FUZZY IDEALS GENERATED BY FUZZY SUBSETS IN SEMIGROUPS

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Abstract. We characterize the fuzzy ideal generated by a fuzzy subset in a semigroup and the fuzzy interior ideal generated by a fuzzy subset in a semigroup. Our work generalizes the characterizations ([8]) of those fuzzy ideals generated by fuzzy subsets in semigroups with an identity element.

1. Introduction

The concept of a fuzzy set was first introduced by Zadeh ([13]) and this concept was adapted by Rosenfeld ([9]) to define fuzzy subgroups and fuzzy ideals. Based on this pioneering work, Kuroki ([2, 3, 4, 5, 6]) introduced fuzzy semigroups and various kinds of fuzzy ideals in semigroups and characterized certain semigroups using those fuzzy ideals. On the other hand, Mo and Wang ([8]) defined some fuzzy ideals generated by fuzzy subsets in semigroups with an identity element and Xie ([12]) reproved, by characterizing the fuzzy ideals generated by fuzzy subsets using the level subsets, the results of Mo and Wang. However fuzzy ideals generated by fuzzy subsets in semigroups have not yet been defined and studied. In this note we are able to define some fuzzy ideals generated by fuzzy subsets in semigroups and obtain the same results, as special cases of our main results, that Mo and Wang ([8]) found in semigroups with an identity element.

In Section 2 we give some definitions and propositions which will be used in next sections. In Section 3 we define the fuzzy left (or right) ideal generated by a fuzzy subset in a semigroup and the fuzzy ideal generated by a fuzzy subset in a semigroup, and find, as special cases, those fuzzy ideals generated by fuzzy subsets in semigroups with an identity element. In Section 4 we define the fuzzy interior ideal generated by a fuzzy set in a semigroup and find, as a

special case, the fuzzy interior ideal generated by a fuzzy subset in a semigroup
with an identity element.

2. Preliminaries

In this section, we give some definitions and propositions which will be used
in next sections.

Definition 2.1. A function B from a set X to the closed unit interval $[0, 1]$ in
$\mathbb{R}$ is called a *fuzzy subset* in X. For every $x \in X$, $B(x)$ is called the *membership
grade* of x in B. A fuzzy subset in X is called a *fuzzy point* if and only if it
takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is
α $(0 < \alpha \le 1)$, we denote this fuzzy point by x_α, where the point x is called
its *support*. The fuzzy point x_α is said to be contained in a fuzzy subset A,
denoted by $x_\alpha \in A$, if and only if $\alpha \le A(x)$.

Remark. The crisp set S itself is a fuzzy subset of S such that $S(x) = 1$ for all
$x \in S$ (see Lemma 2.4 of [5] or [11]).

Definition 2.2. A *triangular norm* (briefly *t-norm*) is a function $T : [0, 1] \times
[0, 1] \to [0, 1]$ satisfying, for each p, q, r, s in $[0,1]$,

(1) $T(p, 1) = p$,
(2) $T(p, q) \le T(r, s)$ if $p \le r$ and $q \le s$,
(3) $T(p, q) = T(q, p)$,
(4) $T(p, T(q, r)) = T(T(p, q), r)$.

Definition 2.3. A t-norm $T : [0, 1] \times [0, 1] \to [0, 1]$ is continuous if T is
continuous with respect to the usual topologies.

It is well known ([1]) that the function $T_m : [0, 1] \times [0, 1] \to [0, 1]$ defined
by $T_m(a, b) = \min(a, b)$, the function $T_p : [0, 1] \times [0, 1] \to [0, 1]$ defined by
$T_p(a, b) = ab$, and the fuction $T_M : [0, 1] \times [0, 1] \to [0, 1]$ defined by $T_M(a, b) =
\max(a + b - 1, 0)$ are continuous t-norms.

For fuzzy sets U, V in a set X, Liu ([7]) defined $U \circ V$ by

$$(U \circ V)(x) = \begin{cases} \sup_{ab=x} \ \min(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \ne x. \end{cases}$$

Sessa ([10]) generalized this definition by replacing the minimum operation
with a t-norm as follows.

Definition 2.4. Let X be a set and let U, V be two fuzzy sets in X. $U \circ V$ is
defined by

$$(U \circ V)(x) = \begin{cases} \sup_{ab=x} \ T(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \ne x. \end{cases}$$

We write UV for $U \circ V$ throughout this paper.

