

ERROR ESTIMATES OF SEMIDISCRETE DISCONTINUOUS GALERKIN APPROXIMATIONS FOR THE VISCOELASTICITY-TYPE EQUATION

MI RAY OHM, HYUN YOUNG LEE, AND JUN YONG SHIN

ABSTRACT. In this paper, we adopt symmetric interior penalty discontinuous Galerkin (SIPG) methods to approximate the solution of nonlinear viscoelasticity-type equations. We construct finite element space which consists of piecewise continuous polynomials. We introduce an appropriate elliptic-type projection and prove its approximation properties. We construct semidiscrete discontinuous Galerkin approximations and prove the optimal convergence in L^2 normed space.

1. Introduction

Let Ω be an open bounded convex domain in \mathbb{R}^d , $d = 2, 3$ with polygonal/ or polyhedral boundary $\partial\Omega$ and let $0 < T < \infty$ be given. In this paper we consider the following nonlinear viscoelasticity-type problems,

$$(1.1) \quad \begin{aligned} u_{tt} - \nabla \cdot \{a(x, u)\nabla u + b(x, u)\nabla u_t\} &= f(x, u) && \text{in } \Omega \times (0, T] \\ (a(x, u)\nabla u + b(x, u)\nabla u_t) \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \times (0, T] \\ u(x, 0) = u_0(x), \quad u_t(x, 0) &= u_1(x) && \text{on } \Omega \end{aligned}$$

where \mathbf{n} denotes the unit outward normal vector to $\partial\Omega$ and $u_0(x)$ and $u_1(x)$ are given functions defined on Ω . The initial data $u_0(x)$, $u_1(x)$, $a(x, u)$, $b(x, u)$ and $f(x, u)$ are assumed to be such that (1.1) has a sufficiently smooth solution enough to guarantee the regularity conditions appearing in convergence results to be presented below. For the details of the physical significance and various properties of existence and uniqueness of the viscoelasticity-type equations, we refer to [7, 8, 10, 14, 16] and references cited there in.

Early Nitsche [11] introduced penalty terms on the boundary of the domain to treat Dirichlet boundary conditions. Douglas and Dupont [5] and Wheeler

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[17] generalized Nitsche's method and introduced discontinuous Galerkin methods using interior penalties for elliptic and parabolic equations. Darlow et al. [4], and Douglas et al. [6] applied these methods to approximate the behavior of the flow in porous media. These methods which are referred to as interior penalty Galerkin methods are not locally mass conservative. On the other hand, Oden, Babuska and Baumann [12] introduced and analyzed a new type of discontinuous Galerkin method for diffusion problem which was shown to be elementwise conservative. For the polynomials of degree at least 3 and for one dimensional problems, a priori error estimates were proved. Rivière and Wheeler [15] introduced a locally conservative discontinuous Galerkin formulation for nonlinear parabolic equations and derived a priori $L^\infty(L^2)$ and $L^2(H^1)$ error estimates. However, they achieved suboptimal convergence in $L^\infty(L^2)$ norm. Ohm, Lee and Shin [13] constructed a discontinuous Galerkin approximation using interior penalty terms for nonlinear parabolic partial differential equations and proved an optimal $L^\infty(L^2)$ error estimate. Compared to the classical Galerkin method, the discontinuous Galerkin method is very well suited for adaptive control of error and can provide high order of accuracy provided that the solution of the model problem is sufficiently smooth.

In [10], Lin and Zhang proved the global L^∞ -convergence of semidiscrete Galerkin approximation of the solutions to the Sobolev and viscoelasticity type equations using an interpolation postprocessing technique. In this paper we adopt a symmetric discontinuous Galerkin method with interior penalties to construct semidiscrete approximate solutions. We apply symmetric interior penalty discontinuous Galerkin methods to approximate solutions of (1.1) and we obtain the hp convergence in $L^\infty(L^2)$ norm. To our knowledge, this paper appears to be the first trial to construct semidiscrete discontinuous Galerkin approximations of viscoelasticity-type equations using symmetric interior penalty method and prove the hp -convergence in $L^\infty(L^2)$ norm. This paper is organized as follows. In Section 2 we introduce several notations and preliminaries. In Section 3 we construct finite element spaces and introduce auxiliary projection \tilde{u} of the solution u of (1.1) onto finite element spaces. We prove the projection \tilde{u} of u converges optimally in h for the L^2 norm. In Section 4, we construct semidiscrete discontinuous Galerkin approximations and obtain the h -optimal convergence and p -suboptimal convergence in $L^\infty(L^2)$ norm.

2. Notations and basis assumptions

Now we make the following assumptions:

Condition (A)

there exist positive constants \underline{k} and \bar{k} such that $\underline{k} \leq a(x, y) \leq \bar{k}$ and $\underline{k} \leq b(x, y) \leq \bar{k}$, $\forall (x, y) \in \Omega \times \mathbb{R}$,

Condition (B)

there exists positive constant \tilde{k} such that $|\frac{\partial a(x, y)}{\partial y}| \leq \tilde{k}$ and $|\frac{\partial b(x, y)}{\partial y}| \leq \tilde{k}$, $\forall (x, y) \in \Omega \times \mathbb{R}$.

We denote the usual inner product in $L^2(\Omega)$ by (\cdot, \cdot) and the norm by $\|\cdot\|$. For an $s \geq 0$, $1 \leq p \leq \infty$ and $E \subset \mathbb{R}^d$ we denote the classical Sobolev spaces by $W^{s,p}(E)$ with norm $\|\cdot\|_{W^{s,p}(E)}$. When $E = \Omega$ we simply write $\|\cdot\|_{W^{s,p}(\Omega)}$ as $\|\cdot\|_{s,p}$ and if $p = 2$ we write $\|\cdot\|_{W^{s,p}}$ as $\|\cdot\|_s$. And also the usual seminorm of a function defined on E is denoted by $|\cdot|_{s,E}$ and we denote simply $|\cdot|_s$ instead of $|\cdot|_{s,\Omega}$ if $E = \Omega$. Since we deal with time dependent problems we need to introduce the norm of a function v mapped from $[0, t]$ to some underlying Banach space X , as follows

$$\|v\|_{L^p([0,t];X)} = \left(\int_0^t \|v(\tau)\|_X^p d\tau \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and

$$\|v\|_{L^\infty([0,t];X)} = \operatorname{ess\,sup}_{\tau \in [0,t]} \|v(\tau)\|_X.$$

We shall abbreviate the notation $L^p([0, T]; X)$ to $L^p(X)$.

