Bull. Korean Math. Soc. **49** (2012), No. 4, pp. 863–874
http://dx.doi.org/10.4134/BKMS.2012.49.4.863

ON LIGHTLIKE HYPERSURFACES OF A GRW SPACE-TIME

Tae Ho Kang

Abstract. We provide a study of lightlike hypersurfaces of a generalized Robertson-Walker (GRW) space-time. In particular, we investigate lightlike hypersurfaces with curvature invariance, parallel second fundamental forms, totally umbilical second fundamental forms, null sectional curvatures and null Ricci curvatures, respectively.

1. Introduction

In general relativity, a space time is a four-dimensional differentiable manifold equipped with a Lorentzian metric. One important cosmological models in general relativity is the family of Robertson-Walker space-times:

$$L_1^4(c,f) := (I \times_f F, \bar{g}),\ \bar{g} = -dt^2 + f^2(t)g_c.$$

Explicitly, $L_1^4(c,f)$ is a warped product with Lorentzian metric $\bar{g}$ of an open interval I and a three-dimensional Riemannian manifold (F, g_c) of constant curvature c with a warping function $f > 0$, which is defined on an open interval I in $\mathbb{R}$.

Recently B. Y. Chen and J. Van der Veken ([4]) studied nondegenerate surfaces (i.e., spatial or Lorentzian) of Robertson-Walker space-times from differential geometric view point. And also B. Y. Chen and S. W. Wei ([5]) provided a general study of submanifolds in the Riemannian warped product $I \times_f F$, $\bar{g} = dt^2 + f^2(t)g_c$, where F is an n-dimensional Riemannian manifold of constant sectional curvature. A generalized Robertson-Walker spacetimes (GRW) is defined as a warped product $L_1^{n+1} = I \times_f F$, where $I \subset \mathbb{R}$ is an interval with the metric $-dt^2$, F is an n-dimensional Riemannian manifold. As far as I know, there are no articles which provide study on degenerate (lightlike or null) surfaces (resp. submanifolds) of Robertson-Walker space-times (resp. GRW space-times). In this article we give a study of degenerate hypersurfaces of a GRW space-time whose fibres are constant curvatures. In particular, we investigate degenerate hypersurfaces with curvature invariance and parallel

Received April 25, 2011; Revised January 11, 2012.
2010 *Mathematics Subject Classification.* 53C25, 53C25.
Key words and phrases. lightlike hypersurface, spacelike slice, generalized Robertson-Walker space-times, curvature invariance, totally umbilical second fundamental form, null sectional curvature, null Ricci curvature.

second fundamental forms (Section 4), totally umbilical second fundamental forms (Section 5), null sectional curvatures and null Ricci curvatures (Section 6), respectively.

2. Basics on GRW space-times

In this section, we review some results of the connection and curvature of a GRW space-time, which follow from general results on warped product ([10]).

Consider a GRW space-time

$$L_1^{n+1}(c,f) = (I \times_f F, \bar{g}),\ \bar{g} = -dt^2 + f^2(t)g_c,$$

where f is a smooth positive function on I, and (F, g_c) is an n-dimensional Riemannian manifold of constant sectional curvature c. The standard choices for F are S^n, E^n and H^n, with curvature 1, 0, -1, respectively.

Let π and σ be the natural projections of $I \times F$ onto I and F, respectively. Let $\mathfrak{L}(I)$ and $\mathfrak{L}(F)$ be the set of horizontal and vertical lifts of vector fields on I and F to $I \times_f F$, respectively. Let $\partial_t \in \mathfrak{L}(I)$ denote the horizontal lift vector field to $I \times_f F$ of the standard vector field $\frac{d}{dt}$ on I.

By a *spacelike slice* of $L_1^{n+1}(c,f) = (I \times_f F, \bar{g})$ we mean a hypersurface of $L_1^{n+1}(c,f)$ given by a fibre $S(t_0) := \pi^{-1}(t_0)$ with metric $f^2(t_0)g_c$.

For each vector X tangent to $L_1^{n+1}(c,f)$, we put

$$X = \phi_X \partial_t + \hat{X}, \tag{2.1}$$

where $\phi_X = -\bar{g}(X, \partial_t)$ and $\hat{X}$ is the vertical component of X.

The following two lemmas are well-known ([10]).