Proposition 2.5. *Let $A_1, A_2, \ldots, A_n$ be fuzzy subsets of a set S. Then*

   (1) $S(A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq SA_1 \cup SA_2 \cup \cdots \cup SA_n$,

   (2) $(A_1 \cup A_2 \cup \cdots \cup A_n)S \subseteq A_1 S \cup A_2 S \cup \cdots \cup A_n S$.

Proof. (1) Since $S(a) = 1$,

$$[S(A_1 \cup A_2 \cup \cdots \cup A_n)](x) = \sup_{ab=x} \ T(S(a), (A_1 \cup A_2 \cup \cdots \cup A_n)(b))$$
$$= \sup_{ab=x} \ \max[A_1(b), A_2(b), \ldots, A_n(b)].$$

Since $S(a) = 1$,

$$(SA_1 \cup SA_2 \cup \cdots \cup SA_n)(x)$$
$$= \max \left[\sup_{ab=x} T(S(a), A_1(b)), \ldots, \sup_{ab=x} T(S(a), A_n(b)) \right]$$
$$= \max \left[\sup_{ab=x} A_1(b), \sup_{ab=x} A_2(b), \ldots, \sup_{ab=x} A_n(b) \right].$$

Thus $S(A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq SA_1 \cup SA_2 \cup \cdots \cup SA_n$.

   (2) Similarly, we may prove $(A_1 \cup A_2 \cup \cdots \cup A_n)S \subseteq A_1 S \cup A_2 S \cup \cdots \cup A_n S$. $\square$

Liu ([7]) proved Proposition 2.6, Proposition 2.7, and Proposition 2.8 for the case that the t-norm is a minimum function.

Proposition 2.6. *Let A, B be fuzzy sets in a set X and let x_p, y_q be fuzzy points in X. Then*

   (1) $x_p y_q = (xy)_{T(p,q)}$,

   (2) $AB = \bigcup_{x_p \in A, y_q \in B} x_p y_q$, *where* $(x_p y_q)(z) = \sup_{cd=z} \ T(x_p(c), y_q(d))$.

Proof. (1) Straightforward.

(2) Since $s_{A(s)} \in A$ and $t_{B(t)} \in B$,

$$\left(\bigcup_{x_p \in A, y_q \in B} x_p y_q \right)(z) = \sup_{x_p \in A, y_q \in B} \ \sup_{st=z} T(x_p(s), y_q(t))$$
$$\geq \sup_{st=z} \ T \left(s_{A(s)}(s), \ t_{B(t)}(t) \right)$$
$$= \sup_{st=z} \ T \left(A(s), \ B(t) \right)$$
$$= (AB)(z).$$

For $x_p \in A$ and $y_q \in B$, $A(s) \geq x_p(s)$ and $B(t) \geq y_q(t)$. Thus

$$(AB)(z) = \sup_{st=z} \ T(A(s), B(t))$$
$$\geq \sup_{st=z} \ T(x_p(s), y_q(t))$$

for all $x_p \in A$ and all $y_q \in B$. Let

$$C = \left\{ c \in \mathbb{R} : c \leq \sup_{st=z} T(A(s), B(t)) \right\},$$

$$D = \left\{ \sup_{st=z} T(x_p(s), y_q(t)) : x_p \in A, \ y_q \in B \right\}.$$

Then $D \subseteq C$ and $\sup_{x_p \in A, y_q \in B} D \in \overline{D} \subseteq \overline{C}$. Since C is closed, $\sup_{x_p \in A, y_q \in B} D \in C$. Thus

$$(AB)(z) \geq \sup_{x_p \in A, y_q \in B} \ \sup_{st=z} T (x_p(s), \ y_q(t))$$

$$= \left(\bigcup_{x_p \in A, y_q \in B} x_p y_q \right) (z). \qquad \square$$

Proposition 2.7. *Let A be a fuzzy set of a groupoid X. Then the followings are equivalent.*

    (1) *A is a fuzzy groupoid, that is, $A(xy) \geq T(A(x), A(y))$.*
    (2) *For any $x_p, y_q \in A$, $x_p y_q \in A$.*
    (3) *$AA \subseteq A$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $A(xy) \geq T(A(x), A(y))$. By Proposition 2.6,

$$(x_p y_q)(z) = [(xy)_{T(p,q)}](z) = \begin{cases} T(p, q), & \text{if } z = xy \\ 0, & \text{if } z \neq xy. \end{cases}$$

Let $x_p, y_q \in A$. Then $A(x) \geq p$ and $A(y) \geq q$. If $z = xy$, $A(z) = A(xy) \geq T(A(x), A(y)) \geq T(p, q) = (x_p y_q)(z)$, and hence $x_p y_q \in A$. If $z \neq xy$, $A(z) \geq (x_p y_q)(z) = 0$, and hence $x_p y_q \in A$.