Let $\mathcal{E}_h = \{E_1, E_2, \dots, E_{N_h}\}$ be a regular quasi-uniform subdivision of Ω where E_i is a triangle or a quadrilateral if $d = 2$ and E_i is a 3-simplex or 3-rectangle if $d = 3$. We let $h_i = \operatorname{diam}(E_i)$ be the diameter of E_i and we let $h = \max\{h_i \mid 1 \leq i \leq N_h\}$. We assume that \mathcal{E}_h satisfies the following regularity condition: there exists a constant $\alpha > 0$ independent of the subdivision such that each E_i contains a ball of radius αh_i . And also we assume that \mathcal{E}_h satisfies the following quasiuniformity requirement condition: there is a constant $\gamma > 0$ such that $\frac{h}{h_i} \leq \gamma$, $\forall i = 1, 2, \dots, N_h$.

3. Finite element spaces and an auxiliary projection

For an $s \geq 0$ and a given subdivision of Ω , $\mathcal{E}_h = \{E_1, E_2, \dots, E_{N_h}\}$, we define the following space

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) \mid v|_{E_i} \in H^s(E_i), \quad i = 1, 2, \dots, N_h\}.$$

Let the edges of \mathcal{E}_h be denoted by $\{e_1, e_2, \dots, e_{L_h}, e_{L_h+1}, \dots, e_{M_h}\}$ where $e_k \subset \Omega$, $1 \leq k \leq L_h$ and $e_k \subset \partial\Omega$, $L_h + 1 \leq k \leq M_h$. With each edge e_k , $1 \leq k \leq L_h$, we associate a unit outward normal vector \mathbf{n}_k to E_i if $e_k = E_{ij}$ where $E_{ij} = \partial E_i \cap \partial E_j$ and $i < j$. For $L_h + 1 \leq k \leq M_h$, we define $\mathbf{n}_k = \mathbf{n}$ the unit outward normal vector to $\partial\Omega$.

To present the discontinuous Galerkin scheme, we need some functions defined on edges. For $\phi \in H^s(\mathcal{E}_h)$ with $s > \frac{1}{2}$, we define the following average function $\{\phi(x)\}$,

$$\{\phi(x)\} = \frac{1}{2}((\phi|_{E_i})(x) + (\phi|_{E_j})(x)), \quad \forall x \in e_k, \quad 1 \leq k \leq L_h$$

and jump function $[\phi(x)]$

$$[\phi(x)] = (\phi|_{E_i})(x) - (\phi|_{E_j})(x), \quad \forall x \in e_k, \quad 1 \leq k \leq L_h,$$

where $e_k = \partial E_i \cap \partial E_j$, $i < j$. We associate the following discontinuous norms with the space $H^s(\mathcal{E}_h)$, $s \geq 2$

$$\|\phi\|^2 = \sum_{i=1}^{N_h} \|\phi\|_{0,E_i}^2 \text{ and } \|\phi\|_1^2 = \sum_{i=1}^{N_h} (\|\phi\|_{1,E_i}^2 + h_i^2 \|\nabla^2 \phi\|_{0,E_i}^2) + J_\beta^\sigma(\phi, \phi),$$

where

$$J_\beta^\sigma(\phi, \psi) = \sum_{k=1}^{L_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\phi][\psi] ds, \quad \beta \geq \frac{1}{2}$$

is an interior penalty term and each σ_k , $\forall 1 \leq k \leq L_h$ is positive constant. We choose $\{\sigma_k\}_{1 \leq k \leq L_h}$ such that $\underline{\sigma} < \sigma_k < \tilde{\sigma}$, $\forall 1 \leq k \leq L_h$ holds for some positive constants $\underline{\sigma}$ and $\tilde{\sigma}$.

Let r be a positive integer. The finite element space is taken to be

$$D_r(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in P_r(E_j), \quad j = 1, 2, \dots, N_h\},$$

where $P_r(E_j)$ denotes the set of polynomials of total degree $\leq r$ on E_j . Now we state the following trace inequalities proved in [1]. In what follows, we shall denote by C a generic positive constant depending on Ω , the subdivision \mathcal{E}_h of Ω , the sobolev norms of u or the constants \bar{k} , \underline{k} , \tilde{k} but independent of h and r , attaining in general different values in different places.

Lemma 3.1. *For each $E_j \in \mathcal{E}_h$, there exists a positive constant C depending only on α and γ such that the two following trace inequalities hold:*

$$\begin{aligned} \|\phi\|_{L^2(e_j)}^2 &\leq C \left(\frac{1}{h_j} |\phi|_{0,E_j}^2 + h_j |\phi|_{1,E_j}^2 \right), \quad \forall \phi \in H^1(E_j), \\ \left\| \frac{\partial \phi}{\partial \mathbf{n}_j} \right\|_{L^2(e_j)}^2 &\leq C \left(\frac{1}{h_j} |\phi|_{1,E_j}^2 + h_j |\phi|_{2,E_j}^2 \right), \quad \forall \phi \in H^2(E_j), \end{aligned}$$

where e_j is an edge of E_j and \mathbf{n}_j is the unit outward normal vector to E_j .

Now we state the following hp -approximation properties whose proofs can be found in [2, 3, 9].

