

## STATISTICAL $A$ -SUMMABILITY OF DOUBLE SEQUENCES AND A KOROVKIN TYPE APPROXIMATION THEOREM

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ABSTRACT. In this paper, we define the notion of statistical  $A$ -summability for double sequences and find its relation with  $A$ -statistical convergence. We apply our new method of summability to prove a Korovkin-type approximation theorem for a function of two variables. Furthermore, through an example, it is shown that our theorem is stronger than classical and statistical cases.

### 1. Introduction

The statistical version of the classical Korovkin approximation theorem (see [11]) was studied for the first time in 2002 by Gadjiev and Orhan [8]. Later, statistical type Korovkin theorems have been developed by many authors. For instance, see [2], [5], [4], [15]. The main idea of this paper is to define statistical  $A$ -summability of a double sequence and obtain a Korovkin-type approximation theorem for double sequences of positive linear operators defined on the space of all functions which are continuous on a compact subset of the real two-dimensional space.

We now recall some basic definitions and notations used in this paper.

The concept of statistical convergence of a sequence of reals  $x := (x_k)$  was first studied by Fast [7]. The sequence  $x = (x_k)$  is said to converge statistically to the number  $L$  if for each  $\varepsilon > 0$ ,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

where  $|A|$  denotes the cardinality of the set  $A$ . In this case we write  $st\text{-}\lim x = L$ .

Let  $A = (a_{jk})$  be an infinite summability matrix. For a given number sequence  $x = (x_k)$  the  $A$ -transform of  $x$ , denoted by  $Ax := (y_j) := ((Ax)_j)$  is

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given by

$$(Ax)_j = \sum_{k=1}^{\infty} a_{jk}x_k,$$

provided the series on the right converges for each  $j$ . We say that  $A$  is regular if  $\lim_j (Ax)_j = L$  whenever  $\lim_k x_k = L$  [1].

A double sequence  $x = (x_{ij})$  of real numbers,  $i, j \in \mathbb{N}$ , the set of all positive integers, is said to be convergent in the Pringsheim sense or  $P$ -convergent if for each  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $|x_{ij} - L| < \epsilon$  whenever  $i, j \geq N$  and  $L$  is called Pringsheim limit (denoted by  $P\text{-}\lim x = L$ ) (cf. [16]).

A double sequence  $x$  is bounded if there exists a positive number  $M$  such that  $|x_{ij}| < M$  for all  $i$  and  $j$ , i.e., if

$$\|x\| = \sup_{i,j} |x_{ij}| < \infty.$$

Let  $A = (a_{ij}^{mn})$ ,  $m, n, i, j \in \mathbb{N}$ , be a four dimensional matrix and  $x = (x_{ij})$  be a double sequence. Then the double (transformed) sequence,  $Ax := (y_{mn})$ , is denoted by

$$(1.1) \quad y_{mn} := \sum_{i=1, j=1}^{\infty, \infty} a_{ij}^{mn} x_{ij},$$

where it is assumed that the summation exists as a Pringsheim limit (of the partial sums) for each  $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ . Also the sums  $y_{mn}$  are called  $A$ -means of the double sequence  $x$ . We say that a sequence  $x$  is  $A$ -summable to the limit  $\ell$  if the  $A$ -means exist for all  $m, n \in \mathbb{N}$  in the sense of Pringsheim convergence,

$$\lim_{p, q \rightarrow \infty} \sum_{i, j}^{p, q} a_{ij}^{mn} x_{ij} = y_{mn}$$

and

$$\lim_{m, n \rightarrow \infty} y_{mn} = \ell.$$

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known conditions for two dimensional matrix to be regular are known as Silverman-Toeplitz conditions (see, for instance, [10]).

In 1926, Robison [17] presented a four dimensional analogue of the regularity by considering an additional assumption of boundedness: This assumption was made because a double  $P$ -convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton regularity, or briefly,  $RH$ -regularity (see [17], [9]).