    (2) $\Rightarrow$ (3) Suppose that for any $x_p, y_q \in A$, $x_p y_q \in A$. By Proposition 2.6,

$$(AA)(z) = \left[\bigcup_{x_p \in A, y_q \in A} x_p y_q \right] (z) = \sup_{x_p \in A, y_q \in A} (x_p y_q)(z).$$

Let $C = \{ p \in \mathbb{R} : \ p \leq A(z) \}$ and $D = \{ (x_p y_q)(z) : x_p \in A, \ y_q \in A \}$. Then $D \subseteq C \subseteq \mathbb{R}$ and $\sup_{x_p \in A, y_q \in A} D \in \overline{D} \subseteq \overline{C}$. Since C is closed, $\sup_{x_p \in A, y_q \in A} D \in C$. Thus $AA(z) \leq A(z)$.

    (3) $\Rightarrow$ (1) Straightforward. $\qquad \square$

Proposition 2.8. *Let A, B, and C be fuzzy sets in a groupoid X and let T be a continuous t-norm. If X is associative, then $(AB)C = A(BC)$.*

Proof. Let $S = \{ T(A(p), B(q)) : pq = a \} \subseteq \mathbb{R}$ and let $\sup S = \alpha$. Then α is an upper bound of S and there exists a sequence $s_n \in S$ such that $s_n \to \alpha$. Since α is an upper bound of S and T is an increasing function, $T(s, r) \leq T(\alpha, r)$ for all $s \in S$. Since T is continuous, $\lim_{n \to \infty} T(s_n, r) = T(\alpha, r)$. Since $T(\alpha, r)$

is an upper bound of $T(S,r)$ and there exists $T(s_n, r) \in T(S,r)$ such that $\lim_{n\to\infty} T(s_n, r) = T(\alpha, r)$, $\sup T(S,r) = T(\alpha, r) = T(\sup S, r)$. That is,

$$\sup_{pq=a} T\left[T(A(p), B(q)), C(b)\right] = T\left[\sup_{pq=a} T(A(p), B(q)), C(b)\right].$$

Thus

$$\sup_{ab=z} T\left[\sup_{pq=a} T(A(p), B(q)), C(b)\right] = \sup_{ab=z}\sup_{pq=a} T\left[T(A(p), B(q)), C(b)\right]$$
$$= \sup_{(pq)b=z} T\left[T(A(p), B(q)), C(b)\right].$$

Similarly we may show that

$$\sup_{pa=z} T\left[A(p), \sup_{qb=a} T(B(q), C(b))\right] = \sup_{p(qb)=z} T\left[A(p), T(B(q), C(b))\right].$$

Since X is associative,

$$
\begin{aligned}
[(AB)C]\,(z) &= \sup_{ab=z} T\left[\sup_{pq=a} T(A(p), B(q)), C(b)\right] \\
&= \sup_{(pq)b=z} T\left[T(A(p), B(q)),\ C(b)\right] \\
&= \sup_{p(qb)=z} T\left[A(p),\ T(B(q), C(b))\right] \\
&= \sup_{pa=z} T\left[A(p), \sup_{qb=a} T(B(q), C(b))\right] \\
&= \sup_{pa=z} T[A(p), (BC)(a)] = [A(BC)](z).
\end{aligned}
$$
$\square$

Definition 2.9. Let A be a fuzzy set in a semigroup S. Then A is called a *fuzzy semigroup* in S if $A(xy) \geq T(A(x), A(y))$ for all $x, y \in S$.

From now on, we assume that every t-norm in this paper is continuous.

3. Fuzzy ideals generated by fuzzy subsets in semigroups

In this section, we define the fuzzy left (or right) ideal generated by a fuzzy subset in a semigroup and the fuzzy ideal generated by a fuzzy subset in a semigroup, and find, as special cases, those fuzzy ideals generated by fuzzy subsets in semigroups with an identity element, that were found originally by Mo and Wang ([8]).

