\alpha) \| x - y \|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; Further, in a Hilbert space, we have that

$$2\langle x - y, z - w\rangle = \| x - w \|^2 + \| y - z \|^2 - \| x - z \|^2 - \| y - w \|^2 . \tag{2.2}$$

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$;
(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemmas that will be used for our main result in the next section.

Lemma 2.1. *Let C be a nonempty closed convex subset of H and f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*:

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.2. *Assume that* $f : C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows*:

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \ \forall x \in H.$$

Then,

(1) T_r *is single-valued*;

(2) T_r *is firmly nonexpansive, that is,* $\forall x, y \in H$,

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ *is nonempty, closed and convex.*

Lemma 2.3. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* S *be a nonspreading mapping of* C *into itself such that* $F(S) \neq \emptyset$. *Then* S *is demiclosed, i.e.,* $x_n \rightharpoonup u$ *and* $x_n - Sx_n \to 0$ *imply* $u \in F(S)$.

Lemma 2.4. *Suppose that* $\{s_n\}$ *and* $\{e_n\}$ *are sequences of nonnegative real numbers such that* $s_{n+1} \leq s_n + e_n$ *for all* $n \in \mathbb{N}$. *If* $\sum_{n=1}^{\infty} e_n < \infty$, *then* $\lim_{n\to\infty} s_n$ *exists.*

3. Main results

We are now in a position to prove our theorem for finding common solutions of equilibrium problem and fixed problems of nonexpansive mappings and nonspreading mappings.

Theorem 3.1. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H *and* $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4). *Let* S *be a nonspreading mapping of* C *into itself and let* T *be a nonexpansive mapping of* C *into itself such that* $F(S) \cap F(T) \cap EP(f) \neq \emptyset$. *Let* x_n *and* u_n *be sequences generated initially by an arbitrary element* $x_1 \in H$ *and then by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{\beta_n Su_n + (1 - \beta_n)Tu_n\}, & \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy the following conditions*: $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *convergence weakly to an element of* $F(S) \cap F(T) \cap EP(f)$.

Proof. First, we show that $\lim_{n\to\infty} \| u_n - Su_n \| = 0$. Note that u_n can be rewritten as $u_n = T_{r_n} x_n$ for all $n \in \mathbb{N}$, take $z \in F(S) \cap F(T) \cap EP(f)$, we know that

$$\| u_n - z \| = \| T_{r_n} x_n - T_{r_n} z \| \leq \| x_n - z \| .$$

We obtain from (3.1) that

$$x_{n+1} = \beta_n\{(1 - \alpha_n)u_n + \alpha_n Su_n\} + (1 - \beta_n)\{(1 - \alpha_n)u_n + \alpha_n Tu_n\} \tag{3.2}$$

for all $n \geq 1$. Further, putting $V_n = \beta_n\{(1-\alpha_n)I + \alpha_n S\} + (1-\beta_n)\{(1-\alpha_n)I + \alpha_n T\}$, we can rewrite (3.2) by $x_{n+1} = V_n u_n$, we have that for any $z \in F(S) \cap F(T) \cap EP(f)$

$$\begin{aligned}
\| x_{n+1} - z \|^2 =& \ \| V_n u_n - z \|^2 \\
&\leq \beta_n \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2 \\
&\quad + (1-\beta_n) \| (1-\alpha_n)u_n + \alpha_n T u_n - z \| \\
&\leq \beta_n \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2 + (1-\beta_n) \| u_n - z \|^2 \\
&\leq \ \| u_n - z \|^2 \\
&\leq \ \| x_n - z \|^2
\end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$. Therefore there exists

$$\lim_{n\to\infty} \| x_n - z \|^2 = \lim_{n\to\infty} \| u_n - z \|^2 = \lim_{n\to\infty} \| V_n u_n - z \|^2 \tag{3.4}$$

hence $\{x_n\}$ and $\{u_n\}$ is bounded.
From (3.3) we get

$$\begin{aligned}
0 &\leq \| u_n - z \|^2 - \beta_n \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2 - (1-\beta_n) \| u_n - z \|^2 \\
&= \beta_n (\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2) \\
&\leq \| u_n - z \|^2 - \| V_n u_n - z \|^2 .
\end{aligned}$$