Recall that a four dimensional matrix  $A$  is said to be  $RH$ -regular (or bounded-regular) if it maps every bounded  $P$ -convergent sequence into a  $P$ -convergent

sequence with the same  $P$ -limit. The Robison-Hamilton conditions state that a four dimensional matrix  $A = (a_{ij}^{mn})$  is  $RH$ -regular if and only if

$$(RH_1) \quad P\text{-}\lim_{m,n} a_{ij}^{mn} = 0 \text{ for each } (i, j) \in \mathbb{N}^2,$$

$$(RH_2) \quad P\text{-}\lim_{m,n} \sum_{(i,j) \in \mathbb{N}^2} a_{ij}^{mn} = 1,$$

$$(RH_3) \quad P\text{-}\lim_{m,n} \sum_{j \in \mathbb{N}} |a_{ij}^{mn}| = 0 \text{ for each } i \in \mathbb{N},$$

$$(RH_4) \quad P\text{-}\lim_{m,n} \sum_{i \in \mathbb{N}} |a_{ij}^{mn}| = 0 \text{ for each } j \in \mathbb{N},$$

$$(RH_5) \quad \sum_{(i,j) \in \mathbb{N}^2} |a_{ij}^{mn}| \text{ is } P\text{-convergent},$$

$(RH_6)$  there exist finite positive integers  $A$  and  $B$  such that  $\sum_{i,j > B} |a_{ij}^{mn}| < A$  holds for every  $(m, n) \in \mathbb{N}^2$ .

Now, let  $A = (a_{ij}^{mn})$  be a nonnegative  $RH$ -regular summability matrix, and let  $K \subset \mathbb{N}^2$ . Then the  $A$ -density of  $K$  is given by

$$\delta_A^{(2)} \{K\} := P\text{-}\lim_{m,n} \sum_{(i,j) \in K} a_{ij}^{mn}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real double sequence  $x = (x_{ij})$  is said to be  $A$ -statistically convergent to a number  $L$  if for every  $\epsilon > 0$ ,

$$\delta_A^{(2)} \{(i, j) \in \mathbb{N}^2 : |x_{ij} - L| \geq \epsilon\} = 0.$$

In this case, we write  $st_A^{(2)}\text{-}\lim_{m,n} x_{m,n} = L$ . Clearly, a  $P$ -convergent double sequence is  $A$ -statistically convergent to the same value but its converse is not always true. For example, take  $A = C(1, 1)$ , which is the double Cesàro matrix, and define the double sequence  $x = (x_{ij})$  by

$$x_{ij} = \begin{cases} 1 & \text{if } i = j = n^2, n \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

Then this sequence is statistically convergent (that is,  $C(1, 1)$ -statistically convergent) to 0 but not  $P$ -convergent, since  $A$ -density coincides with double natural density and  $C(1, 1)$ -statistical convergence coincides with the notion of statistical convergence for double sequences (see [13], [12]), i.e., the double natural density of  $K$  is given by

$$\delta_{C(1,1)}^{(2)} \{K\} := \delta_2 \{K\} := P\text{-}\lim_{m,n} \frac{1}{mn} |\{(m, n) \in \mathbb{N}^2 : (m, n) \in K\}|,$$

and  $x$  is statistically convergent to  $L$  if for each  $\epsilon > 0$

$$P\text{-}\lim_{m,n} \frac{1}{mn} |\{(i, j) \in \mathbb{N}^2, i \leq m, j \leq n : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case we will write  $st_{C(1,1)}^{(2)}\text{-}\lim x = L$  or briefly  $st_2\text{-}\lim x = L$ .

If  $A = I$ , the four dimensional identity matrix, then  $A$ -statistical convergence coincides with Pringsheim's convergence.

### 2. Statistical A-summability

In this section we define statistical  $A$ -summability of a double sequence for a nonnegative  $RH$ -regular summability matrix and prove that it is stronger than  $A$ -statistical convergence for bounded double sequences. In [6], Edely and Mursaleen have given the notion of statistical  $A$ -summability for single sequences: Let  $A = (a_{jk})$  be a nonnegative regular summability matrix and  $x = (x_k)$  be a sequence. Then  $x$  is said to be statistically  $A$ -summable to  $L$  if for every  $\epsilon > 0$

$$\lim_n \frac{1}{n} |\{j \leq n : |y_j - L| \geq \epsilon\}| = 0,$$

where  $y_j = (Ax)_j$ . Thus  $x$  is statistically  $A$ -summable to  $L$  if and only if  $Ax$  is statistically convergent to  $L$ .