Lemma 2.1. *Let $\bar{\nabla}$ be the Levi Civita connection of $L_1^{n+1}(c,f)$. For vectors fields $X, Y \in \mathfrak{L}(F)$ we have*

(1) $\bar{\nabla}_{\partial_t}\partial_t = 0$,
(2) $\bar{\nabla}_{\partial_t}X = \bar{\nabla}_X \partial_t = (\ln f)'X$,
(3) $\bar{g}(\bar{\nabla}_X Y, \partial_t) = -\bar{g}(X,Y)(\ln f)'$,
(4) $\widehat{\bar{\nabla}_X Y}$ *is the vertical lift of* $\nabla^F_X Y$, *where* ∇^F *is the Levi Civita connection of* F.

Lemma 2.2. *Let $\bar{R}$ be the curvature tensor of $L_1^{n+1}(c,f)$. If $X, Y, Z \in \mathfrak{L}(F)$, then*

(1) $\bar{R}(\partial_t, X)\partial_t = \frac{f''}{f}X$,
(2) $\bar{R}(X, \partial_t)Y = -\bar{g}(X,Y)\frac{f''}{f}\partial_t$,
(3) $\bar{R}(X,Y)\partial_t = 0$,
(4) $\bar{R}(X,Y)Z = \frac{(f')^2+c}{f^2}(\bar{g}(Y,Z)X - \bar{g}(X,Z)Y)$.

We can agglomerate (1) $\sim$ (4) in Lemma 2.2 together into a single form (2.2).

Proposition 2.3. *For any vector fields* X, Y, Z *on* $L_1^{n+1}(c, f)$

$$\bar{R}(X,Y)Z = \alpha\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \beta\{\phi_X\phi_Z Y - \phi_Y\phi_Z X + (\phi_X\bar{g}(Y,Z) - \phi_Y\bar{g}(X,Z))\partial_t\}, \tag{2.2}$$

where $\alpha = \frac{f'^2+c}{f^2}$, $\beta = \frac{ff''-(f'^2+c)}{f^2}$.

Proof. Making use of (2.1) for any vector fields X, Y, Z on $L_1^{n+1}(c, f)$, we have from Lemma 2.2 and the linearity of curvature tensor

$$\bar{R}(X,Y)Z = \frac{(f')^2+c}{f^2}\{\bar{g}(\hat{Y},\hat{Z})\hat{X} - \bar{g}(\hat{X},\hat{Z})\hat{Y}\} + \frac{f''}{f}\{\phi_X\phi_Z\hat{Y} - \phi_Y\phi_Z\hat{X} + (\phi_X\bar{g}(\hat{Y},\hat{Z}) - \phi_Y\bar{g}(\hat{X},\hat{Z}))\partial_t\}.$$

Rewriting this equation by substituting $\hat{X} = X - \phi_X\partial_t$ for any vector field X, we get the desired form (2.2). □

From (2.2) we have:

Corollary 2.4. $L_1^{n+1}(c, f)$ *is of constant curvature if and only if* $\beta = 0$.

Proof. Note that if $\beta = 0$, then α is constant. □

Remark 2.5. The following facts follow from solutions of the differential equation $\beta = 0$.

(i) $L_1^{n+1}(c, f)$ is flat if and only if $f(t) = at + b(c = -a^2)$,
(ii) $L_1^{n+1}(c, f)$ has constant curvature $k^2 > 0$ if and only if $f(t) = ae^{kt} + be^{-kt}$, $c = 4k^2ab$,
(iii) $L_1^{n+1}(c, f)$ has constant curvature $-k^2 < 0$ if and only if $f(t) = a\sin(kt) + b\cos(kt)$, $c = -4k^2(a^2 + b^2)$.

3. Basics on lightlike hypersurfaces

In this section, we review some results from the general theory of lightlike hypersurfaces ([6], [7], [8]).

Let (M, g) be a lightlike hypersurface of an $(n + 1)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ with constant index q $(1 \leq q \leq n)$. Then the so called *radical distribution* $Rad(TM) = TM \cap TM^\perp$ is of rank one, and the induced metric g on M is degenerate and has constant rank $n-1$, where $TM^\perp$ denotes the normal bundle over M. Also, a complementary vector bundle of $Rad(TM)$ in TM is a non-degenerate distribution of rank $n-1$ (called a *screen distribution*) over M, denoted by $S(TM)$. Thus we have the orthogonal direct sum

$$TM = S(TM) \perp Rad(TM). \tag{3.1}$$

Let $tr(TM)$ be a complementary (but not orthogonal) vector bundle (called a *transversal vector bundle*) to TM in $T\bar{M} \mid M$. It is known that for any non-zero section $\xi \in \Gamma(TM^{\perp})$ on a coordinate neighborhood $\mathcal{U} \subset M$ there exists a unique null section N of the transversal vector bundle $tr(TM)$ on $\mathcal{U}$ such that

$$\bar{g}(N, \xi) = 1, \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM) \mid_{\mathcal{U}}). \tag{3.2}$$