So we have

$$\begin{aligned}
0 &\leq (1-\beta_n)\beta_n (\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2) \\
&\leq (1-\beta_n)(\| u_n - z \|^2 - \| V_n u_n - z \|^2).
\end{aligned}$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, it follows from (3.4) that

$$\lim_{n\to\infty} (\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2) = 0.$$

From (2.1), we have

$$\begin{aligned}
&\| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2 \\
&= (1-\alpha_n) \| u_n - z \|^2 + \alpha_n \| S u_n - z \| - \alpha_n(1-\alpha_n) \| u_n - S u_n \|^2
\end{aligned}$$

and hence

$$\begin{aligned}
&\alpha_n(1-\alpha_n) \| u_n - S u_n \|^2 \\
&= (\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2) \\
&\quad - \alpha_n \| u_n - z \|^2 + \alpha_n \| S u_n - z \|^2 \\
&\leq (\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2) \\
&\quad - \alpha_n \| u_n - z \|^2 + \alpha_n \| u_n - z \|^2 \\
&= \ \| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n S u_n - z \|^2 .
\end{aligned}$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we get

$$\lim_{n\to\infty} \| u_n - Su_n \| = 0.$$

Since $\{u_n\}$ is a bounded sequence, there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$, such that $\{u_{n_i}\}$ converges weakly to v. From Lemma 2.3 we obtain $v \in F(S)$. We also show that $v \in F(T)$. In fact we have that for any $z \in F(S) \cap F(T) \cap EP(f)$

$$\begin{aligned}
&\| V_n u_n - z \|^2 \\
&\leq \beta_n \| (1-\alpha_n)u_n + \alpha_n Su_n - z \|^2 + (1-\beta_n) \| (1-\alpha_n)u_n + \alpha_n Tu_n - z \|^2 \\
&\leq \beta_n \| u_n - z \|^2 + (1-\beta_n) \| (1-\alpha_n)u_n + \alpha_n Tu_n - z \|^2 \\
&\leq \| u_n - z \|^2
\end{aligned}$$

and hence

$$\begin{aligned}
0 &\leq (1-\beta_n)(\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n Tu_n - z \|^2) \\
&\leq \| u_n - z \|^2 - \| V_n u_n - z \|^2
\end{aligned}$$

so, we have

$$\begin{aligned}
0 &\leq \beta_n(1-\beta_n)(\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n Tu_n - z \|^2) \\
&\leq \beta_n(\| u_n - z \|^2 - \| V_n u_n - z \|^2).
\end{aligned}$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, it follows from (3.4) that

$$\lim_{n\to\infty} (\| u_n - z \|^2 - \| (1-\alpha_n)u_n + \alpha_n Tu_n - z \|^2) = 0.$$

So we obtain from (2.1)

$$\lim_{n\to\infty} \| u_n - Tu_n \| = 0.$$

Since $\{u_{n_i}\}$ converges weakly to v, we have $v \in F(T)$. Next, we shall show $v \in EP(f)$. Since $u_n = Tr_n x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

Note that by (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n} \rangle \geq F(y, u_{n_i}). \tag{3.5}$$

By condition (A4), $F(y, \cdot)$ is lower semicontinuous and convex, and thus weakly semicontinuous. Since $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$ in norm. Therefore, letting $i \to \infty$ in (3.5) yields

$$F(y, v) \leq \lim_{i\to\infty} F(y, u_{n_i}) \leq 0, \quad \forall y \in C.$$

Replacing y with $y_t := ty + (1-t)v, t \in [0,1]$ and using (A1) and (A4), we obtain

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, v) \leq tF(y_t, y).$$

Hence

$$F(ty + (1-t)v, y) \geq 0, \ t \in [0,1], \quad y \in C.$$

Letting $t \to 0^+$ and using assumption (A3), we conclude

$$F(v, y) \geq 0, \quad \forall y \in C.$$

Therefore, $v \in EP(f)$. Then, we conclude that $v \in F(S) \cap F(T) \cap EP(f)$. Next, we shall show that $\lim_{n\to\infty} \| x_n - u_n \| = 0$.