Definition 3.1. Let X be a groupoid and let A, B, C be fuzzy sets in X. A is called a *fuzzy left ideal* if and only if $A(xy) \geq A(y)$. B is called a *fuzzy right ideal* if and only if $B(xy) \geq B(x)$. C is called a *fuzzy ideal* if and only if $C(xy) \geq \max(C(x), C(y))$.

Definition 3.2. Let A be a fuzzy set of a groupoid X. The smallest fuzzy left (or right) ideal of X containing A is called the *fuzzy left (or right) ideal generated by A*. The smallest fuzzy ideal of X containing A is called the *fuzzy ideal generated by A*.

It is easy to see that the intersection of any set of fuzzy ideals is a fuzzy ideal (see Proposition 3.3 of [9]). Thus the intersection of all fuzzy ideals of X containing A is the fuzzy ideal generated by A.

Theorem 3.3. *Let A be a fuzzy subset in a semigroup S. Then the fuzzy left (or right) ideal L (or R) generated by A is $A \cup SA$ (or $A \cup AS$). That is, $L(x) = \max[A(x), \sup_{ab=x} A(b)]$ and $R(x) = \max[A(x), \sup_{ab=x} A(a)]$.*

Proof. Let $\{J_i : i \in I\}$ be the collection of all fuzzy left ideals of S containing A. Since $J_i(\beta) \leq J_i(\alpha\beta)$ and $S(a) = 1$,

$$SJ_i(x) = \sup_{ab=x} T(S(a), J_i(b)) \leq \sup_{ab=x} T(S(a), J_i(ab)) = J_i(x)$$

for each $i \in I$. Thus $SA \subseteq \bigcap_{i \in I} J_i$. Hence $A \cup SA \subseteq \bigcap_{i \in I} J_i$.

By Proposition 2.5, $S(A \cup SA) \subseteq SA \cup S(SA)$. By Proposition 2.8, $S(A \cup SA) \subseteq SA \cup (SS)A$. Since S is a semigroup, $SS \subseteq S$ and hence

$$S(A \cup SA) \subseteq SA \cup SSA \subseteq SA \cup SA \subseteq A \cup SA.$$

Since $S(x) = 1$,

$$(A \cup SA)(xy) \geq [S(A \cup SA)](xy) = \sup_{ab=xy} T(S(a), (A \cup SA)(b))$$

$$\geq T(S(x), (A \cup SA)(y)) = (A \cup SA)(y).$$

Thus $A \cup SA$ is a fuzzy left ideal of S containing A, that is, $\cap J_i \subseteq A \cup SA$. Hence $\cap J_i = A \cup SA$. Also $SA(x) = \sup_{ab=x} T(S(a), A(b)) = \sup_{ab=x} A(b)$. Similarly we may prove that $A \cup AS$ is the fuzzy right ideal generated by A. $\square$

Corollary 3.4 is due to Mo and Wang (see Theorem 3.1 and Theorem 3.2 of [8]). We obtain it as a special case of our more general approach.

Corollary 3.4. *Let A be a fuzzy subset in a semigroup S with an identity element. Then the fuzzy left (or right) ideal L (or R) generated by A is SA (or AS). That is, $L(x) = SA(x) = \sup_{ab=x} A(b)$ and $R(x) = AS(x) = \sup_{ab=x} A(a)$.*

Proof. Let e be the identity of S. Since $S(e) = 1$, $SA(x) = \sup_{ab=x} T(S(a), A(b)) \geq T(S(e), A(x)) = A(x)$. We may show that SA is the smallest fuzzy left ideal containing A by the same way as shown in Theorem 3.3. Similarly we may prove that AS is the fuzzy right ideal of S generated by A. $\square$

Theorem 3.5. *Let A be a fuzzy subset in a semigroup S. Then the fuzzy ideal F generated by A is $A \cup SA \cup AS \cup SAS$. That is,*

$$F(x) = \max \left[A(x), \sup_{ab=x} A(b), \sup_{ab=x} A(a), \sup_{cdb=x} A(d) \right].$$