Thus we have the decomposition.

$$T\bar{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{3.3}$$

Throughout the paper $\Gamma(\bullet)$ denotes the module of smooth sections of the vector bundle $\bullet$.

Now let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$.

According to the decomposition (3.3) and (3.1), we write the local Gauss and Weingarten formulas for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) = \nabla_X Y + B(X, Y)N, B(X, Y) := \bar{g}(h(X, Y), \xi), \tag{3.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^t_X N = -A_N X + \tau(X)N, \ \tau(X) := \bar{g}(\nabla^t_X N, \xi), \tag{3.5}$$

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) = \nabla^*_X PY + C(X, PY)\xi, \tag{3.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{3.7}$$

where h and h^* are the second fundamental forms of M and $S(TM)$, B and C are the local second fundamental forms on $\Gamma(TM)$ and $\Gamma(S(TM))$, respectively, ∇^* is a metric connection on $\Gamma(S(TM))$, A^*_ξ the local shape operator on $\Gamma(S(TM))$ and τ is a 1-form on TM.

The two local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{3.8}$$

$$C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{3.9}$$

Note that in general, A_N is not symmetric with respect to g, the local second fundamental form B is independent of the choice of screen distribution $S(TM)$ and satisfies

$$B(X, \xi) = 0, \quad \forall X \in \Gamma(TM). \tag{3.10}$$

Furthermore, the induced linear connection ∇ is not a metric connection. Indeed we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \tag{3.11}$$

for any $X, Y \in \Gamma(TM)$, where η is a differential 1-form locally defined on M by

$$\eta(X) = \bar{g}(X, N), \quad \forall X \in \Gamma(TM).$$

Denote by $\bar{R}$ and R the curvature tensor of $\bar{\nabla}$ and ∇, respectively.
Then we have the Gauss-Codazzi equations of the lightlike hypersurface

$$\bar{g}(\bar{R}(X, Y)Z, PW) = g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW) \tag{3.12}$$

$$- B(Y,Z)C(X,PW),$$

$$\bar{g}(\bar{R}(X,Y)Z,\xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y), \tag{3.13}$$

$$\bar{g}(\bar{R}(X,Y)Z,N) = \bar{g}(R(X,Y)Z,N) \tag{3.14}$$

for any $X,Y,Z,W \in \Gamma(TM)$, respectively, where we set

$$(\nabla_X B)(Y,Z) = XB(Y,Z) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z).$$

Also, from the right hand side of (3.14) with (3.6) and (3.7) we get

$$\bar{g}(\bar{R}(X,Y)PZ,N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X), \tag{3.15}$$

$$\bar{g}(\bar{R}(X,Y)\xi,N) = C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\tau(X,Y) \tag{3.16}$$

for any $X,Y,Z \in \Gamma(TM)$, where we set

$$(\nabla_X C)(Y,Z) = XC(Y,PZ) - C(\nabla_X Y,PZ) - C(Y,\nabla^*_X Z). \tag{3.17}$$

On the other hand, using again the formulas (3.4) and (3.5) of Gauss and Weingarten, we obtain

$$\bar{R}(X,Y)N = R^t(X,Y)N - h(X,A_N Y) + h(Y,A_N X) - (\nabla_X A)_N Y + (\nabla_Y A)_N X, \tag{3.18}$$

where

$$R^t(X,Y)N = \nabla^t_X\nabla^t_Y N - \nabla^t_Y\nabla^t_X N - \nabla^t_{[X,Y]}N, \tag{3.19}$$

is the curvature tensor of the transversal vector bundle $tr(TM)$ with respect to the transversal connection ∇^t, and

$$(\nabla_X A)_N Y = \nabla_X(A_N Y) - A_N(\nabla_X Y) - A_{\nabla^t_X N}Y. \tag{3.20}$$

4. Curvature invariance and parallel second fundamental forms

Contrary to the case of nondegenerate hypersurfaces ([5]), in the case of lightlike hypersurfaces we have the following lemma.