Lemma 3.2. *Let $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$. Then there exist a positive constant C depending on s , α and γ but independent of ϕ , r and h and a sequence $P_h \phi \in P_r(E_j)$, $r = 1, 2, \dots$, such that for any $0 \leq q \leq s$,*

$$\begin{aligned} \|\phi - P_h \phi\|_{W^{q,p}(E_j)} &\leq C \frac{h_j^{\mu-q}}{r^{s-q}} \|\phi\|_{W^{s,p}(E_j)} \quad s \geq 0, \quad 1 \leq p \leq \infty, \\ \|\phi - P_h \phi\|_{L^2(e_j)} &\leq C \frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}} \|\phi\|_{s,E_j} \quad s > \frac{1}{2}, \\ \|\phi - P_h \phi\|_{H^1(e_j)} &\leq C \frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}} \|\phi\|_{s,E_j} \quad s > \frac{3}{2}, \end{aligned}$$

where $\mu = \min(r + 1, s)$ and e_j is an edge or a face of E_j . Moreover for $e_k = E_{ij}$,

$$\|\nabla(P_h\phi)\|_{L^\infty(e_k)} \leq C\|\nabla\phi\|_{L^\infty(E_i \cup E_j)}.$$

Remark. From Lemma 3.2, we may assume that there exists a constant K^* such that $\|u - P_h u\|_{L^\infty} < K^*$ where we choose K^* sufficiently large so that $C(\|u_0\|_s + \|u\|_{L^\infty(H^s)} + \|u\|_{L^\infty(W^{1,\infty})} + \|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_t\|_{L^2(H^2)}) < K^*$.

Now we introduce the following bilinear mappings $A(\rho; \cdot, \cdot), B(\rho; \cdot, \cdot), A_t(\rho; \cdot, \cdot)$ and $B_t(\rho; \cdot, \cdot)$ defined on $H^1(\mathcal{E}_h) \times H^1(\mathcal{E}_h)$

$$\begin{aligned} A(\rho; \phi, \psi) &= (a(x, \rho)\nabla\phi, \nabla\psi) - \sum_{k=1}^{L_h} \int_{e_k} \{a(x, \rho)\nabla\phi \cdot \mathbf{n}_k\}[\psi] ds \\ &\quad - \sum_{k=1}^{L_h} \int_{e_k} \{a(x, \rho)\nabla\psi \cdot \mathbf{n}_k\}[\phi] ds + J_\beta^\sigma(\phi, \psi), \\ B(\rho; \phi, \psi) &= (b(x, \rho)\nabla\phi, \nabla\psi) - \sum_{k=1}^{L_h} \int_{e_k} \{b(x, \rho)\nabla\phi \cdot \mathbf{n}_k\}[\psi] ds \\ &\quad - \sum_{k=1}^{L_h} \int_{e_k} \{b(x, \rho)\nabla\psi \cdot \mathbf{n}_k\}[\phi] ds + J_\beta^\sigma(\phi, \psi), \\ A_t(\rho; \phi, \psi) &= \left(\left(\frac{\partial}{\partial t} a(x, \rho) \right) \nabla\phi, \nabla\psi \right) - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} a(x, \rho) \right) \nabla\phi \cdot \mathbf{n}_k \right\} [\psi] ds \\ &\quad - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} a(x, \rho) \right) \nabla\psi \cdot \mathbf{n}_k \right\} [\phi] ds, \\ B_t(\rho; \phi, \psi) &= \left(\left(\frac{\partial}{\partial t} b(x, \rho) \right) \nabla\phi, \nabla\psi \right) - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} b(x, \rho) \right) \nabla\phi \cdot \mathbf{n}_k \right\} [\psi] ds \\ &\quad - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} b(x, \rho) \right) \nabla\psi \cdot \mathbf{n}_k \right\} [\phi] ds. \end{aligned}$$

And we define the following weak formulation of the problem (1.1): Find $u \in H^1(\mathcal{E}_h)$ such that

$$(3.1) \quad (u_{tt}, v) + A(u; u, v) + B(u; u_t, v) = (f(x, u), v), \quad \forall v \in H^1(\mathcal{E}_h).$$

For a $\lambda > 0$ we define the following bilinear forms $A_\lambda(\rho; \cdot, \cdot)$ and $B_\lambda(\rho; \cdot, \cdot)$ on $H^1(\mathcal{E}_h) \times H^1(\mathcal{E}_h)$ by

$$\begin{aligned} A_\lambda(\rho; \phi, \psi) &= A(\rho; \phi, \psi) + \lambda(\phi, \psi), \\ B_\lambda(\rho; \phi, \psi) &= B(\rho; \phi, \psi) + \lambda(\phi, \psi). \end{aligned}$$

Then A_λ and B_λ satisfy the following boundedness and coercivity properties. The proof of Lemmas 3.4 and 3.4 with $d = 1, 2$ are given in [13]. For $d \geq 1$, the proofs can be obtained similarly, but for the completeness of description we provide the proof of Lemma 3.3. Lemma 3.4 can be proved similarly so we omit the proof.

Lemma 3.3. *For a $\lambda > 0$ and a $\beta \geq \frac{1}{d-1}$ if the functions $a(x, y)$ and $b(x, y)$ satisfy Condition (A), then there exists a constant $C > 0$ independent of h and r satisfying*

$$\begin{aligned} |A_\lambda(\rho; \phi, \psi)| &\leq C \|\phi\|_1 \|\psi\|_1, \quad \forall \phi, \psi \in H^2(\mathcal{E}_h), \\ |B_\lambda(\rho; \phi, \psi)| &\leq C \|\phi\|_1 \|\psi\|_1, \quad \forall \phi, \psi \in H^2(\mathcal{E}_h). \end{aligned}$$

