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A NOTE ON SKEW DERIVATIONS IN PRIME RINGS

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Abstract. Let m, n, r be nonzero fixed positive integers, R a 2-torsion free prime ring, Q its right Martindale quotient ring, and L a non-central Lie ideal of R. Let $D : R \longrightarrow R$ be a skew derivation of R and $E(x) = D(x^{m+n+r}) - D(x^m)x^{n+r} - x^m D(x^n)x^r - x^{m+n} D(x^r)$. We prove that if $E(x) = 0$ for all $x \in L$, then D is a usual derivation of R or R satisfies $s_4(x_1, \ldots, x_4)$, the standard identity of degree 4.

1. Introduction

Throughout, R will represent an associative ring with a center $Z(R)$, Q its right Martindale quotient ring, and C its extended centroid. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$, $x, y \in R$. An additive mapping $D : R \to R$ is called a derivation on R if $D(xy) = D(x)y + xD(y)$ for all pairs $x, y \in R$. Let $a \in R$ be a fixed element. Then a map $D : R \to R$ defined by $D(x) = [a, x] = ax - xa$, $x \in R$, is a derivation on R. Such derivation is usually called an *inner derivation* defined by a.

Let α be an automorphism of a ring R. An additive mapping $D : R \to R$ is called an *α-derivation* (or a *skew derivation*) on R if $D(xy) = D(x)y + \alpha(x)D(y)$ for all pairs $x, y \in R$. In this case α is called an *associated automorphism* of D. Basic examples of α-derivations are usual derivations and the map $\alpha - 1$, where 1 denotes the identity map. Let $b \in Q$ be a fixed element. Then it is easy to see that a map $D : R \to R$ defined by $D(x) = bx - \alpha(x)b$, $x \in R$, is an α-derivation called an *inner α-derivation* (an *inner skew derivation*) defined by b. If a skew derivation D is not inner, then it is *outer*.

An additive mapping $F : R \to R$ is called a *generalized derivation* on R if there exists a derivation D on R such that $F(xy) = F(x)y + xD(y)$ for all pairs $x, y \in R$. Basic examples of generalized derivations are usual derivations on R, left R-module mappings from R into itself, and so called *generalized inner*

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derivations, i.e., maps of the form $x \mapsto ax + xb$, $x \in R$, where $a, b \in Q$ are fixed elements. Note also that generalized derivations and skew derivations are two natural generalizations of usual derivations.

We say that an automorphism $\alpha : R \to R$ is *inner* if there exists an invertible $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. If an automorphism $\alpha \in \mathrm{Aut}(R)$ is not inner, then it is called *outer*.

Recently the following result was proved.

Theorem 1.1 ([6])**.** *Let m and n be two fixed positive integers, R a 2-torsion free prime ring, and L a non-central Lie ideal of R. If*

$$F(x^{m+n+1}) = F(x)x^{m+n} + x^m D(x)x^n$$

is an identity for L, where both F and D are generalized derivations of R, then either $D = 0$ or R satisfies the standard identity $s_4(x_1, \ldots, x_4)$ and D is a usual derivation of R.

Let us point out that in [6, Theorem 1] the authors also considered the form of a generalized derivation F.

This result motivated us to investigate similar identity involving a skew derivation of a prime ring. More precisely, our aim is to prove the following theorem.

Theorem 1.2. *Let m, n, r be nonzero fixed positive integers, R a 2-torsion free prime ring, L a non-central Lie ideal of R, $D : R \longrightarrow R$ a skew derivations of R, and*

$$E(x) = D(x^{m+n+r}) - D(x^m)x^{n+r} - x^m D(x^n)x^r - x^{m+n}D(x^r), \ x \in R.$$

If $E(x) = 0$ for all $x \in L$, then D is a usual derivation of R or R satisfies $s_4(x_1, \ldots, x_4)$, the standard identity of degree 4.

2. Preliminaries

In this section we will write down some known results which we will need in the following.

Let R be a prime ring and I a two-sided ideal of R. Then I, R, and Q satisfy the same generalized polynomial identities with coefficients in Q (see [2]). Furthermore, I, R, and Q satisfy the same generalized polynomial identities with automorphisms (Theorem 1 in [4]). Recall that in case $\mathrm{char}(R) = 0$ an automorphism α of Q is called *Frobenius* if $\alpha(x) = x$ for all $x \in C$. Moreover, in case $\mathrm{char}(R) = p \geq 2$ an automorphism α is *Frobenius* if there exists a fixed integer t such that $\alpha(x) = x^{p^t}$ for all $x \in C$. In [4, Theorem 2] Chuang proved that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and $\alpha \in \mathrm{Aut}(R)$ an automorphism of R which is not Frobenius, then R also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

Now, let R be a domain and $\alpha \in \mathrm{Aut}(R)$ an automorphism of R which is outer. In [8] Kharchenko proved that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

In [5] Chuang and Lee investigated polynomial identities with skew derivations. They proved that if $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Furthermore, they also proved [5, Theorem 1] that in the case $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for R, where R is a prime ring, D an outer skew derivation of R, and α an outer automorphism of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where x_i, y_i, and z_i are distinct indeterminates.

Let us also mention that if R is a prime ring satisfying a non-trivial generalized polynomial identity and α an automorphism of R such that $\alpha(x) = x$ for all $x \in C$, then α is an inner automorphism of R [1, Theorem 4.7.4].

For proving our main theorem we will also need the following lemma.

Lemma 2.1. *Let R be a prime ring of characteristic different from 2, m, n, r positive integers, and $0 \neq b \in R$ such that*

$$[r_1, r_2]^m (b[r_1, r_2]^n + [r_1, r_2]^n b)[r_1, r_2]^r = 0$$

for all $r_1, r_2 \in R$. Then R is commutative.

Proof. Firstly, assume that $b \in Z(R)$. In this case R satisfies the generalized identity $2b[x_1, x_2]^{m+n+r} = 0$. Moreover, since $0 \neq b \in Z(R)$, R satisfies the polynomial identity $[x_1, x_2]^{m+n+r} = 0$. By the result in [7] (for a bounded index on nilpotency), we conclude that R must be commutative.

