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$\mathcal{N}$-SUBALGEBRAS OF TYPE $(\in, \in\vee q)$ BASED ON POINT $\mathcal{N}$-STRUCTURES IN BCK/BCI-ALGEBRAS

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Abstract. Characterizations of $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ are provided. The notion of $\mathcal{N}$-subalgebras of type $(\overline{\in}, \overline{\in}\vee\overline{q})$ is introduced, and its characterizations are discussed. Conditions for an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ (resp. $(\overline{\in}, \overline{\in}\vee\overline{q})$) to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$ are considered.

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [5] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [6] considered closed ideals in BCH-algebras based on $\mathcal{N}$-structures. Also, using $\mathcal{N}$-structures, Jun and Lee introduced the notion of an $\mathcal{N}$-essence in a subtraction algebra, and investigated related properties. They discussed relations among an $\mathcal{N}$-ideal, an $\mathcal{N}$-subalgebra and an $\mathcal{N}$-essence (see [4]). To obtain more general form of an $\mathcal{N}$-subalgebra in BCK/BCI-algebras, Jun et al. defined the notions of $\mathcal{N}$-subalgebras of types $(\in, \in)$, $(\in, q)$, $(\in, \in\vee q)$, $(q, \in)$, (q, q) and $(q, \in\vee q)$, and investigated related properties. They provided a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$. They also gave conditions for an $\mathcal{N}$-structure to

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be an $\mathcal{N}$-subalgebra of type $(q, \in\vee q)$ (see [2]). As a continuation of the paper [2], we give characterizations of $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$. We introduced the notion of $\mathcal{N}$-subalgebras of type $(\overline{\in}, \overline{\in}\vee\overline{q})$, and discuss its characterizations. We provide conditions for an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ (resp. $(\overline{\in}, \overline{\in}\vee\overline{q})$) to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \Longrightarrow x = y$,

where x, y and z are elements of X. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \iff x * y = 0).$$

In a BCK/BCI-algebra X, the following hold:

(b1) $x * 0 = x$,
(b2) $(x * y) * z = (x * z) * y$,

where x, y and z are elements of X.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [1] and [8] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote a BCK/BCI-algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

Definition 2.1 (See [5]). By a *subalgebra* of X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-subalgebra* of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following condition:

$$f(x * y) \leq \vee\{f(x), f(y)\}, \tag{2.1}$$

where x and y are elements of X.

For any $\mathcal{N}$-structure (X,f) and $\alpha \in [-1,0)$, the set $C(f;\alpha) := \{x \in X \mid f(x) \le \alpha\}$ is called the *closed support* of (X,f) related to α, and the set $O(f;\alpha) := \{x \in X \mid f(x) < \alpha\}$ is called the *open support* of (X,f) related to α.

Using the similar method to the transfer principle in fuzzy theory (see [3, 7]), Jun et al. [6] considered transfer principle in $\mathcal{N}$-structures as follows.

Theorem 2.2 ($\mathcal{N}$-transfer principle, See [6])**.** *An $\mathcal{N}$-structure (X,f) satisfies the property $\overline{\mathcal{P}}$ if and only if for all $\alpha \in [-1,0]$,*

$$C(f;\alpha) \ne \emptyset \;\Rightarrow\; C(f;\alpha) \text{ satisfies the property } \mathcal{P}.$$

Lemma 2.3 (See [5])**.** *An $\mathcal{N}$-structure (X,f) is an $\mathcal{N}$-subalgebra of X if and only if every open support of (X,f) related to α is a subalgebra of X for all $\alpha \in [-1,0)$.*

3. $\mathcal{N}$-subalgebras of types $(\in, \in\vee q)$ and $(\overline{\in}, \overline{\in}\vee\overline{q})$

Let (X,f) be an $\mathcal{N}$-structure in which f is given by

$$f(y) = \begin{cases} 0 & \text{if } y \ne x, \\ \alpha & \text{if } y = x, \end{cases}$$

where $\alpha \in [-1,0)$. In this case, f is denoted by x_α and we call (X, x_α) a *point $\mathcal{N}$-structure*. For any $\mathcal{N}$-structure (X,g), we say that a point $\mathcal{N}$-structure (X, x_α) is an *$\mathcal{N}_\in$-subset* (resp. *$\mathcal{N}_q$-subset*) of (X,g) if $g(x) \le \alpha$ (resp. $g(x)+\alpha+1 < 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_\in$-subset of (X,g) or an $\mathcal{N}_q$-subset of (X,g), we say (X, x_α) is an *$\mathcal{N}_{\in\vee q}$-subset* of (X,g).