Indeed, let z be an arbitrary element of $F(S) \cap F(T) \cap EP(f)$. Then as above $u_n = Tr_n x_n$, and from (2.2) we have

$$\begin{aligned} \| u_n - z \|^2 &= \| Tr_n x_n - Tr_n z \|^2 \\ &\leq \langle Tr_n x_n - Tr_n z, x_n - z \rangle \\ &= \langle u_n - z, x_n - z \rangle \\ &= \frac{1}{2}(\| u_n - z \|^2 + \| x_n - z \|^2 - \| x_n - u_n \|^2) \end{aligned}$$

and hence

$$\| u_n - z \|^2 \leq \| x_n - z \|^2 - \| x_n - u_n \|^2 .$$

From (3.3), we have

$$\| x_{n+1} - z \|^2 \leq \| u_n - z \|^2 \leq \| x_n - z \|^2 - \| x_n - u_n \|^2$$

and hence

$$\| x_n - u_n \|^2 \leq \| x_n - z \|^2 - \| x_{n+1} - z \|^2 .$$

Since $\lim_{n\to\infty} \| x_n - z \|$ exists, we have

$$\lim_{n\to\infty} \| x_n - u_n \| = 0. \tag{3.6}$$

Following we will prove $\{x_n\}$ converges weakly to v. Since $\{u_{n_i}\} \rightharpoonup v$, from (3.6) we have $\{x_{n_i}\} \rightharpoonup v$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$. Then, we have $v = w$. In fact, if $v \neq w$, then we have that

$$\begin{aligned} \lim_{n\to\infty} \| x_n - v \| &= \lim_{i\to\infty} \| x_{n_i} - v \| \\ &< \lim_{i\to\infty} \| x_{n_i} - w \| = \lim_{n\to\infty} \| x_n - w \| \\ &= \lim_{j\to\infty} \| x_{n_j} - w \| < \lim_{j\to\infty} \| x_{n_j} - v \| \\ &= \lim_{n\to\infty} \| x_n - v \| . \end{aligned}$$

This is a contradiction, so we have $v = w$. Therefore we conclude that $\{x_n\}$ converges weakly to $v \in F(S) \cap F(T) \cap EP(f)$.

From (3.6) we conclude $\{u_n\}$ also converges to $v \in F(S) \cap F(T) \cap EP(f)$. □

4. Corollary

As direct consequences of Theorem 3.1, we can obtain two corollaries.

Corollary 4.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let S be a nonspreading mapping of C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let x_n and u_n be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n S u_n, & \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0 \text{ and } \liminf_{n\to\infty} r_n > 0.$$

Then the sequences $\{x_n\}$ and $\{u_n\}$ convergence weakly to an element of $F(S) \cap EP(f)$.

Proof. Putting $\beta_n = 1$ for $n \in \mathbb{N}$ in Theorem 3.1, we get the conclusion. □

Corollary 4.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let S be a nonspreading mapping of C into itself and let T be a nonexpansive mapping of C into itself such that $F(T) \cap EP(f) \neq \emptyset$. Let x_n and u_n be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, & \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0 \text{ and } \liminf_{n\to\infty} r_n > 0.$$

Then the sequences $\{x_n\}$ and $\{u_n\}$ convergence weakly to an element of $F(T) \cap EP(f)$.

Proof. Putting $\beta_n = 0$ for $n \in \mathbb{N}$ in Theorem 3.1, we get the conclusion. □

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