Now suppose that $b \notin Z(R)$. Then

$$[x_1, x_2]^m (b[x_1, x_2]^n + [x_1, x_2]^n b)[x_1, x_2]^r = 0$$

is a non-trivial generalized polynomial identity for R. By Martindale's theorem [12], R is a primitive ring having a nonzero socle with C as the associated division ring. In light of Jacobson's theorem [9, p. 75], R is isomorphic to a dense ring of linear transformations on some vector space V over C. Let $\dim_C V \geq 3$. Since $b \notin C$, there exists $v \in V$ such that $\{v, bv\}$ are linearly C-independent. Moreover, because of the dimension of V over C, there exists $w \in V$ such that $\{v, bv, w\}$ are linearly C-independent. By the density of R, there exist $r_1, r_2 \in R$ such that

$$r_1 v = 0,\ r_2 v = w,\ r_1 w = v,\ r_1 bv = 0,\ r_2 bv = w.$$

By calculation we obtain the contradiction

$$0 = [r_1, r_2]^m (b[r_1, r_2]^n + [r_1, r_2]^n b)[r_1, r_2]^r v = 2v \neq 0.$$

Hence, we may assume that $\dim_C V \leq 2$. So, either R is commutative, or $R \cong M_2(C)$, i.e., the 2×2 matrix ring over C. We have to prove that if $R \cong M_2(C)$, then a contradiction follows.

Denote by e_{ij} the usual unit matrix with 1 in the (i,j)-entry and zero elsewhere. Let $b = \sum_{1 \leq i,j \leq 2} b_{ij} e_{ij}$, where $b_{ij} \in C$. Recall that in case $[r_1, r_2] \neq 0$ for some $r_1, r_2 \in M_2(C)$, then $[r_1, r_2]^2 \in Z(R)$. More precisely, if $[r_1, r_2]$ is an invertible matrix, we have

$$[r_1, r_2]b + b[r_1, r_2] = 0. \tag{1}$$

Now consider $[r_1, r_2] = e_{11} - e_{22}$ in (1). By calculation we get $b_{11} = b_{22} = 0$. Analogously, for $[r_1, r_2] = e_{12} + e_{21}$ in (1) we obtain $b_{12} + b_{21} = 0$. On the other hand, for $[r_1, r_2] = e_{21} - e_{12}$ in (1) we have $b_{12} - b_{21} = 0$. It follows that $b_{12} = b_{21} = 0$. Thus, $b = 0$, a contradiction. □

We will end this section with one basic remark.

Remark 2.2. Our main assumption in Theorem 1.2 is

$$D(x^{m+n+r}) = D(x^m)x^{n+r} + x^m D(x^n)x^r + x^{m+n} D(x^r) \tag{2}$$

for all $x \in L$. On the other hand, the skew-derivation rule says that

$$D(x^{m+n+r}) = D(x^m)x^{n+r} + \alpha(x^m)D(x^n)x^r + \alpha(x^{m+n})D(x^r) \tag{3}$$

for all $x \in R$. Therefore, by comparing (2) and (3) we get

$$(\alpha(x^m) - x^m)D(x^n)x^r + (\alpha(x^{m+n}) - x^{m+n})D(x^r) = 0 \tag{4}$$

for all $x \in L$.

3. The case of inner skew derivations

In this section we will consider the case when $D : R \to R$ is a nonzero inner skew derivation on a prime ring R induced by the element $b \in Q$ and an automorphism $\alpha \in \mathrm{Aut}(R)$, that is, $D(x) = bx - \alpha(x)b$ for all $x \in R$. In this sense, our aim will be to prove the following proposition.

Proposition 3.1. *Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R, $m, n, r \geq 1$ fixed integers, b a nonzero element of Q, and $\alpha \in \mathrm{Aut}(R)$ an automorphism of R. If*

$$(\alpha(u^m) - u^m)(bu^n - \alpha(u^n)b)u^r + (\alpha(u^{m+n}) - u^{m+n})(bu^r - \alpha(u^r)b) = 0$$

for all $u \in L$, then one of the following holds:

(a) $\alpha = 1$, *the identity map on R;*
(b) $bx - \alpha(x)b = 0$ *for all $x \in R$;*
(c) *R satisfies $s_4(x_1, \ldots, x_4)$.*

We begin with the following lemma.

Lemma 3.2. *Let R be a prime ring of characteristic different from 2, I a two-sided ideal of R, $m, n, r \geq 1$ fixed integers, b a nonzero element of Q, q an invertible element of Q, and $\alpha(x) = qxq^{-1}$ for all $x \in R$. If*

$$(\alpha(u^m) - u^m)(bu^n - \alpha(u^n)b)u^r + (\alpha(u^{m+n}) - u^{m+n})(bu^r - \alpha(u^r)b) = 0$$

for all $u \in [I, I]$, then one of the following holds:

(a) $q \in C$ *and hence* $\alpha = 1$, *the identity map on* R;
(b) $q^{-1}b \in C$ *and hence* $bx - \alpha(x)b = 0$ *for all* $x \in R$;
(c) R *satisfies* $s_4(x_1, \ldots, x_4)$.

Proof. By our assumption, I satisfies

$$(5) \qquad (q[x_1,x_2]^m q^{-1} - [x_1,x_2]^m)(b[x_1,x_2]^n - q[x_1,x_2]^n q^{-1}b)[x_1,x_2]^r$$
$$+(q[x_1,x_2]^{m+n} q^{-1} - [x_1,x_2]^{m+n})(b[x_1,x_2]^r - q[x_1,x_2]^r q^{-1}b) = 0.$$

Since I and Q satisfy the same generalized polynomial identities with automorphisms Q also satisfies (5). Note that if $\{q^{-1}b, 1\}$ are linearly C-dependent, then $q^{-1}b \in C$ and we are done. Hence, consider the case when $\{q^{-1}b, 1\}$ are linearly C-independent. Then (5) is a non-trivial generalized polynomial identity for Q. By Martindale's theorem [12], Q is a primitive ring having a nonzero socle with C as the associated division ring. In a light of Jacobson's theorem [9, p. 75], Q is isomorphic to a dense ring of linear transformations on some vector space V over C. Of course, we may assume that $\dim_C V \geq 2$.