Proof. Let $\{J_i : i \in I\}$ be the collection of all fuzzy ideals of S containing A. We may show $AS \subseteq \bigcap_{i \in I} J_i$ and $SA \subseteq \bigcap_{i \in I} J_i$ by the same way as shown in Theorem 3.3. Since $S(b) = S(c) = 1$ and $SJ_iS = (SJ_i)S$ from Proposition 2.8,

$$
\begin{aligned}
(SJ_iS)(x) &= \sup_{ab=x} T(SJ_i(a), S(b)) = \sup_{ab=x} SJ_i(a) \\
&= \sup_{ab=x} \sup_{cd=a} T(S(c), J_i(d)) = \sup_{ab=x} \sup_{cd=a} J_i(d) \\
&\leq \sup_{ab=x} \sup_{cd=a} J_i(cd) = \sup_{ab=x} J_i(a) \\
&\leq \sup_{ab=x} J_i(ab) = J_i(x)
\end{aligned}
$$

for each $i \in I$. Thus $SAS \subseteq \bigcap_{i \in I} J_i$. Hence $A \cup SA \cup AS \cup SAS \subseteq \bigcap_{i \in I} J_i$. By Proposition 2.5, $S(A \cup SA \cup AS \cup SAS) \subseteq SA \cup S(SA) \cup S(AS) \cup S(SAS)$. By Proposition 2.8, $S(SA) = (SS)A$, $S(AS) = SAS$, and $S(SAS) = (SS)AS$. Thus $S(A \cup SA \cup AS \cup SAS) \subseteq SA \cup (SS)A \cup SAS \cup (SS)AS$. Since S is a semigroup, $S(A \cup SA \cup AS \cup SAS) \subseteq A \cup SA \cup AS \cup SAS$. Since $S(x) = 1$,

$$
\begin{aligned}
(A \cup SA \cup AS \cup SAS)(xy) &\geq [S(A \cup SA \cup AS \cup SAS)](xy) \\
&= \sup_{ab=xy} T(S(a), (A \cup SA \cup AS \cup SAS)(b)) \\
&\geq T(S(x), (A \cup SA \cup AS \cup SAS)(y)) \\
&= (A \cup SA \cup AS \cup SAS)(y).
\end{aligned}
$$

Thus $A \cup SA \cup AS \cup SAS$ is a fuzzy left ideal of S. Similarly we may show that $A \cup SA \cup AS \cup SAS$ is a fuzzy right ideal of S. Thus $A \cup SA \cup AS \cup SAS$ is a fuzzy ideal of S containing A, that is, $\cap J_i \subseteq A \cup SA \cup AS \cup SAS$. Hence $\cap J_i = A \cup SA \cup AS \cup SAS$. Also

$$
SAS(x) = \sup_{ab=x} T(SA(a), S(b)) = \sup_{ab=x} SA(a) = \sup_{ab=x} \sup_{cd=a} A(d) = \sup_{cdb=x} A(d).
$$

$\square$

Example of Theorem 3.5. Let $S = \{a, b, c, d, f\}$. We define a binary operation on S by means of the following table:

$*$	a	b	c	d	f
a	a	d	a	d	f
b	f	b	c	f	f
c	c	b	c	b	f
d	f	d	a	f	f
f	f	f	f	f	f

Then S is a noncommutative semigroup. Let A be a fuzzy set in S such that $A(a) = 0.3$, $A(b) = 0.2$, $A(c) = 0.1$, $A(d) = 0.5$, $A(f) = 0.7$. Since $S(x) = 1$ for all $x \in S$,

$$
SA(a) = \sup_{xy=a} T(S(x), A(y)) = \max [A(a), A(c)] = 0.3.
$$

Similarly we may show that

$$SA(c) = 0.3, \ SA(b) = SA(d) = 0.5, \ AS(a) = AS(d) = 0.5,$$
$$AS(b) = AS(c) = 0.2, \ SA(f) = AS(f) = 0.7.$$