Definition 3.1 (See [2])**.** An $\mathcal{N}$-structure (X,f) is called an *$\mathcal{N}$-subalgebra* of type $(\in, \in)$ (resp. type $(\in, \in\vee q)$) if whenever two point $\mathcal{N}$-structures (X, x_{α_1}) and (X, y_{α_2}) are $\mathcal{N}_\in$-subsets of (X,f), then the point $\mathcal{N}$-structure $\big(X, (x*y)_{\vee\{\alpha_1,\alpha_2\}}\big)$ is an $\mathcal{N}_\in$-subset (resp. $\mathcal{N}_{\in\vee q}$-subset) of (X,f).

Lemma 3.2 (See [2])**.** *An $\mathcal{N}$-structure (X,f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ if and only if it satisfies:*

$$(3.1) \qquad (\forall x,y \in X)\ \big(f(x*y) \le \vee\{f(x), f(y), -0.5\}\big).$$

Theorem 3.3. *An $\mathcal{N}$-structure (X,f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ if and only if for every $\alpha \in [-0.5,0]$ the nonempty closed support of (X,f) related to α is a subalgebra of X.*

Proof. Assume that (X,f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ and let $\alpha \in [-0.5,0]$ be such that $C(f;\alpha) \ne \emptyset$. Let $x,y \in C(f;\alpha)$. Then $f(x) \le \alpha$ and $f(y) \le \alpha$. It follows from Lemma 3.2 that $f(x*y) \le \vee\{f(x), f(y), -0.5\} \le \vee\{\alpha, -0.5\} = \alpha$ so that $x*y \in C(f;\alpha)$. Therefore $C(f;\alpha)$ is a subalgebra of X.

Conversely, let (X,f) be an $\mathcal{N}$-structure such that the non-empty closed support of (X,f) related to α is a subalgebra of X for all $\alpha \in [-0.5,0]$. If there exist $a, b \in X$ such that $f(a * b) > \vee\{f(a), f(b), -0.5\}$, then we can take $\beta \in [-1, 0]$ such that $f(a * b) > \beta \geq \vee\{f(a), f(b), -0.5\}$. Thus $a, b \in C(f; \beta)$ and $\beta \geq -0.5$, and so $a * b \in C(f; \beta)$, i.e., $f(a * b) \leq \beta$. This is a contradiction, and therefore $f(x * y) \leq \vee\{f(x), f(y), -0.5\}$ for all $x, y \in X$. Using Lemma 3.2, we conclude that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$. □

Theorem 3.4. *Let S be a subalgebra of X. For any $\alpha \in [-0.5, 0)$, there exists an $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in\vee q)$ for which S is represented by the closed support of (X, f) related to α.*

Proof. Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(x) = \begin{cases} \alpha & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases}$$

for all $x \in X$ where $\alpha \in [-0.5, 0)$. Assume that $f(a*b) > \vee\{f(a), f(b), -0.5\}$ for some $a, b \in X$. Since the cardinality of the image of f is 2, we have $f(a * b) = 0$ and $\vee\{f(a), f(b), -0.5\} = \alpha$. Since $\alpha \geq -0.5$, it follows that $f(a) = \alpha = f(b)$ so that $a, b \in S$. Since S is a subalgebra of X, we obtain $a * b \in S$ and so $f(a * b) = \alpha < 0$. This is a contradiction. Therefore $f(x * y) \leq \vee\{f(x), f(y), -0.5\}$ for all $x, y \in X$. Using Lemma 3.2, we conclude that (X, f) is an $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in\vee q)$. Obviously, S is represented by the closed support of (X, f) related to α. □

Note that every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$, but the converse is not true in general (see [2]). Now, we give a condition for an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$.