First, suppose that the vector space V is finite dimensional over C, i.e., $\dim_C V = k \geq 2$. Then $Q \cong M_k(C)$, the ring of $k \times k$ matrices over C. We will denote by $b = \sum_{1\leq i,j\leq k} b_{ij}e_{ij}$ and by $c = q^{-1}b = \sum_{1\leq i,j\leq k} c_{ij}e_{ij}$ for $b_{ij}, c_{ij} \in C$.

Let $i \neq j$ and choose $[x_1, x_2] = e_{ii} - e_{jj}$ in (5). For all $t \neq i, j$ we have

$$e_{tt}(q(e_{ii}-e_{jj})^m q^{-1} - (e_{ii}-e_{jj})^m)(b(e_{ii}-e_{jj})^n - q(e_{ii}-e_{jj})^n q^{-1}b)(e_{ii}-e_{jj})^r e_{tt}$$
$$+e_{tt}(q(e_{ii}-e_{jj})^{m+n} q^{-1} - (e_{ii}-e_{jj})^{m+n})(b(e_{ii}-e_{jj})^r - q(e_{ii}-e_{jj})^r q^{-1}b)e_{tt} = 0.$$

Then

$$(6) \qquad q_{ti}c_{it} + \gamma q_{tj}c_{jt} = 0$$

for all $i \neq j$, $t \neq i, j$, and $\gamma = (-1)^{m+n+r}$. Recall that for any $\varphi \in \mathrm{Aut}(Q)$

$$(\varphi(q)[x_1,x_2]^m\varphi(q)^{-1} - [x_1,x_2]^m)(\varphi(b)[x_1,x_2]^n - \varphi(q)[x_1,x_2]^n\varphi(q)^{-1}\varphi(b))[x_1,x_2]^r$$
$$+(\varphi(q)[x_1,x_2]^{m+n}\varphi(q)^{-1} - [x_1,x_2]^{m+n})(\varphi(b)[x_1,x_2]^r - \varphi(q)[x_1,x_2]^r\varphi(q)^{-1}\varphi(b)) = 0$$

is also an identity for Q. Therefore, the matrices $\varphi(q)$ and $\varphi(c)$ must satisfy the condition (6). In order to finish our proof we will use this argument a number of times.

In particular, let

$$\varphi_0(x) = (1 + e_{ti})x(1 - e_{ti}) = x + e_{ti}x - xe_{ti} - e_{ti}xe_{ti},$$
$$\varphi_1(x) = (1 - e_{ti})x(1 + e_{ti}) = x - e_{ti}x + xe_{ti} - e_{ti}xe_{ti},$$

and apply (6) to $\varphi_0(q)$ and $\varphi_0(c)$. Then we have

$$(q_{ii} - q_{tt} - q_{it})c_{it} + \gamma q_{ij}c_{jt} = 0 \tag{7}$$

for all $i \neq j$ and $t \neq i, j$. Analogously, applying (6) to $\varphi_1(q)$ and $\varphi_1(c)$ we obtain

$$(-q_{ii} + q_{tt} - q_{it})c_{it} - \gamma q_{ij}c_{jt} = 0 \tag{8}$$

for all $i \neq j$ and $t \neq i, j$. Hence, by (7) and (8), and since $\mathrm{char}(R) \neq 2$, we have

$$q_{it}c_{it} = 0 \tag{9}$$

for all $i \neq t$. In the next step we will show that either q is a diagonal matrix or c is a diagonal matrix. So, suppose that q is not diagonal. Then there exist integers $i \neq t$ such that $q_{it} \neq 0$. By (9) it follows that $c_{it} = 0$.

Now, let $j \neq i, t$ and

$$\chi_0(x) = (1 + e_{ij})x(1 - e_{ij}) = x + e_{ij}x - xe_{ij} - e_{ij}xe_{ij},$$

$$\chi_1(x) = (1 - e_{ij})x(1 + e_{ij}) = x - e_{ij}x + xe_{ij} - e_{ij}xe_{ij}.$$

Denote $\chi_0(q) = \sum \chi(q)'_{hl}e_{hl}$, $\chi_1(q) = \sum \chi(q)''_{hl}e_{hl}$, $\chi_0(c) = \sum \chi(c)'_{hl}e_{hl}$, and $\chi_1(c) = \sum \chi(c)''_{hl}e_{hl}$. Here, $\chi(q)'_{hl}, \chi(q)''_{hl}, \chi(c)'_{hl}, \chi(c)''_{hl} \in C$. If both $\chi(q)'_{it} = 0$ and $\chi(q)''_{it} = 0$, then $q_{it} + q_{jt} = 0 = q_{it} - q_{jt}$, which implies $q_{it} = 0$, a contradiction. Thus, at least one of $\chi(q)'_{it}$ and $\chi(q)''_{it}$ is not zero. By applying (9), we have that either $\chi(c)'_{it} = 0$ or $\chi(c)''_{it} = 0$. So, $0 = c_{it} \pm c_{jt} = c_{jt}$ and hence,

$$q_{it} \neq 0 \quad \Longrightarrow \quad c_{rt} = 0 \tag{10}$$

for all $r \neq t$.

Consider $m \neq i, t$ and

$$\mu_0(x) = (1 + e_{tm})x(1 - e_{tm}) = x + e_{tm}x - xe_{tm} - e_{tm}xe_{tm},$$

$$\mu_1(x) = (1 - e_{tm})x(1 + e_{tm}) = x - e_{tm}x + xe_{tm} - e_{tm}xe_{tm}.$$

Denote $\mu_0(q) = \sum \mu(q)'_{hl}e_{hl}$, $\mu_1(q) = \sum \mu(q)''_{hl}e_{hl}$, $\mu_0(c) = \sum \mu(c)'_{hl}e_{hl}$, and $\mu_1(c) = \sum \mu(c)''_{hl}e_{hl}$ with $\mu(q)'_{hl}, \mu(q)''_{hl}, \mu(c)'_{hl}, \mu(c)''_{hl} \in C$. Then we can observe the following by (10).