Lemma 4.1. *Let M be a lightlike hypersurface of $L^{n+1}_1(c,f)$. Then we have*
(i) *∂_t can not be tangent to M, i.e., $\partial^{tr}_t \neq 0$,*
(ii) *∂_t can not be orthogonal to M,*
(iii) *$\phi_U \neq 0$ for any nonzero null vector U on $L^{n+1}_1(c,f)$,*
where ∂^{tr}_t denotes the transversal projection of ∂_t with respect to the decomposition (3.3).

Proof. (i) Assume that ∂_t is tangent to M. Then by decomposition (3.1), $\partial_t = w + \xi$, where $w \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$. Then we get $-1 = \bar{g}(\partial_t,\partial_t) = \bar{g}(w,w) > 0$, since any screen distribution $S(TM)$ on a lightlike hypersurface of a Lorentzian manifold is Riemannian, i.e., the induced metric on $S(TM)$ is positive definite. This is a contradiction. Hence ∂_t can not be tangent to M.

(ii) If ∂_t is orthogonal to M, then $\bar{g}(\partial_t, \xi) = 0$ for any $\xi \in \Gamma(Rad(TM))$, i.e., $\phi_\xi = 0$. This means that ξ is spacelike, which contradicts.

(iii) By (2.1) $U = \phi_U \partial_t + \hat{U}$. If $\phi_U = 0$, U is spacelike, which leads to a contradiction. □

Let (M, g) be a submanifold (degenerate or nondegenerate) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Let $\mathcal{V}$ be a vector bundle over M. If for any vectors X and Y tangent to M, $\bar{R}(X, Y)\mathcal{V}_p \subset \mathcal{V}_p$ for each $p \in M$, then the vector bundle $\mathcal{V}$ is said to be *curvature invariant.* In particular, in case $\mathcal{V} = TM$, M is said to be *curvature invariant* ([11]).

Proposition 4.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $L_1^{n+1}(c, f)$ and X, Y be any vector fields tangent to M. Then we have*

(i) *M is curvature invariant if and only if $L_1^{n+1}(c, f)$ is of constant curvature,*
(ii) *If $S(TM)$ is curvature invariant, then $L_1^{n+1}(c, f)$ is flat,*
(iii) *If $Rad(TM)$ is curvature invariant, then $L_1^{n+1}(c, f)$ is of constant curvature,*
(iv) *If $tr(TM)$ is curvature invariant and* $\mathrm{rank}(S(TM)) > 1$, *then $L_1^{n+1}(c, f)$ is flat or the screen distribution $S(TM)$ is tangent to spacelike slices.*

Proof. Let $\{\xi, N\}$ be a pair satisfying (3.2).

(i) M is curvature invariant if and only if

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = 0, \quad \forall Z \in \Gamma(TM). \tag{4.1}$$

From which, using (2.1) and (2.2) we obtain

$$\beta\phi_\xi\{\phi_X \bar{g}(Y, Z) - \phi_Y \bar{g}(X, Z)\} = 0.$$

Putting $X = \xi$ gives $\beta\phi_\xi^2 \bar{g}(Y, Z) = 0$. Again, substituting $Y = Z = PY (\neq 0)$ gives $\beta = 0$, since $S(TM)$ is Riemannian and $\phi_\xi \neq 0$ (Lemma 4.1(iii)). The converse is clear.

(ii) $S(TM)$ is curvature invariant if and only if

$$\bar{g}(\bar{R}(X, Y)PZ, \xi) = 0 \text{ and } \bar{g}(\bar{R}(X, Y)PZ, N) = 0. \tag{4.2}$$

The first equation and (2.2) show that

$$0 = \beta\phi_\xi\{\phi_X \bar{g}(Y, PZ) - \phi_Y \bar{g}(X, PZ)\}.$$

Putting $X = \xi, Y = PZ \neq 0$ in this equation yields

$$\beta\phi_\xi^2 \bar{g}(PZ, PZ) = 0,$$

which means that $\beta = 0$. The second equation of $(4, 2)$ with $\beta = 0$ gives

$$0 = \alpha\{\bar{g}(Y, PZ)\bar{g}(X, N) - \bar{g}(X, PZ)\bar{g}(Y, N)\}.$$

From this equation with $X = \xi$ and $Y = PZ$, we get $\alpha = 0$. Therefore $L_1^{n+1}(c, f)$ is flat.