Now we give our definition for a double sequence.

**Definition 2.1.** Let  $A = (a_{ij}^{mn})$  be a nonnegative  $RH$ -regular summability matrix and  $x = (x_{ij})$  be a double sequence. We say that  $x$  is statistically  $A$ -summable to  $L$  if for every  $\epsilon > 0$ ,

$$\delta_2 (\{(m, n) \in \mathbb{N}^2 : |y_{mn} - L| \geq \epsilon\}) = 0,$$

where  $y_{mn}$  is as in (1.1). So, if  $x$  is statistically  $A$ -summable to  $L$ , then for every  $\epsilon > 0$ ,

$$P\text{-}\lim_{m,n} \frac{1}{mn} |\{(i, j), i \leq m, j \leq n : |y_{ij} - L| \geq \epsilon\}| = 0.$$

Thus, the double sequence  $x$  is statistically  $A$ -summable to  $L$  if and only if  $Ax$  is statistically convergent to  $L$ .

Now we prove the following relation between  $A$ -statistical convergence and statistical  $A$ -summability for a double sequence.

**Theorem 2.2.** *If a double sequence is bounded and  $A$ -statistically convergent to  $L$ , then it is  $A$ -summable to  $L$ ; hence it is statistically  $A$ -summable to  $L$  but not conversely.*

*Proof.* Let  $x = (x_{ij})$  be bounded and  $A$ -statistically convergent to  $L$ , and  $K(\epsilon) = \{(i, j), i \leq m, j \leq n : |x_{ij} - L| \geq \epsilon\}$ . Then,

$$\begin{aligned} |y_{mn} - L| &= \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} x_{ij} - L \right| \\ &= \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} (x_{ij} - L) + L \left( \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} - 1 \right) \right| \\ &\leq \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} (x_{ij} - L) \right| + |L| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} - 1 \right| \end{aligned}$$

$$\begin{aligned} &\leq \left| \sum_{(i,j) \in K(\epsilon)} a_{ij}^{mn} (x_{ij} - L) \right| + \left| \sum_{(i,j) \notin K(\epsilon)} a_{ij}^{mn} (x_{ij} - L) \right| \\ &\quad + |L| \left| \sum_{i,j=1,1}^{\infty, \infty} a_{ij}^{mn} - 1 \right| \\ &\leq \sup_{i,j} |x_{ij} - L| \sum_{(i,j) \in K(\epsilon)} a_{ij}^{mn} + \epsilon \sum_{(i,j) \notin K(\epsilon)} a_{ij}^{mn} + |L| \left| \sum_{i,j=1,1}^{\infty, \infty} a_{ij}^{mn} - 1 \right|. \end{aligned}$$

Using the definition of  $A$ -statistical convergence and the conditions of  $RH$ -regularity of  $A$ , we get  $P\text{-}\lim_{m,n} |y_{mn} - L| = 0$  from the arbitrariness of  $\epsilon > 0$ . Hence,  $st_2\text{-}\lim_{m,n} |y_{mn} - L| = 0$ .  $\square$

To show that the converse is not true in general, we give the following examples:

(i) Let  $A = (a_{ij}^{mn})$  be  $C(1, 1)$ , the four dimensional Cesàro matrix, i.e.,

$$a_{ij}^{mn} = \begin{cases} 1/mn, & \text{if } i \leq m \text{ and } j \leq n \\ 0, & \text{otherwise} \end{cases}$$

and let  $x = (x_{ij})$  be defined

$$x_{ij} = (-1)^i \text{ for all } j.$$

Then  $x$  is  $C(1, 1)$ -summable (and hence statistical  $C(1, 1)$ -summable) to zero but not  $C(1, 1)$ -statistically convergent.