- If $0 = \mu(q)'_{im} = q_{im} - q_{it}$, then $q_{im} \neq 0$ and, by (10), $c_{rm} = 0$ for all $r \neq m$.
- If $0 = \mu(q)''_{im} = q_{im} + q_{it}$, then $q_{im} \neq 0$ and, by (10), $c_{rm} = 0$ for all $r \neq m$.
- If both $\mu(q)'_{im} \neq 0$ and $\mu(q)''_{im} \neq 0$, then by (10), both $\mu(c)'_{rm} = 0$ and $\mu(c)''_{rm} = 0$ for all $r \neq m$. In particular, for $r \neq t$ this means that $0 = c_{rm} - c_{rt}$, and since $c_{rt} = 0$ from (10) we have $c_{rm} = 0$. On the other hand, for $r = t$ we have both $c_{tm} - c_{mm} + c_{tt} - c_{mt} = 0$ and $c_{tm} + c_{mm} - c_{tt} - c_{mt} = 0$. Since $c_{mt} = 0$, by (10) and $\mathrm{char}(R) \neq 2$, it follows that $c_{tm} = 0$.

Therefore, the previous step says that

$$q_{it} \neq 0 \implies c_{rm} = 0 \tag{11}$$

for all $r \neq m$ and $m \neq i$. In other words, if $q_{it} \neq 0$, then the nonzero entries of the matrix c are just on the i-th column and on the main diagonal.

Finally, let $j \neq i, t$ and

$$\eta_0(x) = (1 + e_{ji})x(1 - e_{ji}) = x + e_{ji}x - xe_{ji} - e_{ji}xe_{ji},$$

$$\eta_1(x) = (1 - e_{ji})x(1 + e_{ji}) = x - e_{ji}x + xe_{ji} - e_{ji}xe_{ji}.$$

Denote $\eta_0(q) = \sum \eta(q)'_{hl}e_{hl}$, $\eta_1(q) = \sum \eta(q)''_{hl}e_{hl}$, $\eta_0(c) = \sum \eta(c)'_{hl}e_{hl}$, and $\eta_1(c) = \sum \eta(c)''_{hl}e_{hl}$ with $\eta(q)'_{hl}, \eta(q)''_{hl}, \eta(c)'_{hl}, \eta(c)''_{hl} \in C$. Also we can observe the following by (11).

- If $0 = \eta(q)'_{jt} = q_{jt} + q_{it}$, then $q_{jt} \neq 0$ and, by (11), $c_{ri} = 0$ for all $r \neq i$.
- If $0 = \eta(q)''_{jt} = q_{jt} - q_{it}$, then $q_{jt} \neq 0$ and, by (11), $c_{ri} = 0$ for all $r \neq i$.
- If both $\eta(q)'_{jt} \neq 0$ and $\eta(q)''_{jt} \neq 0$, then by (11) all the entries in the i-th column of $\eta_0(c)$ are zero. The same is true for the i-th column of $\eta_1(c)$. In particular, for $m \neq j$ this means that $0 = \eta(c)'_{mi} = c_{mi} - c_{mj}$, and since $c_{mj} = 0$ from (11) we have $c_{mi} = 0$. On the other hand, for $m = j$ we have both $0 = \eta(c)'_{ji} = c_{ji} + c_{ii} - c_{jj} - c_{ij}$ and $0 = \eta(c)''_{ji} = c_{ji} - c_{ii} + c_{jj} - c_{ij}$. Since $c_{ij} = 0$, by (11) and $\mathrm{char}(R) \neq 2$, it follows that $c_{ji} = 0$.

This yields that if $q_{it} \neq 0$, then the nonzero entries of the matrix c are just on the main diagonal. The previous argument says that either q is a diagonal matrix or c is a diagonal matrix.

In the next step we will prove that either q is a central matrix or c is a central matrix. To do this, we assume first that q is not a diagonal matrix. So, suppose that $q_{ji} \neq 0$ for some $i \neq j$. As above, we introduce some suitable automorphisms of $M_k(C)$. More precisely, let $m \neq i, j$ and

$$\lambda_0(x) = (1 + e_{im})x(1 - e_{im}) = x + e_{im}x - xe_{im} - e_{im}xe_{im},$$

$$\lambda_1(x) = (1 - e_{mj})x(1 + e_{mj}) = x - e_{mj}x + xe_{mj} - e_{mj}xe_{mj}.$$

Denote $\lambda_0(q) = \sum \lambda(q)'_{hl}e_{hl}$, $\lambda_1(q) = \sum \lambda(q)''_{hl}e_{hl}$, $\lambda_0(c) = \sum \lambda(c)'_{hl}e_{hl}$, and $\lambda_1(c) = \sum \lambda(c)''_{hl}e_{hl}$ with $\lambda(q)'_{hl}, \lambda(q)''_{hl}, \lambda(c)'_{hl}, \lambda(c)''_{hl} \in C$. Note that both $\lambda(q)'_{ji} = q_{ji} \neq 0$ and $\lambda(q)_{ji} = q_{ji} \neq 0$. Therefore, both $\lambda_0(c)$ and $\lambda_1(c)$ are diagonal matrices. In particular,

$$0 = \lambda(c)'_{im} = c_{mm} - c_{ii},$$

$$0 = \lambda(c)''_{mj} = c_{jj} - c_{mm},$$

and hence, $c_{ii} = c_{jj} = c_{mm}$ and c is a central matrix in $M_k(C)$.