(iii) $Rad(TM)$ is curvature invariant if and only if

$$\bar{g}(\bar{R}(X,Y)\xi, PZ) = 0,$$

which is the first equation of (4.2).

(iv) $tr(TM)$ is curvature invariant if and only if

$$\bar{g}(\bar{R}(X,Y)N, PZ) = 0. \tag{4.3}$$

It is clear from this equation and (2.2) that

$$\begin{aligned} 0 = {} & \alpha\{\bar{g}(Y,N)\bar{g}(X,PZ) - \bar{g}(X,N)\bar{g}(Y,PZ)\} \\ & + \beta\{\phi_X\phi_N\bar{g}(Y,PZ) - \phi_Y\phi_N\bar{g}(X,PZ) - \phi_{PZ}(\phi_X\bar{g}(Y,N) - \phi_Y\bar{g}(X,N))\}, \end{aligned}$$

Putting $X = \xi$ and $Y = PY$ gives

$$-\alpha\bar{g}(PY,PZ) + \beta\{\phi_\xi\phi_N\bar{g}(PY,PZ) + \phi_{PY}\phi_{PZ}\} = 0. \tag{4.4}$$

In (4.4) taking PY and PZ to be orthogonal in $S(TM)$ yields $\beta\phi_{PY}\phi_{PZ} = 0$. Hence $\beta = 0$ or $\phi_{PY}\phi_{PZ} = 0$. In case of $\beta = 0$, it is clear from (4.4) that $\alpha = 0$, i.e., $L_1^{n+1}(c,f)$ is flat. For the case which $\phi_{PY}\phi_{PZ} = 0$ for any orthogonal pair $\{PY, PZ\}$ in $S(TM)$, we show that $\phi_{PY} = 0$ for any $PY \in \Gamma(S(TM))$. Fix a point $p \in M$ and suppose that $\phi_{PY} \neq 0$, $PY \in S(T_pM)$. Then $\phi_{PZ} = 0$ for any $PZ \in \{PY\}^\perp$ where $\{PY\}^\perp$ denotes the orthogonal complement of the linear span of {PY} in $S(T_pM)$. By continuity we can choose $PY'(\neq 0)$ sufficiently near to PY. Then $\phi_{PW} = 0$ for any $PW \in \{PY'\}^\perp$. Now consider that $S(T_pM)$ is spanned by $\{PY\}^\perp$ and $\{PY'\}^\perp$, so PY is a linear combination of $PZ's$ in $\{PY\}^\perp$ and $PW's$ in $\{PW\}^\perp$. Then $\phi_{PY} = 0$, which leads to a contradiction. Thus $\phi_{PY}\phi_{PZ} = 0$ for any orthogonal pair $\{PY, PZ\}$ in $S(TM)$ implies that $\phi_{PY} = 0$. □

Proposition 4.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $L_1^{n+1}(c,f)$. If the second fundamental form h is parallel, then $L_1^{n+1}(c,f)$ has constant curvature.*

Proof. Assume that the second fundamental form h is parallel, i.e., $(\nabla_X h)(Y,Z) = 0$, $\forall X, Y, Z \in \Gamma(TM)$, which is equivalent to

$$(\nabla_X B)(Y,Z) = -\tau(X)B(Y,Z). \tag{4.5}$$

From which and (3.13) we get $\bar{g}(\bar{R}(X,Y)Z,\xi) = 0$, i.e., M is curvature invariant. Hence from Proposition 4.5(i), $L_1^{n+1}(c,f)$ has constant curvature. □

Proposition 4.4. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $L_1^{n+1}(c,f)$ with non-constant curvature and* rank$(S(TM)) > 1$. *Assume that one of the conditions* (i) $\sim$ (iii) *is satisfied*:

(i) *η is parallel,*

(ii) *The screen second fundamental form h^* is parallel,*

(iii) $(\nabla_X A)_N Y = (\nabla_Y A)_N X$ $\forall X, Y \in \Gamma(TM)$ *and* $\forall N \in tr(TM)$.

Then the screen distribution $S(TM)$ is tangent to spacelike slices.