(ii) Define  $A = (a_{ij}^{mn})$  by

$$a_{ij}^{mn} = \begin{cases} 1/m^2, & \text{if } m = n, i, j \leq m \text{ and } m \text{ is even square} \\ 1/(m^2 - m), & \text{if } m = n, i \neq j, i, j \leq m \text{ and } m \text{ is odd square} \\ 0, & \text{otherwise} \end{cases}$$

and define the double sequence  $x = (x_{ij})$  by

$$x_{ij} = \begin{cases} 1, & \text{if } i \text{ is odd and for all } j \\ 0, & \text{otherwise.} \end{cases}$$

We can easily verify that  $A$  is  $RH$ -regular, that is, conditions  $(RH_1)$ - $(RH_6)$  hold. Moreover, for the sequence defined above, we have (see [14])

$$\sum_{i,j=1,1}^{\infty, \infty} a_{ij}^{mn} x_{ij} = \begin{cases} 1/2, & \text{if } m \text{ is even square} \\ (m + 1)/2m, & \text{if } m \text{ is odd square} \\ 0, & \text{otherwise.} \end{cases}$$

Then it is clear that  $x$  is not  $A$ -summable and hence is not  $A$ -statistically convergent but  $st_2\text{-}\lim_{m,n} y_{mn} = 0$ , i.e.,  $x$  is statistically  $A$ -summable to zero.

### 3. Korovkin-type approximation theorem

In this section by  $C(K)$  we denote the space of all continuous real valued functions on any compact subset of the real two dimensional space. Then  $C(K)$  is a Banach space with the norm  $\|\cdot\|_{C(K)}$  by

$$\|f\|_{C(K)} := \sup_{(x,y) \in K} |f(x,y)| \quad (f \in C(K)).$$

Let  $T$  be a linear operator from  $C(K)$  into  $C(K)$ . Then as usual, we say that  $T$  is a positive linear operator provided that  $f \geq 0$  implies  $Tf \geq 0$ . Also, we denote the value of  $Tf$  at a point  $(x,y)$  by  $T(f;x,y)$ .

Before proceeding further, we recall the classical and statistical forms of Korovkin-type theorems studied in [19] and [3].

**Theorem 3.1** ([19]). *Let  $\{L_{ij}\}$  be a double sequence of positive linear operators acting from  $C(K)$  into itself. Then for all  $f \in C(K)$ ,*

$$P\text{-}\lim_{m,n} \|L_{ij}(f) - f\|_{C(K)} = 0$$

*if and only if*

$$P\text{-}\lim_{m,n} \|L_{ij}(f_r) - f_r\|_{C(K)} = 0 \quad (r = 0, 1, 2, 3),$$

where  $f_0(x,y) = 1$ ,  $f_1(x,y) = x$ ,  $f_2(x,y) = y$ ,  $f_3(x,y) = x^2 + y^2$ .

**Theorem 3.2** ([3]). *Let  $A = (a_{ij}^{mn})$  be a nonnegative RH-regular summability matrix. Let  $\{L_{ij}\}$  be a double sequence of positive linear operators acting from  $C(K)$  into itself. Then, for all  $f \in C(K)$ ,*

$$st_A^{(2)}\text{-}\lim_{m,n} \|L_{ij}(f) - f\|_{C(K)} = 0$$

*if and only if*

$$st_A^{(2)}\text{-}\lim_{m,n} \|L_{ij}(f_r) - f_r\|_{C(K)} = 0 \quad (r = 0, 1, 2, 3),$$

where  $f_0(x,y) = 1$ ,  $f_1(x,y) = x$ ,  $f_2(x,y) = y$ ,  $f_3(x,y) = x^2 + y^2$ .

By using the concept of statistical  $A$ -summability for single sequences, a Korovkin-type theorem is proved in [2]. Now, we prove the following.