Thus, we assume that q is a diagonal matrix. Moreover, if there exists an automorphism θ of $M_k(C)$ such that $\theta(q)$ is not diagonal, then, by the previous argument, we can prove that $\theta(c)$ is central as well as c. Therefore, we may assume that $\theta(q)$ is a diagonal matrix for all $\theta \in \mathrm{Aut}(M_k(C))$. In particular, let $l \neq t$ and

$$\theta(x) = (1 + e_{lt})x(1 - e_{lt}) = x + e_{lt}x - xe_{lt} - e_{lt}xe_{lt}.$$

Denote $\theta(q) = \sum \theta(q)'_{ij} e_{hl}$ with $\theta(q)'_{ij} \in C$. Since $\theta(q)$ is diagonal, we have $\theta(q)'_{lt} = 0$. Hence, $q_{tt} - q_{ll} = 0$. In this case we conclude that q is a central matrix and we are done.

At the end, suppose that $\dim_C V = \infty$ and assume that $q \notin C$ and $c \notin C$. Under this assumption there exist $r_1, r_2 \in Q$ such that $qr_1 \neq r_1 q$ and $cr_2 \neq r_2 c$. By Litoff's Theorem (see, for example, [10, p. 280]) there exist $e^2 = e \in Q$ and a positive integer $k = \dim_C(Ve)$ such that

$$q, c, qr_1, r_1q, cr_2, r_2c, b, r_1, r_2 \in eQe \cong M_k(C).$$

Moreover, eQe satisfies the identity

$$\big((eqe)[x_1,x_2]^m(eq^{-1}e) - [x_1,x_2]^m\big)\big((ebe)[x_1,x_2]^n - (eqe)[x_1,x_2]^n(eq^{-1}be)\big)[x_1,x_2]^r$$
$$+\big((eqe)[x_1,x_2]^{m+n}(eq^{-1}e) - [x_1,x_2]^{m+n}\big)\big((ebe)[x_1,x_2]^r - (eqe)[x_1,x_2]^r(eq^{-1}be)\big) = 0.$$

By the arguments in the previous case we have that either $eqe \in Z(eQe)$ or $ece = eq^{-1}be \in Z(eQe)$. Hence, one of the following holds:

- $qr_1 = eqr_1 = eqer_1 = r_1eqe = r_1qe = r_1q$,
- $cr_2 = ecr_2 = ecer_2 = r_2ece = r_2ce = r_2c$.

In both cases we have a contradiction. The proof of lemma is completed. □

Proof of Proposition 3.1. Set $I = R[L,L]R$. Then $0 \neq [I,R] \subseteq L$. Therefore, by our hypothesis

$$(\alpha(u^m) - u^m)(bu^n - \alpha(u^n)b)u^r + (\alpha(u^{m+n}) - u^{m+n})(bu^r - \alpha(u^r)b) = 0$$

for all $u \in [I,R]$. Since I, R, and Q satisfy the same generalized polynomial identities with automorphisms it follows that Q satisfies

$$(\alpha([x_1,x_2]^m) - [x_1,x_2]^m)(b[x_1,x_2]^n - \alpha([x_1,x_2]^n)b)[x_1,x_2]^r$$
$$+(\alpha([x_1,x_2]^{m+n}) - [x_1,x_2]^{m+n})(b[x_1,x_2]^r - \alpha([x_1,x_2]^r)b) = 0. \tag{12}$$

In the case α is inner, then there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. Hence, by Lemma 3.2 the result follows.

Next, suppose that α is outer. Since $b \neq 0$, by the main theorem in [3], Q satisfies a non-trivial generalized polynomial identity (Q is a GPI-ring). Therefore, by [12, Theorem 3] Q is a primitive ring and it is a dense subring of the ring of linear transformations of a vector space V over a division ring D. Moreover, Q contains nonzero linear transformations of finite rank.

If α is not Frobenius, then by [4, Theorem 2] and (12) we have that Q satisfies

$$([y_1,y_2]^m - [x_1,x_2]^m)(b[x_1,x_2]^n - [y_1,y_2]^n b)[x_1,x_2]^r$$
$$+([y_1,y_2]^{m+n} - [x_1,x_2]^{m+n})(b[x_1,x_2]^r - [y_1,y_2]^r b) = 0 \tag{13}$$

and, in particular, Q satisfies

$$[x_1,x_2]^m(b[x_1,x_2]^n-[x_1,x_2]^n b)[x_1,x_2]^r=0. \tag{14}$$

Thus, Q must be commutative from Lemma 2.1. On the other hand, if Q is a domain, Q satisfies both (13) and (14), and, as above, we conclude that Q is commutative. In the light of previous arguments we assume that α is Frobenius and $\dim_D V\geq 2$. Note that if $\mathrm{char}(R)=0$, we have $\alpha(x)=x$ for all $x\in R$ since α is Frobenious. By [1, Theorem 4.7.4] this implies that α is inner, a contradiction. Thus, we may assume that $\mathrm{char}(R)=p>2$ and $\alpha(\gamma)=\gamma^{p^t}$ for all $\gamma\in C$ and some nonzero fixed integer t. In particular, $\alpha([\gamma x_1,x_2])=\gamma^{p^t}\alpha([x_1,x_2])$. Hence, by replacing $[x_1,x_2]$ with $[\gamma x_1,x_2]$ in (12) we obtain that Q satisfies

$$\begin{aligned}\gamma^{m+n+r}(\gamma^{m(p^t-1)}\alpha([x_1,x_2]^m)-[x_1,x_2]^m)(b[x_1,x_2]^n-\gamma^{n(p^t-1)}\alpha([x_1,x_2]^n)b)[x_1,x_2]^r\\ +\gamma^{m+n+r}(\gamma^{(m+n)(p^t-1)}\alpha([x_1,x_2]^{m+n}))(b[x_1,x_2]^r-\gamma^{r(p^t-1)}\alpha([x_1,x_2]^r)b)\\ -\gamma^{m+n+r}[x_1,x_2]^{m+n}(b[x_1,x_2]^r-\gamma^{r(p^t-1)}\alpha([x_1,x_2]^r)b)=0\end{aligned}$$

for all $0\neq\gamma\in C$. Since $\gamma\neq 0$, Q satisfies

$$\begin{aligned}(\gamma^{m(p^t-1)}\alpha([x_1,x_2]^m)-[x_1,x_2]^m)(b[x_1,x_2]^n-\gamma^{n(p^t-1)}\alpha([x_1,x_2]^n)b)[x_1,x_2]^r\\ +(\gamma^{(m+n)(p^t-1)}\alpha([x_1,x_2]^{m+n}))(b[x_1,x_2]^r-\gamma^{r(p^t-1)}\alpha([x_1,x_2]^r)b)\\ -[x_1,x_2]^{m+n}(b[x_1,x_2]^r-\gamma^{r(p^t-1)}\alpha([x_1,x_2]^r)b)=0.\end{aligned} \tag{15}$$