Proof. (i) Differentiating $\eta(Y)=\bar{g}(Y,N)$ in the direction X and our assumption yield

$$-\bar{g}(Y,A_NX)+\tau(X)\bar{g}(Y,N)=0,\quad \forall X,Y\in\Gamma(TM).$$

Putting $Y=PY$ in this equation yields $C(X,PY)=0$ with the aid of (3.9). It follows from (3.15) that $\bar{g}(\bar{R}(X,Y)PZ,N)=0$, so the transversal bundle $tr(TM)$ is curvature invariant. Thus Proposition 4.2(iv) shows that $S(TM)$ is tangent to spacelike slices.

(ii) Assume that the screen second fundamental form h^* is parallel, i.e., $(\nabla_X h^*)(Y,PZ)=0$, $\forall X,Y,Z\in\Gamma(TM)$, which is equivalent to

$$(\nabla_X C)(Y,PZ)=\tau(X)C(Y,PZ), \tag{4.6}$$

where $(\nabla_X h^*)(Y,PZ)=\nabla^{*t}_X(h^*(Y,PZ))-h^*(\nabla_X Y,PZ)-h^*(Y,\nabla^*_X PZ)$. From this and (3.15) we obtain $\bar{g}(\bar{R}(X,Y)PZ,N)=0$. By the same argument $S(TM)$ is tangent to spacelike slices.

(iii) From our assumption and (3.18) we also have $\bar{g}(\bar{R}(X,Y)PZ,N)=0$. Therefore we complete the proof. □

5. Totally umbilical lightlike hypersurfaces

Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M},\bar{g})$.

If on any coordinate neighborhood $\mathcal{U}$ in M there is a smooth function ρ such that

$$B(X,Y)=\rho g(X,Y),\ \forall X,Y\in\Gamma(TM), \tag{5.1}$$

then M is said to be *totally umbilical.* In case $\rho=0$ on $\mathcal{U}$ we say that M is *totally geodesic*.

A screen distribution $S(TM)$ is called *totally umbilical* in M if there exists a smooth function λ on any coordinate neighborhood $\mathcal{U}$ in M such that

$$C(X,PY)=\lambda g(X,Y),\ \forall X,Y\in\Gamma(TM). \tag{5.2}$$

In case $\lambda=0$ (resp. $\lambda\neq 0$) we say that $S(TM)$ is *totally geodesic* (resp. *proper totally umbilical*) ([6], [7], [8]).

Theorem 5.1. *Let $(M,g,S(TM))$ be a totally umbilical lightlike hypersurface of $L_1^{n+1}(c,f)$. Then ρ satisfies the partial differential equations*

$$\xi(\rho)-\rho^2+\rho\tau(\xi)+\beta\phi_\xi^2=0, \tag{5.3}$$

$$PX(\rho)+\rho\tau(PX)+\beta\phi_\xi\phi_{PX}=0,\ \forall X\in\Gamma(TM). \tag{5.4}$$

In case of (5.4) *we have assumed* $\operatorname{rank}(S(TM))>1$.

Proof. Substituting (5.1) and (2.2) into (3.13) yields

$$\begin{aligned}(5.5)\quad &\beta\phi_\xi\{\phi_Y g(X,Z)-\phi_X g(Y,Z)\}\\ &=\{X(\rho)-\rho^2\eta(X)+\rho\tau(X)\}g(Y,Z)-\{Y(\rho)-\rho^2\eta(Y)+\rho\tau(Y)\}g(X,Z)\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, where we have used (3.10) and (3.11). Putting $X = \xi$ and $Y = Z$ in this equation, we get (5.3).

Next, putting $X = PX, Y = PY$ and $Z = PZ$ in (5.5), and remembering that $S(TM)$ is nondegenerate, we also have

$$\{PX(\rho) + \rho\tau(PX) + \beta\phi_\xi\phi_{PX}\}PY = \{PY(\rho) + \rho\tau(PY) + \beta\phi_\xi\phi_{PY}\}PX.$$

Taking PX and PY to be linearly independent (rank$(S(TM)) > 1$) yields (5.4). □

Theorem 5.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $L_1^{n+1}(c, f)$ such that the screen distribution $S(TM)$ whose rank > 1 is totally umbilical. Then λ satisfies the partial differential equation*

$$PX(\lambda) + \lambda\tau(PX) - \beta\phi_N\phi_{PX} = 0, \ \forall X \in \Gamma(TM). \tag{5.6}$$