**Theorem 3.3.** *Let  $A = (a_{ij}^{mn})$  be a nonnegative RH-regular summability matrix. Let  $\{L_{ij}\}$  be a double sequence of positive linear operators acting from  $C(K)$  into itself. Then for all  $f \in C(K)$*

$$(3.1) \quad st_2\text{-}\lim_{m,n} \left\| \sum_{i,j=1,1}^{\infty, \infty} a_{ij}^{mn} L_{ij}(f) - f \right\|_{C(K)} = 0$$

if and only if

$$(3.2) \quad st_2\text{-}\lim_{m,n} \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (f_r) - f_r \right\|_{C(K)} = 0 \quad (r = 0, 1, 2, 3),$$

where  $f_0(x, y) = 1$ ,  $f_1(x, y) = x$ ,  $f_2(x, y) = y$ ,  $f_3(x, y) = x^2 + y^2$ .

*Proof.* Condition (3.2) follows immediately from condition (3.2) since each  $f_r \in C(K)$  ( $r = 0, 1, 2, 3$ ). Let us prove the converse. By the continuity of  $f$  on compact set  $K$ , we can write  $|f(x, y)| \leq M$ , where  $M = \|f\|_{C(K)}$ . Also since  $f \in C(K)$ , for every  $\epsilon > 0$ , there is a number  $\delta > 0$  such that  $|f(u, v) - f(x, y)| < \epsilon$  for all  $(u, v) \in K$  satisfying  $|u - x| < \delta$  and  $|v - y| < \delta$ . Hence we get

$$(3.3) \quad |f(u, v) - f(x, y)| < \epsilon + \frac{2M}{\delta^2} \left\{ (u - x)^2 + (v - y)^2 \right\}.$$

Since  $L_{ij}$  is linear and positive, from (3.3) we obtain for any  $m, n \in \mathbb{N}$  that

$$\begin{aligned} & \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (f; x, y) - f(x, y) \right| \\ & \leq \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (|f(u, v) - f(x, y)|; x, y) \\ & \quad + |f(x, y)| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (f_0; x, y) - f_0(x, y) \right| \\ & \leq \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} \left( \epsilon + \frac{2M}{\delta^2} [(u - x)^2 + (v - y)^2]; x, y \right) \\ & \quad + |f(x, y)| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (f_0; x, y) - f_0(x, y) \right| \\ & \leq \epsilon + (\epsilon + M) \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (f_0; x, y) - f_0 \right| \\ & \quad + \frac{2M}{\delta^2} \left\{ \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (f_3; x, y) - f_3(x, y) \right| \right. \\ & \quad \left. + 2|x| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij} (f_1; x, y) - f_1(x, y) \right| \right\} \end{aligned}$$

$$\begin{aligned}
 &+ 2|y| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_2; x, y) - f_2(x, y) \right| \\
 &+ (x^2 + y^2) \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_0; x, y) - f_0(x, y) \right| \\
 &\leq \epsilon + \left( \epsilon + M + \frac{2M}{\delta^2} (C^2 + D^2) \right) \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_0; x, y) - f_0(x, y) \right| \\
 &+ \frac{2M}{\delta^2} \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_3; x, y) - f_3(x, y) \right| \\
 &+ \frac{4MC}{\delta^2} \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_1; x, y) - f_1(x, y) \right| \\
 &+ \frac{4MD}{\delta^2} \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_2; x, y) - f_2(x, y) \right|,
 \end{aligned}$$

where  $C := \max|x|$ ,  $D := \max|y|$ . Taking supremum over  $(x, y) \in K$  we get

$$\left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f) - f \right\| \leq \epsilon + B \sum_{r=0}^3 \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_r; x, y) - f_r(x, y) \right\|,$$

where

$$B := \max \left\{ \epsilon + M + \frac{2M}{\delta^2} (C^2 + D^2), \frac{2M}{\delta^2}, \frac{4MC}{\delta^2}, \frac{4MD}{\delta^2} \right\}.$$