Proof. Let $\phi, \psi \in H^2(\mathcal{E}_h)$. From the definition of $A_\lambda(\rho, \phi, \psi)$, we have

$$\begin{aligned} |A_\lambda(\rho; \phi, \psi)| &\leq |(a(x, \rho) \nabla \phi, \nabla \psi)| + \left| \sum_{k=1}^{L_h} \int_{e_k} \{a(x, \rho) \nabla \phi \cdot \mathbf{n}_k\} [\psi] ds \right| \\ &\quad + \left| \sum_{k=1}^{L_h} \int_{e_k} \{a(x, \rho) \nabla \psi \cdot \mathbf{n}_k\} [\phi] ds \right| + |J_\beta^\sigma(\phi, \psi)| + \lambda |(\phi, \psi)| \\ &= E_1 + E_2 + E_3 + E_4 + E_5. \end{aligned}$$

By Condition (A), we have $E_1 \leq \bar{k} \|\phi\|_1 \|\psi\|_1$. Applying Lemma 3.1 and the condition $\beta \geq \frac{1}{d-1}$, we estimate E_2 as follows

$$\begin{aligned} E_2 &\leq \bar{k} \sum_{k=1}^{L_h} \int_{e_k} |\{\nabla \phi \cdot \mathbf{n}_k\}| |\psi| ds \\ &\leq \bar{k} \left(\sum_{k=1}^{L_h} \frac{|e_k|^\beta}{\sigma_k} \int_{e_k} \{\nabla \phi \cdot \mathbf{n}_k\}^2 ds \right)^{\frac{1}{2}} \left(\sum_{k=1}^{L_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\psi]^2 ds \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{i=1}^{N_h} \frac{(h^{d-1})^\beta}{\tilde{\sigma}} \left[\frac{1}{h} |\phi|_{1, E_i}^2 + h |\phi|_{2, E_i}^2 \right] \right)^{\frac{1}{2}} \|\psi\|_1 \\ &\leq C \|\phi\|_1 \|\psi\|_1. \end{aligned}$$

Similarly $E_3 \leq C \|\phi\|_1 \|\psi\|_1$ can be obtained. By the definition of $J_\beta^\sigma(\phi, \psi)$, we have

$$\begin{aligned} |J_\beta^\sigma(\phi, \psi)| &= \left| \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\phi][\psi] ds \right| \\ &\leq \left| \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \left(\int_{e_k} [\phi]^2 ds \right)^{\frac{1}{2}} \left(\int_{e_k} [\psi]^2 ds \right)^{\frac{1}{2}} \right| \\ &\leq \left(\sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\phi]^2 ds \right)^{\frac{1}{2}} \left(\sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\psi]^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

$$= \|\phi\|_1 \|\psi\|_1.$$

Trivially $E_5 \leq \lambda \|\phi\|_1 \|\psi\|_1$ holds. Therefore we have $|A_\lambda(\rho; \phi, \psi)| \leq C \|\phi\|_1 \|\psi\|_1$. By the similar method the boundedness result for $B_\lambda(\rho; \phi, \psi)$ can be proved. \square

Lemma 3.4. *For a $\lambda > 0$, a $\beta \geq \frac{1}{d-1}$ and a sufficiently large σ if the functions $a(x, y)$ and $b(x, y)$ satisfy Condition (A), then there exists a constant $\tilde{c} > 0$ independent of h and r satisfying*

$$\begin{aligned} A_\lambda(\rho; \phi, \phi) &\geq \tilde{c} \|\phi\|_1^2, \quad \forall \phi \in D_r(\mathcal{E}_h), \\ B_\lambda(\rho; \phi, \phi) &\geq \tilde{c} \|\phi\|_1^2, \quad \forall \phi \in D_r(\mathcal{E}_h). \end{aligned}$$

The results of Lemma 3.3 and Lemma 3.4 provide the existence and uniqueness of the following auxiliary projection $\tilde{u}(x, t) \in D_r(\mathcal{E}_h)$ of $u(x, t)$: Find $\tilde{u}(t) : [0, T] \rightarrow D_r(\mathcal{E}_h)$ such that

$$(3.2) \quad \begin{cases} A_\lambda(u; u - \tilde{u}, v) + B_\lambda(u; u_t - \tilde{u}_t, v) = 0, & \forall v \in D_r(\mathcal{E}_h), \quad \forall t > 0, \\ \tilde{u}(x, 0) = P_h(u_0(x)), \quad \tilde{u}_t(x, 0) = P_h(u_1(x)) & \forall x \in \Omega, \end{cases}$$

where $P_h(u_0(x))$ and $P_h(u_1(x))$ are the projections of $u_0(x)$ and $u_1(x)$, respectively generated by Lemma 3.2. Now we construct the semidiscrete discontinuous Galerkin approximations as follows: Find $U(\cdot, t) \in D_r(\mathcal{E}_h)$, $\forall t \in (0, T]$ such that

$$(3.3) \quad \begin{cases} (U_{tt}, v) + A(U; U, v) + B(U; U_t, v) = (f(x, U), v), & \forall v \in D_r(\mathcal{E}_h), \\ U(x, 0) = \tilde{u}(x, 0) & \forall x \in \Omega, \\ U_t(x, 0) = \tilde{u}_t(x, 0) & \forall x \in \Omega. \end{cases}$$

To proceed the convergence of the semidiscrete approximation $U(x, t)$ to $u(x, t)$, we let $\eta(x, t) = u(x, t) - \tilde{u}(x, t)$, $\theta(x, t) = P_h u(x, t) - \tilde{u}(x, t)$ and $\xi(x, t) = \tilde{u}(x, t) - U(x, t)$. We let $\tilde{H}(\Omega) = \{\psi \in H^1(\Omega) \mid \nabla \psi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$. By the definition of A_λ , B_λ , A_t and B_t we have the following Lemma 3.5.