Since $S(x) = 1$ for all $x \in S$,

$$SAS(a) = \sup_{xy=a} T(S(x), \ AS(y)) = \max [AS(a), \ AS(c)] = 0.5.$$

Similarly we may show that

$$SAS(b) = SAS(c) = SAS(d) = 0.5, \ SAS(f) = 0.7.$$

Let $H = A \cup SA \cup AS \cup SAS$. Then

$$H(a) = H(b) = H(c) = H(d) = 0.5, \ \ H(f) = 0.7.$$

It is easily checked that $H(\alpha\beta) \geq \max [H(\alpha), \ H(\beta)]$ for every $\alpha, \beta \in S$. That is, H is a fuzzy ideal containing A. Let I be a fuzzy ideal in S containing A. Then $I(a) = I(dc) \geq I(d) \geq A(d) = 0.5 = H(a)$. That is, $H(a) \leq I(a)$. Similarly, $H(b) \leq I(b)$, $H(c) \leq I(c)$, $H(d) \leq I(d)$, and $H(f) \leq I(f)$. That is, $H \subseteq I$. Thus $H = A \cup SA \cup AS \cup SAS$ such that $H(a) = H(b) = H(c) = H(d) = 0.5$ and $H(f) = 0.7$ is the fuzzy ideal generated by A.

Corollary 3.6 is due to Mo and Wang (see Theorem 3.3 of [8]). Also Xie showed it again (see Theorem 4.1 of [12]). We obtain it as a special case of our more general approach.

Corollary 3.6. *Let A be a fuzzy subset in a semigroup S with an identity element. Then the fuzzy ideal F generated by A is SAS. That is, $F(x) = SAS(x) = \sup_{cdb=x} A(d)$.*

Proof. Let e be the identity of S. Then $SAS(x) \geq T(S(e), AS(x)) = (AS)(x) \geq T(A(x), S(e)) = A(x)$. We may show that SAS is the smallest fuzzy ideal containing A by the same way as shown in Theorem 3.5. □

4. Fuzzy interior ideals generated by fuzzy subsets in semigroups

In this section, we define the fuzzy interior ideal generated by a fuzzy subset in a semigroup and find, as a special case, the fuzzy interior ideal generated by a fuzzy subset in a semigroup with an identity element, which was found originally by Mo and Wang ([8]).

Definition 4.1. Let S be a semigroup. A fuzzy semigroup I in S is called a *fuzzy interior ideal* of S if $I(xyz) \geq I(y)$.

Theorem 4.2. *Let A be a fuzzy subset in a semigroup S. Then the fuzzy interior ideal F generated by A is $A \cup A^2 \cup SAS$. That is, $F(x) = \max [A(x), \ AA(x), \sup_{cdb=x} A(d)]$.*

Proof. Let $\{J_i : i \in I\}$ be the collection of all fuzzy interior ideals of S containing A. Since T is an increasing continuous function,

$$
\begin{aligned}
(SJ_iS)(x) &= \sup_{ab=x} T(SJ_i(a), S(b)) = \sup_{ab=x} SJ_i(a) \\
&= \sup_{ab=x} \sup_{cd=a} T(S(c), J_i(d)) = \sup_{ab=x} \sup_{cd=a} J_i(d) \\
&= \sup_{cdb=x} J_i(d) \leq \sup_{cdb=x} J_i(cdb) = J_i(x)
\end{aligned}
$$

for each $i \in I$. Thus $SAS \subseteq SJ_iS \subseteq J_i$ for each $i \in I$. Since J_i is a fuzzy semigroup, $(J_iJ_i)(x) = \sup_{ab=x} T(J_i(a), J_i(b)) \leq \sup_{ab=x} J_i(ab) = J_i(x)$. Thus $A^2 \subseteq J_iJ_i \subseteq J_i$ for each $i \in I$. Hence $A \cup A^2 \cup SAS \subseteq \bigcap_{i \in I} J_i$.

By Proposition 2.5 and Proposition 2.8, $S(A \cup A^2 \cup SAS)S \subseteq (SA \cup SA^2 \cup SSAS)S \subseteq SAS \cup SA^2S \cup SSASS$. Since S is a semigroup, $SAS \cup SA^2S \cup SSASS \subseteq SAS \cup SA(SS) \cup (SS)A(SS) \subseteq SAS$. Thus $S(A \cup A^2 \cup SAS)S \subseteq A \cup A^2 \cup SAS$. Let $H = A \cup A^2 \cup SAS$. Then $SHS \subseteq H$. Since T is an increasing continuous function,

$$
\begin{aligned}
H(xyz) \geq (SHS)(xyz) &= \sup_{ab=xyz} T[SH(a), S(b)] \\
&= \sup_{ab=xyz} SH(a) = \sup_{ab=xyz} \sup_{cd=a} T(S(c), H(d)) \\
&= \sup_{ab=xyz} \sup_{cd=a} H(d) = \sup_{cdb=xyz} H(d) \geq H(y).
\end{aligned}
$$