Now for a given  $\sigma > 0$ , choose  $\epsilon > 0$  such that  $\epsilon < \sigma$  and define

$$\begin{aligned}
 E &:= \left\{ (m, n) \in \mathbb{N}^2 : \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f; x, y) - f(x, y) \right\| \geq \sigma \right\}, \\
 E_r &:= \left\{ (m, n) \in \mathbb{N}^2 : \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_r; x, y) - f_r(x, y) \right\| \geq \frac{\sigma - \epsilon}{4B} \right\}, \quad r = 0, 1, 2, 3.
 \end{aligned}$$

Then  $E \subset \bigcup_{r=0}^3 E_r$  and so  $\delta_2(E) \leq \sum_{r=0}^3 \delta_2(E_r)$ . By considering this inequality and using (3.2) we obtain (3.1), which completes the proof.  $\square$

**Example.** Now, we will show that Theorem 3.3 is stronger than its classical and statistical forms. Let  $A$  be  $C(1, 1)$  and define  $x = (x_{ij})$  by  $x_{ij} = (-1)^j$  for all  $i$ . Then this sequence is neither  $P$ -convergent nor  $A$ -statistically convergent but  $st_2\text{-lim } Ax = 0$ .



Now, consider the following Bernstein operators (see [18]) given by

$$B_{ij}(f; x, y) = \sum_{k=0}^i \sum_{l=0}^j f\left(\frac{k}{i}, \frac{l}{j}\right) C(i, k)x^k(1-x)^{i-k} C(j, l)y^l(1-y)^{j-l},$$

where  $(x, y) \in K = [0, 1] \times [0, 1]$ ;  $f \in C(K)$ . By using these operators, define the following positive linear operators on  $C(K)$ :

$$(3.4) \quad L_{ij}(f; x, y) = (1 + x_{ij}) B_{ij}(f; x, y), \quad (x, y) \in K, \quad f \in C(K).$$

Then observe that

$$\begin{aligned} L_{ij}(f_0; x, y) &= (1 + x_{ij}) f_0(x, y), \\ L_{ij}(f_1; x, y) &= (1 + x_{ij}) f_1(x, y), \\ L_{ij}(f_2; x, y) &= (1 + x_{ij}) f_2(x, y), \\ L_{ij}(f_3; x, y) &= (1 + x_{ij}) \left( f_3(x, y) + \frac{x - x^2}{i} + \frac{y - y^2}{j} \right), \end{aligned}$$

where  $f_0(x, y) = 1$ ,  $f_1(x, y) = x$ ,  $f_2(x, y) = y$ ,  $f_3(x, y) = x^2 + y^2$ . Since  $st_2\text{-lim } Ax = 0$ , we obtain

$$\begin{aligned} &st_2\text{-lim}_{m,n} \left\| \sum_{i,j=1,1}^{\infty, \infty} a_{ij}^{mn} L_{ij}(f_r) - f_r \right\|_{C(K)} \\ &= st_2\text{-lim}_{m,n} \frac{1}{mn} \left\| \sum_{i,j=1,1}^{\infty, \infty} L_{ij}(f_r) - f_r \right\|_{C(K)} = 0 \end{aligned}$$

for  $r = 0, 1, 2, 3$ . Hence, by Theorem 3.3 we conclude that

$$st_2\text{-lim}_{m,n} \left\| \sum_{i,j=1,1}^{\infty, \infty} a_{ij}^{mn} L_{ij}(f) - f \right\|_{C(K)} = 0$$

for any  $f \in C(K)$ .

However, since  $P$ -limit and the statistical limit of the double sequence  $(x_{ij})$  is not zero, then, for  $r = 0, 1, 2, 3$ ,  $\|L_{ij}(f_i) - f_i\|_{C(K)}$  is neither  $P$ -convergent nor statistically convergent to zero. So, Theorem 3.1 and Theorem 3.2 do not work for our operators defined by (3.4).

**Concluding remarks.** Theorem 3.3 motivates to deal with the related problems like Korovkin type approximation theorems for functions of two variables. It is shown that our version is more general to deal with the situation when the operators  $L_{ij}$  do not satisfy the conditions of Theorem 3.1 and Theorem 3.2. Note that Theorem 3.3 can also be proved by using different test functions.

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