Theorem 3.5. *Let (X, f) be an $\mathcal{N}$-structure of type $(\in, \in\vee q)$ such that $f(x) > -0.5$ for all $x \in X$. Then (X, f) is an $\mathcal{N}$-subalgebra of $(\in, \in)$.*

Proof. Let $x, y \in X$ and $\alpha \in [-1, 0)$ be such that (X, x_{α_1}) and (X, y_{α_2}) are $\mathcal{N}_\in$-subsets of (X, f). Then $f(x) \leq \alpha_1$ and $f(y) \leq \alpha_2$. It follows from Lemma 3.2 and the hypothesis that $f(x * y) \leq \vee\{f(x), f(y), -0.5\} = \vee\{f(x), f(y)\} \leq \vee\{\alpha_1, \alpha_2\}$ so that $\left(X, (x * y)_{\vee\{\alpha_1, \alpha_2\}}\right)$ is an $\mathcal{N}_\in$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$. □

For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0)$, the *q-support* and the *$\in\vee q$-support* of (X, f) related to α are defined to be the sets

$$\mathcal{N}_q(f; \alpha) := \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_q\text{-subset of } (X, f)\}$$

and

$$\mathcal{N}_{\in\vee q}(f; \alpha) := \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_{\in\vee q}\text{-subset of } (X, f)\},$$

respectively. Note that the $\in\vee q$-support is the union of the closed support and the q-support, that is, $\mathcal{N}_{\in\vee q}(f; \alpha) = C(f; \alpha) \cup \mathcal{N}_q(f; \alpha),\ \alpha \in [-1, 0)$.

Theorem 3.6. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$ if and only if the $\in\vee q$-support of (X, f) related to α is a subalgebra of X for all $\alpha \in [-1, 0)$.*

Proof. Suppose that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$. Let $x, y \in \mathcal{N}_{\in\vee q}(f; \alpha)$ for $\alpha \in [-1, 0)$. Then (X, x_α) and (X, y_α) are $\mathcal{N}_{\in\vee q}$-subsets of (X, f). Hence $f(x) \leq \alpha$ or $f(x) + \alpha + 1 < 0$, and $f(y) \leq \alpha$ or $f(y) + \alpha + 1 < 0$. Then we consider the following four cases:

(c1) $f(x) \leq \alpha$ and $f(y) \leq \alpha$.
(c2) $f(x) \leq \alpha$ and $f(y) + \alpha + 1 < 0$.
(c3) $f(x) + \alpha + 1 < 0$ and $f(y) \leq \alpha$.
(c4) $f(x) + \alpha + 1 < 0$ and $f(y) + \alpha + 1 < 0$.

Combining (3.1) and (c1), we have $f(x * y) \leq \vee\{\alpha, -0.5\}$. If $\alpha \geq -0.5$, then $f(x * y) \leq \alpha$ and so $(X, (x * y)_\alpha)$ is an $\mathcal{N}_\in$-subset of (X, f). Hence $x * y \in C(f; \alpha) \subseteq \mathcal{N}_{\in\vee q}(f; \alpha)$. If $\alpha < -0.5$, then $f(x*y) \leq -0.5$ and so $f(x*y)+\alpha+1 < -0.5 - 0.5 + 1 = 0$, that is, $(X, (x * y)_\alpha)$ is an $\mathcal{N}_q$-subset of (X, f). Therefore $x * y \in \mathcal{N}_q(f; \alpha) \subseteq \mathcal{N}_{\in\vee q}(f; \alpha)$. For the case (c2), assume that $\alpha < -0.5$. Then

$$\begin{aligned} f(x * y) &\leq \vee\{f(x), f(y), -0.5\} \leq \vee\{\alpha, f(y), -0.5\} = \vee\{f(y), -0.5\} \\ &= \begin{cases} f(y) & \text{if } f(y) > -0.5, \\ -0.5 & \text{if } f(y) \leq -0.5 \end{cases} \\ &< -1 - \alpha, \end{aligned}$$