Furthermore if $S(TM)$ is tangent to spacelike slices and proper totally umbilical, then M is totally umbilical immersed in $L_1^{n+1}(c, f)$. In this case M is totally geodesic if and only if λ is a solution of the partial differential equation

$$\xi(\lambda) - \lambda\tau(\xi) - \alpha + \beta\phi_N\phi_\xi = 0.$$

Proof. Substituting (5.2) and (2.2) into (3.15) gives

$$\begin{aligned}\{X(\lambda) - \lambda\tau(X) - \alpha\eta(X) + \beta\phi_N\phi_X\}g(PY, PZ) - \lambda\eta(X)B(PY, PZ)\\ = \{Y(\lambda) - \lambda\tau(Y) - \alpha\eta(Y) + \beta\phi_N\phi_Y\}g(PX, PZ) - \lambda\eta(Y)B(PX, PZ)\\ + \beta\{\phi_X\phi_{PZ}\eta(Y) - \phi_Y\phi_{PZ}\eta(X)\}\end{aligned} \tag{5.7}$$

for any $X, Y, Z \in \Gamma(TM)$. Putting $X = PX, Y = PY$ and $Z = PZ$ in (5.7), we get

$$\begin{aligned}\{PX(\lambda) - \lambda\tau(PX) + \beta\phi_N\phi_{PX}\}g(PY, PZ)\\ = \{PY(\lambda) - \lambda\tau(PY) + \beta\phi_N\phi_{PY}\}g(PX, PZ).\end{aligned} \tag{5.8}$$

Then by the same argument as in the proof of the previous theorem, we obtain (5.6).

Next, substituting $\phi_{PZ} = 0$ (since $S(TM)$ is tangent to spacelike slices) and $X = \xi$ into (5.7), we have

$$\{\xi(\lambda) - \lambda\tau(\xi) - \alpha + \beta\phi_N\phi_\xi\}g(PY, PZ) = \lambda B(PY, PZ). \tag{5.9}$$

The rest statement follows from this equation. □

Corollary 5.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $L_1^{n+1}(c, f)$ such that the screen distribution $S(TM)$ is proper totally umbilical. If $S(TM)$ is tangent to spacelike slices, then M is either totally umbilical or totally geodesic.*

Proof. The proof follows from (5.9). □

6. Null sectional curvatures and null Ricci curvatures

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold and $p \in \bar{M}$. Given a nonzero null vector $U \in T_p\bar{M}$ and a null plane H of $T_p\bar{M}$ containing U, the *null sectional curvature* at $p \in \bar{M}$ with respect to U in the plane H is defined by

$$\bar{K}_U(p, H) = \frac{\bar{g}(\bar{R}_p(X, U)U, X)}{\bar{g}(X, X)},$$

where X is any non-null vector in H ([3], [6], [7], [8]). In a similar way we define the null sectional curvature on a lightlike hypersurface (M, g) of $(\bar{M}, \bar{g})$ as follows;

$$K_\xi(p, H) = \frac{g(R_p(X, \xi)\xi, X)}{g(X, X)},$$

where H is a null plane of T_pM containing a nonzero null vector ξ and X is any non-null vector in H.

Clearly the null sectional curvature of a null plane H is independent of the choice of non-null vectors in H, but depends quadratically on the null vectors. For a geometric interpretation of the null sectional curvature see [1].

Proposition 6.1. *The null sectional curvature at $p \in L_1^{n+1}(c, f)$ is given by*

$$\bar{K}_U(p, H) = -\alpha\bar{g}(X, U)^2 + \beta[2\phi_X\phi_U\bar{g}(U, X) - \phi_U^2], \tag{6.1}$$

where H is the null plane spanned by a null vector U and a unit spacelike vector X.

Proof. It follows from (2.1) and (2.2). □

Theorem 6.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $L_1^{n+1}(c, f)$. Then $L_1^{n+1}(c, f)$ is of constant curvature if and only if at a single point $p \in M$, either $K_\xi(p, H) = 0$ or $\bar{K}_\xi(p, H) = 0$ where $H \subset T_p(M)$ is a null plane which is spanned by any $\xi \in Rad(T_pM)$ and any non-null vector $X \in T_p(M)$.*