Lemma 3.5. *Suppose that $a(x, y)$ and $b(x, y)$ satisfy Conditions (A) and (B), then there exists a constant $C > 0$ such that*

$$\begin{aligned} |A_\lambda(u; \eta, \psi)| &\leq C \|\eta\| \|\psi\|_2, \\ |B_\lambda(u; \eta, \psi)| &\leq C \|\eta\| \|\psi\|_2, \\ |A_t(u; \eta, \psi)| &\leq C \|\eta\| \|\psi\|_2, \\ |B_t(u; \eta, \psi)| &\leq C \|\eta\| \|\psi\|_2, \end{aligned}$$

hold for $\psi \in H^2(\Omega) \cap \tilde{H}(\Omega)$.

Proof. By the definition of A_λ and the continuity of ψ we have

$$A_\lambda(u; \eta, \psi) = (a(x, u) \nabla \eta, \nabla \psi) - \sum_{k=1}^{L_h} \int_{e_k} \{a(x, u) \nabla \eta \cdot \mathbf{n}_k\} [\psi] ds$$

$$\begin{aligned}
 & - \sum_{k=1}^{L_h} \int_{e_k} \{a(x, u) \nabla \psi \cdot \mathbf{n}_k\} [\eta] \, ds + J_\beta^\sigma(\eta, \psi) + \lambda(\eta, \psi) \\
 & = (a(x, u) \nabla \eta, \nabla \psi) - \sum_{k=1}^{L_h} \int_{e_k} \{a(x, u) \nabla \psi \cdot \mathbf{n}_k\} [\eta] \, ds + \lambda(\eta, \psi).
 \end{aligned}$$

On the other hand, by the continuity of $\nabla \psi \cdot \mathbf{n}_k$ across interior edge e_k and $\nabla \psi \cdot \mathbf{n} = 0$ on $\partial\Omega$, the following holds:

$$\begin{aligned}
 (a(x, u)\eta, \Delta\psi) &= \sum_{j=1}^{N_h} \int_{E_j} (a(x, u)\eta) \Delta\psi \, dx \\
 &= \sum_{j=1}^{N_h} \left[\int_{\partial E_j} a(x, u)\eta \nabla \psi \cdot \mathbf{n}_j \, ds - \int_{E_j} \nabla(a(x, u)\eta) \cdot \nabla \psi \, dx \right] \\
 &= -(a(x, u) \nabla \eta, \nabla \psi) - \sum_{j=1}^{N_h} \int_{E_j} \nabla(a(x, u)\eta) \cdot \nabla \psi \, dx \\
 &\quad + \int_{\partial\Omega} a(x, u)\eta \nabla \psi \cdot \mathbf{n} \, ds + \sum_{k=1}^{L_h} \int_{e_k} \{\nabla \psi \cdot \mathbf{n}_k\} [a(x, u)\eta] \, ds \\
 &= -(a(x, u) \nabla \eta, \nabla \psi) - \sum_{j=1}^{N_h} \int_{E_j} \nabla(a(x, u)\eta) \cdot \nabla \psi \, dx \\
 &\quad + \sum_{k=1}^{L_h} \int_{e_k} \{\nabla \psi \cdot \mathbf{n}_k\} [a(x, u)\eta] \, ds.
 \end{aligned}$$

Therefore

$$A_\lambda(u; \eta, \psi) = -(a(x, u)\eta, \Delta\psi) - \sum_{j=1}^{N_h} \int_{E_j} \nabla(a(x, u)\eta) \cdot \nabla \psi \, dx + \lambda(\eta, \psi),$$

from which we get, $|A_\lambda(u; \eta, \psi)| \leq C \|\eta\| \|\psi\|_2$. Similarly we can obtain the results for B_λ , A_t and B_t . □

As shown in [13] to prove the following lemma we need the regularity property of the elliptic operator $L(u)w = \nabla \cdot (a(x, u)\nabla w) + \lambda w$ with u , the solution of the problem (1.1). The regularity result of the preceding elliptic operator can be sufficiently obtained when Ω is a bounded open convex domain. We state the following lemma whose proof can be found in [13].

Lemma 3.6. *Suppose that $F : H^2(\mathcal{E}_h) \rightarrow \mathbb{R}$ is a linear function and that there exists $\phi \in H^2(\mathcal{E}_h)$ satisfying $B_\lambda(u; \phi, v) = F(v)$, $\forall v \in D_r(\mathcal{E}_h)$. If there exist positive constants M_1 and M_2 satisfying*

$$|F(\omega)| \leq M_1 \|\omega\|_1, \quad \omega \in H^2(\mathcal{E}_h),$$

$$|F(\psi)| \leq M_2 \|\psi\|_2, \quad \psi \in H^2(\Omega) \cap \tilde{H}(\Omega),$$

then the following estimation holds

$$\|\phi\| \leq C(\|\phi\|_1 + M_1)h + M_2.$$

Now we obtain the following approximation properties for η whose proofs can be found in [13]. Hereafter we omit the time variable t if there is no possibility of confusion. In the remaining part of this paper we assume that $a(x, y)$ and $b(x, y)$ satisfy Conditions (A) and (B).

Theorem 3.1. *If $u_0 \in H^s$, $u_t \in L^2(H^s)$ and $\beta = \frac{1}{d-1}$, then there exists a constant $C > 0$ independent of h and r satisfying*

$$(i) \quad \|\eta_t(t)\| + h\|\eta_t(t)\|_1 \leq C \frac{h^\mu}{r^{s-2}} (\|u_0\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)}),$$

$$(ii) \quad \|\eta(t)\| + h\|\eta(t)\|_1 \leq C \frac{h^\mu}{r^{s-2}} (\|u_0\|_s + \|u_t\|_{L^2(H^s)}),$$

where $\mu = \min(r+1, s)$, $r \geq \frac{d}{2}$ and $s \geq 1 + \frac{d}{2}$.