and so $f(x * y) + \alpha + 1 < 0$. Thus $(X, (x * y)_\alpha)$ is an $\mathcal{N}_q$-subset of (X, f). If $\alpha \geq -0.5$, then

$$\begin{aligned} f(x * y) &\leq \vee\{f(x), f(y), -0.5\} \leq \vee\{\alpha, f(y), -0.5\} = \vee\{\alpha, f(y)\} \\ &= \begin{cases} f(y) & \text{if } f(y) > \alpha, \\ \alpha & \text{if } f(y) \leq \alpha, \end{cases} \end{aligned}$$

and thus $x * y \in \mathcal{N}_q(f; \alpha)$ or $x * y \in C(f; \alpha)$. Consequently, $x * y \in \mathcal{N}_{\in\vee q}(f; \alpha)$. For the case (c3), it is similar to the case (c2). Finally, for the case (c4), if $\alpha \geq -0.5$, then $-1 - \alpha \leq -0.5 \leq \alpha$. Hence $f(x * y) \leq \vee\{f(x), f(y), -0.5\} \leq \vee\{-1 - \alpha, -0.5\} = -0.5 \leq \alpha$, which implies that $x * y \in C(f; \alpha)$. If $\alpha < -0.5$, then $\alpha < -0.5 < -1 - \alpha$. Therefore $f(x * y) \leq \vee\{f(x), f(y), -0.5\} \leq \vee\{-1 - \alpha, -0.5\} = -1 - \alpha$, that is, $f(x * y) + \alpha + 1 < 0$, which means that $(X, (x * y)_\alpha)$ is an $\mathcal{N}_q$-subset of (X, f). Consequently, the $\in\vee q$-support of (X, f) related to α is a subalgebra of X for all $\alpha \in [-1, 0)$.

Conversely, let (X, f) be an $\mathcal{N}$-structure for which the $\in\vee q$-support of (X, f) related to α is a subalgebra of X for all $\alpha \in [-1, 0)$. Assume that there exist $a, b \in X$ such that $f(a * b) > \vee\{f(a), f(b), -0.5\}$. Then $f(a * b) > \beta \geq \vee\{f(a), f(b), -0.5\}$ for some $\beta \in [-0.5, 0)$. It follows that $a, b \in C(f; \beta) \subseteq \mathcal{N}_{\in\vee q}(f; \beta)$ but $a * b \notin C(f; \beta)$. Also, $f(a * b) + \beta + 1 > 2\beta + 1 \geq 0$, i.e., $a * b \notin \mathcal{N}_q(f; \beta)$. Thus $a * b \notin \mathcal{N}_{\in\vee q}(f; \beta)$ which is a contradiction. Therefore $f(x * y) \leq \vee\{f(x), f(y), -0.5\}$ for all $x, y \in X$. Using Lemma 3.2, we conclude that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q)$. □

Table 1. $*$-operation

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	a	a	0	0
d	d	b	a	b	0

Table 2. $*$-operation

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_{\overline{\in}}$*-subset* (resp. $\mathcal{N}_{\overline{q}}$*-subset*) of (X, g) if $g(x) > \alpha$ (resp. $g(x)+\alpha+1 \geq 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, g) or an $\mathcal{N}_{\overline{q}}$-subset of (X, g), we say (X, x_α) is an $\mathcal{N}_{\overline{\in}\vee\overline{q}}$-subset of (X, g).

Definition 3.7. An $\mathcal{N}$-structure (X, f) is called an $\mathcal{N}$*-subalgebra* of type $(\overline{\in}, \overline{\in}\vee\overline{q})$ if whenever a point $\mathcal{N}$-structure $\big(X, (x*y)_{\vee\{\alpha,\beta\}}\big)$ is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f), then (X, x_α) is an $\mathcal{N}_{\overline{\in}\vee\overline{q}}$-subset of (X, f) or (X, x_β) is an $\mathcal{N}_{\overline{\in}\vee\overline{q}}$-subset of (X, f).