Proof. Let $\xi \in Rad(T_pM)$ and $X \in T_p(M)$ be a unit spacelike vector. Then we get from (4.1)

$$\bar{K}_\xi(p, H) = -\beta\phi_\xi^2.$$

Combining this with the Gauss equation (3.12) and (3.10) yields

$$\bar{K}_\xi(p, H) = K_\xi(p, H) = -\beta\phi_\xi^2. \tag{6.2}$$

From (6.2) with $\phi_\xi \neq 0$ (Lemma 4.1(iii)) we complete the proof. □

The Ricci tensor on $L_1^{n+1}(c, f)$ is given by

$$\bar{Ric}(X, Y) = \text{trace}\{Z \to \bar{R}(X, Z)Y\} \tag{6.3}$$

for any vector fields X and Y on $L_1^{n+1}(c, f)$. From (3.12) and (3.14) the relation between the Ricci tensor $\bar{Ric}$ of $\bar{M}$ and the induced Ricci tensor Ric on M is given by

(6.4)

$$Ric(X, Y) = \bar{Ric}(X, Y) - B(X, Y)\,TrA_N + g(A_\xi^*Y, A_NX) + \bar{g}(\bar{R}(\xi, Y)X, N),$$

where $\{\xi, N\}$ is a pair satisfying (3.2) and TrA_N denotes the trace of A_N (cf. [2], [9]).

Since the induced connection ∇ on M is not a Levi-Civita connection, Ric is not symmetric. In [9] some geometric objects for the induced Ricci tensor to be symmetric are studied.

Proposition 6.3. *The Ricci tensor $\bar{R}ic$ on $L_1^{n+1}(c, f)$ is given by*

$$\bar{R}ic(X, Y) = -\alpha n \bar{g}(X, Y) + \beta\{(n-1)\phi_X \phi_Y - \bar{g}(X, Y)\} \tag{6.5}$$

for any vector fields X and Y on $L_1^{n+1}(c, f)$.

Proof. It follows from (2.1), (2.2) and (6.3). □

Theorem 6.4. *For any nonzero null direction U on $L_1^{n+1}(c, f)$ $(n > 2)$, $\bar{R}ic(U, U) = 0$ if and only if $L_1^{n+1}(c, f)$ is of constant sectional curvature.*

Proof. It follows from (6.5) that for any nonzero null vector U on $L_1^{n+1}(c, f)$

$$\bar{R}ic(U, U) = (n-1)\beta\phi_U^2,$$

which and $\phi_U \neq 0$ (Lemma 4.1(iii)) complete the proof. □

Theorem 6.5. *Let $(M, g, S(TM))$ be a lightlike hypersurfaces of $L_1^{n+1}(c, f)$ $(n > 2)$. Then $Ric(\xi, \xi) = 0$, $\forall \xi \in \Gamma(Rad(TM))$ if and only if $L_1^{n+1}(c, f)$ is of constant curvature.*

Proof. From (6.4) and (6.5) we get

$$\bar{R}ic(\xi, \xi) = Ric(\xi, \xi) = (n-1)\beta\phi_\xi^2$$

with the aid of (3.8) and (3.10). The proof follows from this equation. □

Remark 6.6. In any two-dimensional Lorentzian manifold Ricci curvature always vanishes in any null direction ([3]).

Acknowledgements. This work was supported by 2010 Research Fund of University of Ulsan.

References

[1] A. L. Albujer and S. Haesen, *A geometric interpretation of the null sectional curvature*, J. Geom. Phys. **60** (2010), no. 3, 471–476.

[2] C. Atindogbe, J.-P. Ezin, and J. Tossa, *Lightlike Einstein hypersurfaces in Lorentzian manifolds with constant curvature*, Kodai Math. J. **29** (2006), no. 1, 58–71.

[3] J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker, INC. New York and Basel, 1981.

[4] B.-Y. Chen and J. Van der Veken, *Spatial and Lorentzian surfaces in Robertson-Walker space-times*, J. Math. Phys. **48** (2007), no. 7, 073509, 12 pp.

[5] B.-Y. Chen and S. W. Wei, *Differential geometry of submanifolds of warped product manifolds $I \times_f S^{m-1}(k)$*, J. Geom. **91** (2009), no. 1-2, 21–42.

[6] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers, Dordrecht, 1996.

[7] K. L. Duggal and D. H. Jin, *Null curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.

[8] K. L. Duggal and B. Sahin, *Differential geometry of lightlike submanifolds*, Birkhäuser Verlag AG, 2010.
[9] T. H. Kang, *On the geometry of lightlike submanifolds*, Kyungpook Math. J. **51** (2011), no. 2, 125–138.
[10] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1982.
[11] K. Yano and M. Kon, *Structures on Manifold*, World Scientific, 1984.

Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail address: thkang@mail.ulsan.ac.kr