Since Q is a primitive ring with a nonzero socle, by [9, p. 79] there exists a semi-linear automorphism $T\in\mathrm{End}(V)$ such that $\alpha(x)=TxT^{-1}$ for all $x\in R$. Hence, by (15), Q satisfies

$$\begin{aligned}(\gamma^{m(p^t-1)}T[x_1,x_2]^mT^{-1}-[x_1,x_2]^m)(b[x_1,x_2]^n-\gamma^{n(p^t-1)}T[x_1,x_2]^nT^{-1}b)[x_1,x_2]^r\\ +(\gamma^{(m+n)(p^t-1)}T[x_1,x_2]^{m+n}T^{-1})(b[x_1,x_2]^r-\gamma^{r(p^t-1)}T[x_1,x_2]^rT^{-1}b)\\ -[x_1,x_2]^{m+n}(b[x_1,x_2]^r-\gamma^{r(p^t-1)}T[x_1,x_2]^rT^{-1}b)=0.\end{aligned} \tag{16}$$

Denote the identity (16) by $\Phi(x_1,x_2)$. Assume first that v and $T^{-1}bv$ are D-dependent for all $v\in V$. More precisely, let $T^{-1}bv=\lambda v$ for $\lambda\in D$. In this case

$$\begin{aligned}(bx-TxT^{-1}b)v&=bxv-TxT^{-1}bv=bxv-T(x(\lambda v))=bxv-T(\lambda(xv))\\ &=bxv-T(T^{-1}b)(xv)=bxv-bxv=0\end{aligned}$$

for all $x\in R$. This yields that $(bx-\alpha(x)b)V=\{0\}$ for all $x\in R$. Since V is faithful it follows that $bx-\alpha(x)b=0$ for all $x\in R$ and we are done.

Thus, there exists $v_0\in V$ such that v_0 and $T^{-1}cv_0$ are linearly D-independent. If $\dim_D V\geq 3$, then there exists $w\in V$ such that w, v, and $T^{-1}bv$ are linearly D-independent. We will denote $T^{-1}bv=u$. By the density of Q there exist $r_1,r_2,r_3\in Q$ such that

$$r_1v=v,\ r_2v=v,\ r_1u=0,\ r_2u=w,\ r_1w=u.$$

Thus, by (16) we have the contradiction

$$0 = \Phi(r_1, r_2)v = (-\gamma)^{r(p^t-1)}bv \neq 0.$$

Hence, we may consider the last case that $\dim_D V = 2$. Then Q is a finite-dimensional central simple algebra over C since D is finite-dimensional over C. Moreover, if C is finite, then D is finite. Thus, D is a commutative field and we are done. So, we may assume that C is infinite. We will denote

$$\begin{aligned}
\Phi_0 &= -[x_1, x_2]^m b[x_1, x_2]^{n+r} - [x_1, x_2]^{m+n} b[x_1, x_2]^r,\\
\Phi_1 &= \alpha([x_1, x_2]^m) b[x_1, x_2]^{n+r},\\
\Phi_2 &= [x_1, x_2]^m \alpha([x_1, x_2]^n) b[x_1, x_2]^r,\\
\Phi_3 &= [x_1, x_2]^{m+n} \alpha([x_1, x_2]^r) b,\\
\Phi_4 &= -\alpha([x_1, x_2]^{m+n+r}) b,
\end{aligned}$$

and from (15) we have that Q satisfies

$$\Phi_0 + \lambda_1\Phi_1 + \lambda_2\Phi_2 + \lambda_3\Phi_3 + \lambda_4\Phi_4 = 0 \tag{17}$$

for all $0 \neq \gamma \in C$. Here, $\lambda_1 = \gamma^{m(p^t-1)}$, $\lambda_2 = \gamma^{n(p^t-1)}$, $\lambda_3 = \gamma^{r(p^t-1)}$, and $\lambda_4 = \gamma^{(m+n+r)(p^t-1)}$. Replacing γ successively by $1, \gamma^2, \gamma^3, \gamma^4$ the identity (17) gives the homogeneous system of equations:

$$\begin{cases}
\Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 0\\
\Phi_0 + \lambda_1\Phi_1 + \lambda_2\Phi_2 + \lambda_3\Phi_3 + \lambda_4\Phi_4 = 0\\
\Phi_0 + \lambda_1^2\Phi_1 + \lambda_2^2\Phi_2 + \lambda_3^2\Phi_3 + \lambda_4^2\Phi_4 = 0\\
\Phi_0 + \lambda_1^3\Phi_1 + \lambda_2^3\Phi_2 + \lambda_3^3\Phi_3 + \lambda_4^3\Phi_4 = 0\\
\Phi_0 + \lambda_1^4\Phi_1 + \lambda_2^4\Phi_2 + \lambda_3^4\Phi_3 + \lambda_4^4\Phi_4 = 0
\end{cases}$$

Moreover, since C is infinite, there exists infinitely many $\gamma \in C$ such that $\gamma^{h(p^t-1)} \neq 1$ for $h = 1, \ldots, m+n+r$. Hence, the Vandermonde determinant

$$\begin{vmatrix}
1 & 1 & 1 & 1 & 1\\
1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\\
1 & \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2\\
1 & \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3\\
1 & \lambda_1^4 & \lambda_2^4 & \lambda_3^4 & \lambda_4^4
\end{vmatrix} = \pm(1-\lambda_1) \cdot \prod_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j)$$

is not zero. Thus, we can solve the above system of equations and obtain $\Phi = 0$. Hence, Q satisfies

$$[x_1, x_2]^m b[x_1, x_2]^{n+r} + [x_1, x_2]^{m+n} b[x_1, x_2]^r = 0$$

and, by Lemma 2.1, Q is commutative, a contradiction. The proof of proposition is completed. □

4. The proof of Theorem 1.2

We are now ready to prove the main result of this paper. So, let m, n, r be nonzero fixed positive integers, R a 2-torsion free prime ring, L a non-central Lie ideal of R, $D : R \longrightarrow R$ a skew derivations of R, and

$$E(x) = D(x^{m+n+r}) - D(x^m)x^{n+r} - x^m D(x^n)x^r - x^{m+n}D(x^r),\ x \in R.$$

We have to prove that if $E(x) = 0$ for all $x \in L$, then D is a usual derivation of R or R satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree 4.