Theorem 3.2. *If $u_0 \in H^s$, $u_t \in L^2(H^s)$, $u_{tt}(t) \in H^s$ and $\beta = \frac{1}{d-1}$, then there exists a constant C independent of h and r such that*

$$(i) \quad \|\eta_{tt}(t)\|_1 \leq C \frac{h^{\mu-1}}{r^{s-2}} (\|u_0\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}(t)\|_s),$$

$$(ii) \quad \|\eta_{tt}(t)\| \leq C \frac{h^\mu}{r^{s-2}} (\|u_0\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}(t)\|_s),$$

where $\mu = \min(r+1, s)$, $r \geq \frac{d}{2}$ and $s \geq 1 + \frac{d}{2}$.

Proof. (i) Differentiating both sides of (3.2) with respect to t , we get

$$(3.4) \quad \begin{aligned} & \left(\left(\frac{\partial}{\partial t} a(x, u) \right) \nabla \eta, \nabla v \right) - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} a(x, u) \right) \nabla \eta \cdot \mathbf{n}_k \right\} [v] ds \\ & - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} a(x, u) \right) \nabla v \cdot \mathbf{n}_k \right\} [\eta] ds + A_\lambda(u; \eta_t, v) + \left(\left(\frac{\partial}{\partial t} b(x, u) \right) \nabla \eta_t, \nabla v \right) \\ & - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} b(x, u) \right) \nabla \eta_t \cdot \mathbf{n}_k \right\} [v] ds - \sum_{k=1}^{L_h} \int_{e_k} \left\{ \left(\frac{\partial}{\partial t} b(x, u) \right) \nabla v \cdot \mathbf{n}_k \right\} [\eta_t] ds \\ & + B_\lambda(u; \eta_{tt}, v) = 0. \end{aligned}$$

By adopting the definitions of A_t and B_t to (3.4), we get

$$(3.5) \quad B_\lambda(u; \eta_{tt}, v) = -A_\lambda(u; \eta_t, v) - A_t(u; \eta, v) - B_t(u; \eta_t, v), \quad \forall v \in D_r(\mathcal{E}_h).$$

Then by (3.5) we have

$$\begin{aligned} B_\lambda(u; \theta_{tt}, \theta_{tt}) &= B_\lambda(u; u_{tt} - \tilde{u}_{tt} - u_{tt} + P_h u_{tt}, \theta_{tt}) \\ &= B_\lambda(u; \eta_{tt}, \theta_{tt}) - B_\lambda(u; u_{tt} - P_h u_{tt}, \theta_{tt}) \\ &= -A_\lambda(u; \eta_t, \theta_{tt}) - A_t(u; \eta, \theta_{tt}) - B_t(u; \eta_t, \theta_{tt}) \\ &\quad - B_\lambda(u; u_{tt} - P_h u_{tt}, \theta_{tt}). \end{aligned}$$

By Lemma 3.3 and Lemma 3.4, we obtain

$$\|\theta_{tt}\|_1 \leq C(\|\eta_t\|_1 + \|\eta\|_1 + \|u_{tt} - P_h u_{tt}\|_1)$$

which implies that

$$(3.6) \quad \begin{aligned} \|\eta_{tt}\|_1 &\leq \|u_{tt} - P_h u_{tt}\|_1 + \|P_h u_{tt} - \tilde{u}_{tt}\|_1 \\ &\leq \|u_{tt} - P_h u_{tt}\|_1 + C(\|\eta_t\|_1 + \|\eta\|_1 + \|u_{tt} - P_h u_{tt}\|_1). \end{aligned}$$

From Lemma 3.1 we have the following estimation

$$(3.7) \quad \begin{aligned} &\|u - P_h u\|_1^2 \\ &= \sum_{i=1}^{N_h} \|u - P_h u\|_{1,E_i}^2 + \sum_{i=1}^{N_h} h_i^2 \|\nabla^2(u - P_h u)\|_{0,E_i}^2 + J_\beta^\sigma(u - P_h u, u - P_h u) \\ &\leq C \left(\frac{h^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_s^2 + h^{-(d-1)\beta} \sum_{k=1}^{L_h} \|[u - P_h u]\|_{0,e_k}^2 \right) \\ &\leq C \left[\frac{h^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_s^2 + \frac{h^{-(d-1)\beta-1}}{r^{2(s-1)}} (h^{2\mu} \|u\|_s^2) \right] \\ &\leq C \frac{h^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_s^2. \end{aligned}$$

By applying the inequality above with u_{tt} to (3.6) we obtain the following

$$\begin{aligned} \|\eta_{tt}\|_1^2 &\leq C(\|u_{tt} - P_h u_{tt}\|_1^2 + \|\eta\|_1^2 + \|\eta_t\|_1^2) \\ &\leq C \frac{h^{2(\mu-1)}}{r^{2(s-2)}} (\|u_0\|_s^2 + \|u_t\|_s^2 + \|u_t\|_{L^2(H^s)}^2 + \|u_{tt}\|_s^2), \end{aligned}$$

which implies

$$(3.8) \quad \|\eta_{tt}\|_1 \leq C \frac{h^{\mu-1}}{r^{s-2}} (\|u_0\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_s).$$

(ii) By Lemma 3.3 there exists a constant $C > 0$ satisfying

$$|B_\lambda(u; \eta_{tt}, v)| \leq C \|\eta_{tt}\|_1 \|v\|_1, \quad \forall v \in H^2(\mathcal{E}_h).$$

By applying the definitions of A_t and B_t and Lemma 3.5 we have the following estimation

$$|A_\lambda(u; \eta_t, v)| + |A_t(u; \eta, v)| + |B_t(u; \eta_t, v)| \leq M_2 \|v\|_2 \quad \forall v \in H^2(\Omega) \cap \tilde{H}(\Omega)$$

with $M_2 = C(\|\eta_t\| + \|\eta\|)$. By applying Lemma 3.6 to (3.5) with $M_1 = C\|\eta_{tt}\|_1$, we have

$$\begin{aligned} \|\eta_{tt}\| &\leq C(h\|\eta_{tt}\|_1 + \|\eta_t\| + \|\eta\|) \\ &\leq C \frac{h^\mu}{r^{s-2}} (\|u_0\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_s). \quad \square \end{aligned}$$

4. The convergence of semidiscrete discontinuous Galerkin approximations

Before we prove the convergence of the semidiscrete approximation defined in (3.3), we will show the existence and uniqueness of semidiscrete approximation in the following theorem.