Example 3.8. Let $X = \{0, a, b, c, d\}$ be a set with a $*$-operation which is given by Table 1. Then $(X; *, 0)$ is a BCK-algebra (see [8]). Consider an $\mathcal{N}$-structure (X, g) in which f is defined by

$$g = \begin{pmatrix} 0 & a & b & c & d \\ -0.8 & -0.8 & -0.5 & -0.6 & -0.3 \end{pmatrix}.$$

It is routine to verify that (X, g) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in}\vee\overline{q})$.

Example 3.9. Consider a BCI-algebra $X = \{0, a, b, c\}$ with a $*$-operation which is given by Table 2. Let (X, f) be an $\mathcal{N}$-structure in which f is defined by

$$f = \begin{pmatrix} 0 & a & b & c \\ -0.8 & -0.6 & -0.5 & -0.5 \end{pmatrix}.$$

It is routine to verify that (X, f) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in}\vee\overline{q})$.

Theorem 3.10. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ if and only if the following condition is valid*:

$$\wedge\{f(x*y), -0.5\} \leq \vee\{f(x), f(y)\}, \tag{3.2}$$

where x and y are elements of X.

Proof. Suppose that (X, f) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. If there exist $a, b \in X$ such that $\wedge\{f(a*b), -0.5\} > \alpha = \vee\{f(a), f(b)\}$, then $\alpha \in [-1, -0.5)$. It follows that (X, a_α) and (X, b_α) are $\mathcal{N}_\in$-subsets of (X, f), and $(X, (a*b)_\alpha)$ is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Hence (X, a_α) is an $\mathcal{N}_{\overline{q}}$-subset of (X, f) or (X, b_α) is an $\mathcal{N}_{\overline{q}}$-subset of (X, f). Therefore $f(a)+\alpha+1 \geq 0$ or $f(b)+\alpha+1 \geq 0$, which imply that $2\alpha + 1 \geq 0$, that is, $\alpha \geq -0.5$. This is a contradiction. Consequently, (3.2) is valid.

Conversely assume that an $\mathcal{N}$-structure (X, f) satisfies the condition (3.2). Let $x, y \in X$ and $\alpha, \beta \in [-1, 0]$ such that a point $\mathcal{N}$-structure $\big(X, (x*y)_{\vee\{\alpha,\beta\}}\big)$ is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Then $f(x*y) > \vee\{\alpha, \beta\}$. If $f(x*y) \leq \vee\{f(x), f(y)\}$, then $\vee\{f(x), f(y)\} > \vee\{\alpha, \beta\}$. Hence $f(x) > \alpha$ or $f(y) > \beta$. Thus (X, x_α) is an $\mathcal{N}_{\overline{\in}\vee\overline{q}}$-subset of (X, f) or (X, y_β) is an $\mathcal{N}_{\overline{\in}\vee\overline{q}}$-subset of (X, f). If $f(x*y) > \vee\{f(x), f(y)\}$, then $\wedge\{f(x*y), -0.5\} = -0.5$ by (3.2). Hence $f(x) \geq -0.5$ or $f(y) \geq -0.5$. Suppose that (X, x_α) and (X, y_β) are $\mathcal{N}_\in$-subsets of (X, f). Then $\alpha \geq f(x) \geq -0.5$ or $\beta \geq f(y) \geq -0.5$. It follows that $f(x)+\alpha+1 \geq 2f(x)+1 \geq 0$ or $f(y) + \beta + 1 \geq 2f(y) + 1 \geq 0$ so that (X, x_α) is an $\mathcal{N}_{\overline{q}}$-subset of (X, f) or (X, y_β) is an $\mathcal{N}_{\overline{q}}$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. □

Proposition 3.11. *If (X, f) is an $\mathcal{N}$-structure of type $(\overline{\in}, \overline{\in}\vee\overline{q})$, then $f(x) \geq f(0)$ or $f(x) \geq -0.5$ for all $x \in X$.*