Let $\alpha \in \mathrm{Aut}(R)$ such that $D(xy) = D(x)y + \alpha(x)D(y)$ for all $x, y \in R$. In the case $\alpha = 1$, the identity map of R, there is nothing to prove. Hence, we may assume that $\alpha \neq 1$.

We will divide the proof into two parts. Firstly, consider the case when D is inner, i.e., there exists $b \in Q$ such that $D(x) = bx - \alpha(x)b$ for all $x \in R$. In the light of Proposition 3.1 we have that either $D = 0$ or R satisfies $s_4(x_1, \dots, x_4)$ and we are done.

Now, assume that D is outer. As above, there exists a suitable two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$. Hence, by (4), I satisfies

$$(\alpha([x_1,x_2]^m) - [x_1,x_2]^m)D([x_1,x_2]^n)[x_1,x_2]^r + (\alpha([x_1,x_2]^{m+n}) - [x_1,x_2]^{m+n})D([x_1,x_2]^r) = 0. \tag{18}$$

Since by [5, Theorem 2] I, R, and Q satisfy the same generalized polynomial identities with a single skew derivation, Q satisfies the identity(18) as well. Note that for $1 < n \in \mathbb{N}$

$$D(x^n) = \sum_{i=0}^{n-1} \alpha(x^i)D(x)x^{n-i-1},\ x \in R,$$

and

$$D([x_1,x_2]) = D(x_1)x_2 + \alpha(x_1)D(x_2) - D(x_2)x_1 - \alpha(x_2)D(x_1),\ x_1, x_2 \in R.$$

By (18) we obtain

$$\alpha([x_1,x_2]^m)\Big(\sum_{i=0}^{n-1}\alpha([x_1,x_2]^i)(D(x_1)x_2 + \alpha(x_1)D(x_2) - D(x_2)x_1 - \alpha(x_2)D(x_1))[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^r$$
$$-[x_1,x_2]^m\Big(\sum_{i=0}^{n-1}\alpha([x_1,x_2]^i)(D(x_1)x_2 + \alpha(x_1)D(x_2) - D(x_2)x_1 - \alpha(x_2)D(x_1))[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^r$$
$$+\alpha([x_1,x_2]^{m+n})\Big(\sum_{j=0}^{r-1}\alpha([x_1,x_2]^j)(D(x_1)x_2 + \alpha(x_1)D(x_2) - D(x_2)x_1 - \alpha(x_2)D(x_1))[x_1,x_2]^{r-j-1}\Big)$$
$$-[x_1,x_2]^{m+n}\Big(\sum_{j=0}^{r-1}\alpha([x_1,x_2]^j)(D(x_1)x_2 + \alpha(x_1)D(x_2) - D(x_2)x_1 - \alpha(x_2)D(x_1))[x_1,x_2]^{r-j-1}\Big) = 0$$

for all $x_1, x_2 \in Q$. Since D is outer and by [5], Q satisfies

(19)

$$\alpha([x_1,x_2]^m)\Big(\sum_{i=0}^{n-1}\alpha([x_1,x_2]^i)(y_1x_2 + \alpha(x_1)y_2 - y_2x_1 - \alpha(x_2)y_1)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^r$$

$$-[x_1,x_2]^m\Big(\sum_{i=0}^{n-1}\alpha([x_1,x_2]^i)(y_1x_2+\alpha(x_1)y_2-y_2x_1-\alpha(x_2)y_1)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^r$$
$$+\alpha([x_1,x_2]^{m+n})\Big(\sum_{j=0}^{r-1}\alpha([x_1,x_2]^j)(y_1x_2+\alpha(x_1)y_2-y_2x_1-\alpha(x_2)y_1)[x_1,x_2]^{r-j-1}\Big)$$
$$-[x_1,x_2]^{m+n}\Big(\sum_{j=0}^{r-1}\alpha([x_1,x_2]^j)(y_1x_2+\alpha(x_1)y_2-y_2x_1-\alpha(x_2)y_1)[x_1,x_2]^{r-j-1}\Big)=0.$$

Moreover, if α is outer, by [5] and identity (19), Q satisfies

(20)
$$[z_1,z_2]^m\Big(\sum_{i=0}^{n-1}[z_1,z_2]^i(y_1x_2+z_1y_2-y_2x_1-z_2y_1)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^r$$
$$-[x_1,x_2]^m\Big(\sum_{i=0}^{n-1}[z_1,z_2]^i(y_1x_2+z_1y_2-y_2x_1-z_2y_1)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^r$$
$$+[z_1,z_2]^{m+n}\Big(\sum_{j=0}^{r-1}[z_1,z_2]^j(y_1x_2+z_1y_2-y_2x_1-z_2y_1)[x_1,x_2]^{r-j-1}\Big)$$
$$-[x_1,x_2]^{m+n}\Big(\sum_{j=0}^{r-1}[z_1,z_2]^j(y_1x_2+z_1y_2-y_2x_1-z_2y_1)[x_1,x_2]^{r-j-1}\Big)=0.$$

In particular, if we write $z_1=z_2=0$, $y_1=x_1$, and $y_2=x_2$ in (20), we get that Q satisfies
$$-2[x_1,x_2]^{m+n+r}=0.$$
In other words, Q is commutative (see [7] for a fixed bounded index of nilpotency) and we are done.