By Proposition 2.5 and Proposition 2.8, $HH = (A \cup A^2 \cup SAS)(A \cup A^2 \cup SAS) \subseteq A^2 \cup A^2A \cup SASA \cup AA^2 \cup A^2A^2 \cup SASA^2 \cup ASAS \cup A^2SAS \cup SASSAS$. Since S is a semigroup and $A \subseteq S$, $HH \subseteq A^2 \cup A^2A \cup SASA \cup AA^2 \cup A^2A^2 \cup SASA^2 \cup ASAS \cup A^2SAS \cup SASSAS \subseteq A^2 \cup SAS \cup SA(SS) \cup SAS \cup (SS)AS \cup (SSS)AS \cup (SS)AS \cup (SSS)AS \cup S^4AS \subseteq A \cup A^2 \cup SAS$. By Proposition 2.7, $A \cup A^2 \cup SAS$ is a fuzzy semigroup. Thus $H = A \cup A^2 \cup SAS$ is a fuzzy interior ideal of S containing A. Hence $\bigcap_{i \in I} J_i \subseteq A \cup A^2 \cup SAS$. Also $SAS(x) = \sup_{cdb=x} A(d)$ by the proof of Theorem 3.5. $\qquad \square$

Example of Theorem 4.2. Let $S = \{a, b, c, d, f\}$. We define a binary operation on S by means of the following table:

$*$	a	b	c	d	f
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
f	d	f	f	a	a

It is straightforward to see that S is a noncommutative semigroup. Let A be a fuzzy set in S such that

$$A(a) = 0.3, \ A(b) = 0.1, \ A(c) = 0.5, \ A(d) = 0.9, \ \text{and} \ A(f) = 0.7.$$

Since $A^2(b) = \sup_{xy=b} T(A(x), A(y))$,

$$A^2(b) = \max [\ T(A(b), A(b)),\ T(A(c), A(c))\ ] = T(0.5, 0.5).$$

Similarly we may show that

$$A^2(a) = T(0.9, 0.9),\ A^2(c) = T(0.1, 0.5),\ A^2(d) = T(0.5, 0.9),\ A^2(f) = T(0.7, 0.5).$$

Since $(AS)(b) = \sup_{xy=b} T(A(x), S(y)) = \max[A(b), A(c)] = 0.5$ and $AS(c) = 0.5$,

$$(SAS)(b) = \sup_{xy=b} T(S(x), AS(y)) = \max [AS(b),\ AS(c)] = \max [0.5,\ 0.5] = 0.5.$$

Similarly we may show $(SAS)(a) = (SAS)(d) = 0.9$, $SAS(c) = 0.5$, $SAS(f) = 0.5$. Let $H = A \cup A^2 \cup SAS$. Then

$$H(a) = H(d) = 0.9,\ H(b) = H(c) = 0.5,\ H(f) = 0.7.$$

It is easily checked that $H(\alpha\beta) \geq T(H(\alpha),\ H(\beta))$ and $H(\alpha\beta\gamma) \geq H(\beta)$ for every $\alpha, \beta, \gamma \in S$. That is, H is a fuzzy interior ideal in S. Let I be a fuzzy interior ideal containing A. Then $I(a) = I(add) \geq I(d) \geq A(d) = 0.9 = H(a)$, $I(b) = I(bcc) \geq I(c) \geq A(c) = 0.5 = H(b)$, $I(c) \geq A(c) = H(c)$, $I(d) \geq A(d) = H(d)$, and $I(f) \geq A(f) = H(f)$. That is, $H \subseteq I$. Thus $H = A \cup A^2 \cup SAS$ is the fuzzy interior ideal generated by A.

Corollary 4.3 is due to Mo and Wang (see Theorem 4.2 of [8]). We obtain it as a special case of our more general approach.

Corollary 4.3. *Let A be a fuzzy subset in a semigroup S with an identity element. Then the fuzzy interior ideal F generated by A is SAS. That is, $F(x) = \sup_{cdb=x}\ A(d)$.*

Proof. From the proof of Corollary 3.6, $A \subseteq SAS$. We may show that SAS is the smallest fuzzy interior ideal of S containing A by the same way as shown in Theorem 4.2. $\square$

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