Proof. It is straightforward by (a3) and (3.2). □

Theorem 3.12. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ if and only if for every $\alpha \in [-1, -0.5)$ the nonempty closed support of (X, f) related to α is a subalgebra of X.*

Proof. Suppose that (X, f) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. Let $x, y \in C(f; \alpha)$ where $\alpha \in [-1, -0.5)$. Then $f(x) \leq \alpha$ and $f(y) \leq \alpha$. It follows from (3.2) that $\wedge\{f(x*y), -0.5\} \leq \alpha$. Since $\alpha < -0.5$, we have $f(x*y) \leq \alpha$, and so $x*y \in C(f; \alpha)$. Therefore $C(f; \alpha)$ is a subalgebra of X.

Conversely, let (X, f) be an $\mathcal{N}$-structure such that the non-empty closed support of (X, f) related to α is a subalgebra of X for all $\alpha \in [-1, -0.5)$. Assume that there exist $a, b \in X$ such that $\wedge\{f(a*b), -0.5\} > \vee\{f(a), f(b)\}$. If we take $\beta := \frac{1}{2}\left(\wedge\{f(a*b), -0.5\} + \vee\{f(a), f(b)\}\right)$, then $\beta \in [-1, -0.5)$ and $\wedge\{f(a*b), -0.5\} > \beta > \vee\{f(a), f(b)\}$. Thus $a, b \in C(f; \beta)$ and $a*b \notin C(f; \beta)$. This is a contradiction, and therefore $\wedge\{f(x*y), -0.5\} \leq \vee\{f(x), f(y)\}$ for all $x, y \in X$. Using Theorem 3.10, we conclude that (X, f) is an $\mathcal{N}$-subalgebra of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. □

Obviously, every $\mathcal{N}$-subalgebra of type $(\in,\in)$ is an $\mathcal{N}$-subalgebra of type $(\overline{\in},\overline{\in}\vee\overline{q})$, but the converse is not true in general as seen in the following example.

Example 3.13. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra which is given in Example 3.8. Consider an $\mathcal{N}$-structure (X, g) in which g is defined by

$$g = \begin{pmatrix} 0 & a & b & c & d \\ -0.7 & -0.6 & -0.2 & -0.2 & -0.4 \end{pmatrix}.$$

Then (X, g) is an $\mathcal{N}$-subalgebra of type $(\overline{\in},\overline{\in}\vee\overline{q})$. But it is not an $\mathcal{N}$-subalgebra of type $(\in,\in)$ since $(X, d_{-0.25})$ and $(X, a_{-0.5})$ are $\mathcal{N}_{\in}$-subsets of (X, g), but $\big(X, (d * a)_{\vee\{-0.25,-0.5\}}\big) = (X, b_{-0.25})$ is not an $\mathcal{N}_{\in}$-subset of (X, g).

Finally, we give a condition for an $\mathcal{N}$-subalgebra of type $(\overline{\in},\overline{\in}\vee\overline{q})$ to be an $\mathcal{N}$-subalgebra of type $(\in,\in)$.

Theorem 3.14. *Let (X, f) be an $\mathcal{N}$-structure of type $(\overline{\in},\overline{\in}\vee\overline{q})$ such that $f(x) \leq -0.5$ for all $x \in X$. Then (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

Proof. Let $x, y \in X$ and $\alpha \in [-1, 0)$ be such that (X, x_{α_1}) and (X, y_{α_2}) are $\mathcal{N}_{\in}$-subsets of (X, f). Then $f(x) \leq \alpha_1$ and $f(y) \leq \alpha_2$. Since $f(x) \leq -0.5$ for all $x \in X$, it follows from (3.2) that

$$f(x * y) = \wedge\{f(x * y), -0.5\} \leq \vee\{f(x), f(y)\} \leq \vee\{\alpha_1, \alpha_2\}$$

so that $\big(X, (x * y)_{\vee\{\alpha_1,\alpha_2\}}\big)$ is an $\mathcal{N}_{\in}$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$. □

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