At the end we have to consider the case when α is inner. So, there exists an invertible element $q\in Q$ such that $\alpha(x)=qxq^{-1}$ for all $x\in R$. Writing $y_1=0$ and $y_2=qy_3$ in (19) we obtain

(21)
$$\Big(q[x_1,x_2]^mq^{-1}-[x_1,x_2]^m\Big)\Big(\sum_{i=0}^{n-1}(q[x_1,x_2]^iq^{-1})(q[x_1,y_3])[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^r$$
$$\Big(q[x_1,x_2]^{m+n}q^{-1}-[x_1,x_2]^{m+n}\Big)\Big(\sum_{j=0}^{r-1}(q[x_1,x_2]^jq^{-1})(q[x_1,y_3])[x_1,x_2]^{r-j-1}\Big)=0$$

for all $x_1,x_2,y_3\in Q$. Denote the left hand side of the identity (21) by $P(x_1,x_2,y_3)$. Note that $q\notin C$ since $\alpha\neq 1$. Therefore, (21) is a non-trivial generalized polynomial identity for Q. By Martindale's theorem [12], Q is a primitive ring having a nonzero socle with C as the associated division ring. In the light of Jacobson's theorem [9, p. 75] a ring R is isomorphic to a dense ring of linear transformations on some vector space V over C. Of course, we may assume that $\dim_C V\geq 3$.

Since $q \notin C$ there exists $v \in V$ such that v and qv are linearly C-independent. Moreover, since $\dim_C V \geq 3$ we can find $w \in V$ such that $\{v, qv, w\}$ are linearly C-independent.

Assume first that $r \geq 2$. By the density of Q, there exist $r_1, r_2, r_3 \in Q$ such that

$$r_1 v = 0, \quad r_2 v = w,$$

$$r_1 qv = w, \quad r_2 qv = 0, \quad r_3 qv = 0,$$

$$r_1 w = v, \quad r_2 w = 0, \quad r_3 w = -v.$$

Then $[r_1, r_2]v = 0$, $[r_1, r_2]qv = 0$, $[r_1, r_3]qv = v$. This yields that

$$0 = P(r_1, r_2, r_3)qv = qv \neq 0,$$

a contradiction. On the other hand, if $r = 1$, we can write (21) as follows

$$\left(q[x_1, x_2]^m q^{-1} - [x_1, x_2]^m\right)\left(\sum_{i=0}^{n-1}(q[x_1, x_2]^i q^{-1})(q[x_1, y_3])[x_1, x_2]^{n-i-1}\right)[x_1, x_2]$$

$$+\left(q[x_1, x_2]^{m+n} q^{-1} - [x_1, x_2]^{m+n}\right)q[x_1, y_3] = 0.$$

By the density of Q, there exist $r_1, r_2, r_3 \in Q$ such that

$$r_1 v = 0, \quad r_2 v = 0, \quad r_3 v = w,$$

$$r_1 qv = w, \quad r_2 qv = 0, \quad r_3 qv = 0,$$

$$r_1 w = v, \quad r_2 w = -qv.$$

Then $[r_1, r_2]v = 0$, $[r_1, r_2]qv = qv$, $[r_1, r_3]v = v$. This yields

$$0 = P(r_1, r_2, r_3)v = -qv \neq 0,$$

a contradiction. The proof of Theorem 1.2 is completed.

At the end we will give an example which shows that in our main theorem we can not expect the conclusion that R is a commutative ring.

Example 4.1. Let R be a ring of all 2×2 matrices over the field of complex numbers and let $\alpha : R \to R$ be an automorphism of R defined by $\alpha(x) = qxq^{-1}$ for all $x \in R$ and some fixed invertible element $q \in R$. Let $b \in R$ be a fixed nonzero matrix. Consider a skew derivation $D : R \to R$ defined by $D(x) = bx - \alpha(x)b$ for all $x \in R$. Let $L = [R, R]$. Then $u^2 \in Z(R)$ for all $u \in L$. Hence, for $m = r = 2$ we have $\alpha(u^m) - u^m = qu^2q^{-1} - u^2 = 0$ and $bu^r - \alpha(u^r)b = bu^2 - qu^2q^{-1}b = bu^2 - u^2b = 0$. Therefore, the hypothesis of Theorem 1.2 are satisfied. Note also that R satisfies $s_4(x_1, \ldots, x_4)$ but it is not a commutative ring.

References

[1] K. I. Beidar, W. S. Martindale III, and A. V. Mikhalev, *Rings with Generalized Identities*, Pure and Applied Math., Dekker, New York, 1996.

[2] C.-L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (1988), no. 3, 723–728.

[3] ———, *Differential identities with automorphisms and antiautomorphisms I*, J. Algebra **149** (1992), no. 2, 371–404.

[4] ———, *Differential identities with automorphisms and antiautomorphisms II*, J. Algebra **160** (1993), no. 1, 130–171.

[5] C.-L. Chuang and T.-K. Lee, *Identities with a single skew derivation*, J. Algebra **288** (2005), no. 1, 59–77.

[6] B. Dhara, V. De Filippis, and R. K. Sharma, *Generalized derivations and left multipliers on Lie ideals*, Aequationes Math. **81** (2011), 251–261.

[7] O. M. Di Vincenzo, *A note on k-th commutators in an associative ring*, Rend. Circ. Mat. Palermo (2) **47** (1998), no. 1, 106–112.

[8] V. K. Harčenko, *Generalized identities with automorphisms*, Algebra i Logika **14** (1975), no. 2, 215–237.

[9] N. Jacobson, *Structure of Rings*, Amer. Math. Soc., Providence, 1964.

[10] C. Lanski, *Differential identities, Lie ideals, and Posner's theorems*, Pacific J. Math. **134** (1988), no. 2, 275–297.

[11] T.-K. Lee and K.-S. Liu, *Generalized skew derivations with algebraic values of bounded degree*, preprint.

[12] W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.

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