Theorem 4.1. *The following statements hold:*

- (i) *If f is a continuous function, then there exists a semidiscrete discontinuous Galerkin approximation $U(x, t)$ satisfying (3.3). Furthermore $\|U(t)\|$ and $\|U_t(t)\|$ are continuous with respect to t .*
- (ii) *In addition to the hypothesis of (i), if f is globally Lipschitz continuous, i.e., there exists a constant $L > 0$ such that $|f(x, u) - f(x, v)| \leq L|u - v|$, $\forall (x, u), (x, v) \in \Omega \times \mathbb{R}$, then (3.3) has a unique semidiscrete approximation $U(x, t)$.*

Proof. Let $\{\phi_i(x)\}_{i=1}^m$ be a basis of $D_r(\mathcal{E}_h)$ and $U(x, t) = \sum_{i=1}^m \alpha_i(t)\phi_i(x)$. Then (3.3) reduces to a system of nonlinear ordinary differential equations such that

$$(4.1) \quad \left(\sum_{i=1}^m \alpha_i''(t)\phi_i(x), \phi_j(x) \right) + A \left(U; \sum_{i=1}^m \alpha_i(t)\phi_i(x), \phi_j(x) \right) + B \left(U; \sum_{i=1}^m \alpha_i'(t)\phi_i(x), \phi_j(x) \right) = \left(f \left(x, \sum_{i=1}^m \alpha_i(t)\phi_i(x) \right), \phi_j(x) \right), 1 \leq j \leq m.$$

which implies the following initial value problem

$$(4.2) \quad \begin{cases} \mathbf{S} \boldsymbol{\alpha}''(t) = -\mathbf{T}(\boldsymbol{\alpha})\boldsymbol{\alpha}'(t) - \mathbf{W}(\boldsymbol{\alpha})\boldsymbol{\alpha} + \mathbf{F}(\boldsymbol{\alpha}), \\ \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0, \quad \boldsymbol{\alpha}'(0) = \boldsymbol{\alpha}_1, \end{cases}$$

where $\boldsymbol{\alpha} = (\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t))^T$, $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$ are initial value vectors, $\mathbf{S} = (S_{ij})_{1 \leq i, j \leq m}$, $\mathbf{T}(\boldsymbol{\alpha}) = (T_{ij}(\boldsymbol{\alpha}))_{1 \leq i, j \leq m}$ and $\mathbf{W}(\boldsymbol{\alpha}) = (W_{ij}(\boldsymbol{\alpha}))_{1 \leq i, j \leq m}$ are $m \times m$ symmetric matrices and $\mathbf{F}(\boldsymbol{\alpha}) = (F_1(\boldsymbol{\alpha}), F_2(\boldsymbol{\alpha}), \dots, F_m(\boldsymbol{\alpha}))^T$ is a vector. Matrices S , T and W are defined by $S_{ij} = (\phi_i, \phi_j)$, $T_{ij} = B(U; \phi_i, \phi_j)$, $W_{ij} = A(U; \phi_i, \phi_j)$ and $F_j = (f(x, U), \phi_j)$.

Let $\boldsymbol{\beta}(t) = (\alpha_1'(t), \alpha_2'(t), \dots, \alpha_m'(t))^T$ and let $\tilde{\mathbf{S}}$ be the $2m$ by $2m$ block diagonal matrix with \mathbf{I} , \mathbf{S} as block diagonal elements. $P_h(u_0(x)) = U(x, 0)$ and $P_h(u_1(x)) = U_t(x, 0)$ can be represented by

$$U(x, 0) = \sum_{i=1}^m \alpha_i(0)\phi_i(x) \quad \text{and} \quad U_t(x, 0) = \sum_{i=1}^m \beta_i(0)\phi_i(x).$$

Then (4.2) can be reduced to initial boundary value problem associated with a system of 1st order differential equations.

$$\begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{S} \end{pmatrix} \begin{pmatrix} \frac{d\boldsymbol{\alpha}}{dt} \\ \frac{d\boldsymbol{\beta}}{dt} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}(t) \\ -\mathbf{T}(\boldsymbol{\alpha})\boldsymbol{\beta}(t) - \mathbf{W}(\boldsymbol{\alpha})\boldsymbol{\alpha}(t) + \mathbf{F}(\boldsymbol{\alpha}) \end{pmatrix},$$

with initial condition $(\boldsymbol{\alpha}(0), \boldsymbol{\beta}(0))^T = (\alpha_1(0), \dots, \alpha_m(0), \beta_1(0), \dots, \beta_m(0))^T$. For $\mathbf{v} = (v_1, v_2, \dots, v_m)^T \in \mathbb{R}^m$ and $\mathbf{y} = (y_1, y_2, \dots, y_m)^T \in \mathbb{R}^m$ we let

$$v(x) = \sum_{i=1}^m v_i\phi_i(x) \quad \text{and} \quad y(x) = \sum_{i=1}^m y_i\phi_i(x).$$

Then we get

$$(\mathbf{v}^T, \mathbf{y}^T) \tilde{\mathbf{S}} \begin{pmatrix} \mathbf{v} \\ \mathbf{y} \end{pmatrix} = \sum_{i=1}^m v_i^2 + \sum_{i=1}^m y_i S_{ij} y_j = \sum_{i=1}^m v_i^2 + (y, y) \geq 0.$$

Therefore $\tilde{\mathbf{S}}$ is positive definite. By the continuity condition on f , and the theory of ordinary differential equations, $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ exist which completes the proof of the existence of the semidiscrete approximation. This result suffices to show that $\alpha_i(t)$ and $\beta_i(t)$ are continuous with respect to t so that $\|U(t)\|$ and $\|U_t(t)\|$ are also continuous which completes the proof of the statement (i). If the functions a and b are bounded and f is Lipschitz continuous then there exists a unique pair of $\{\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)\}$ by the theory of ordinary differential equation. This completes the proof of (ii). \square

Lemma 4.1. *Suppose that $u_0 \in H^s, u(t) \in H^s$ and $u_t \in L^2(H^s)$ with $s \geq \frac{d}{2} + 1$. For a $\lambda > 0$ and $\beta = 1/(d - 1)$, there exists a constant C satisfying*

$$\|\theta(t)\|_1 \leq C \left(\frac{h^{\mu-1}}{r^{s-2}} \right) (\|u_0\|_s + \|u\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)})$$

and

$$\|\theta_t(t)\|_1 \leq c \left(\frac{h^{\mu-1}}{r^{s-2}} \right) (\|u_0\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)}),$$

where $\mu = \min(r + 1, s)$ and $r \geq d/2$.

Proof. By Lemma 3.4 and Lemma 3.3 we obtain

$$\begin{aligned} \|\theta\|_1^2 &\leq CA_\lambda(u; \theta, \theta) = C(A_\lambda(u; \theta, \theta) - A_\lambda(u; \eta, \theta) - B_\lambda(u; \eta_t, \theta)) \\ &\leq C(\|u - P_h u\|_1 + \|\eta_t\|_1) \|\theta\|_1 \end{aligned}$$

which implies that by Theorem 3.1 and (3.7)

$$\begin{aligned} \|\theta\|_1 &\leq C(\|u - P_h u\|_1 + \|\eta_t\|_1) \\ &\leq C \left(\frac{h^{\mu-1}}{r^{s-2}} \right) (\|u_0\|_s + \|u\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)}). \end{aligned}$$

By (3.2) we get

$$\begin{aligned} B_\lambda(u; \theta_t, \theta_t) &= B_\lambda(u; u_t + P_h u_t - u_t - \tilde{u}_t, \theta_t) \\ &= B_\lambda(u; \eta_t, \theta_t) - B_\lambda(u; u_t - P_h u_t, \theta_t) \\ &= -A_\lambda(u; \eta, \theta_t) - B_\lambda(u; u_t - P_h u_t, \theta_t). \end{aligned}$$

By Lemma 3.4 and Lemma 3.2, we have

$$\|\theta_t\|_1 \leq c(\|\eta\|_1 + \|u_t - P_h u_t\|) \leq C \frac{h^{\mu-1}}{r^{s-2}} (\|u_0\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)}). \quad \square$$

Lemma 4.2. *For a $\lambda > 0$ and $\beta = 1/(d - 1)$ there exists a constant C satisfying if $u_0 \in H^s, u(t) \in H^s, u(t) \in W^{1, \infty}$ and $u_t \in L^2(H^s)$, then*

- (i) $\|\nabla \tilde{u}\|_{L^\infty} \leq C(\|u_0\|_s + \|u\|_s + \|u\|_{1, \infty} + \|u_t\|_{L^2(H^s)}),$
- (ii) $\|\nabla \tilde{u}\|_{L^\infty(e_k)} \leq C(\|u_0\|_s + \|u\|_s + \|u\|_{1, \infty} + \|u_t\|_s + \|u_t\|_{L^2(H^s)})$

hold and if $u_0 \in H^s$, $u_t \in L^2(H^s)$ and $u_t(t) \in W^{1,\infty}$, then

$$(iii) \quad \|\nabla \tilde{u}_t\|_{L^\infty} \leq C(\|u_0\|_s + \|u_t\|_s + \|u_t\|_{1,\infty} + \|u_t\|_{L^2(H^s)}),$$

$$(iv) \quad \|\nabla \tilde{u}_t\|_{L^\infty(e_k)} \leq C(\|u_0\|_s + \|u_t\|_s + \|u_t\|_{1,\infty} + \|u_t\|_{L^2(H^s)})$$

hold, where $\mu = \min(r+1, s)$, $s \geq \frac{d}{2} + 1$ and $r \geq d/2$.

Proof. By Lemma 3.2 and Theorem 3.1, we note that

$$\begin{aligned} \|\nabla \tilde{u}\|_{L^\infty} &\leq \|\nabla(\tilde{u} - P_h u)\|_{L^\infty} + \|\nabla(P_h u) - \nabla u\|_{L^\infty} + \|\nabla u\|_{L^\infty} \\ &\leq C(h^{-\frac{d}{2}} \|\nabla(\tilde{u} - P_h u)\|_{L^2} + \|u\|_{1,\infty}) \\ &\leq Ch^{-\frac{d}{2}} (\|\nabla \tilde{u} - \nabla u\|_{L^2} + \|\nabla u - \nabla P_h u\|_{L^2}) + C\|u\|_{1,\infty} \\ &\leq Ch^{-\frac{d}{2}} \frac{h^{\mu-1}}{r^{s-2}} (\|u_0\|_s + \|u\|_s + \|u_t\|_{L^2(H^s)}) + C\|u\|_{1,\infty} \\ &\leq C(\|u_0\|_s + \|u\|_s + \|u\|_{1,\infty} + \|u_t\|_{L^2(H^s)}) \end{aligned}$$

which completes the proof of (i). If $e_k = E_i \cap E_j = E_{ij}$, by Lemma 3.2, Lemma 3.1 and Lemma 4.1, we have

$$\begin{aligned} &\|\nabla \tilde{u}\|_{L^\infty(e_k)} \\ &\leq \|\nabla(\tilde{u} - P_h u)\|_{L^\infty(e_k)} + \|\nabla P_h u\|_{L^\infty(e_k)} \\ &\leq Ch^{-\frac{(d-1)}{2}} \|\nabla(\tilde{u} - P_h u)\|_{L^2(e_k)} + C\|\nabla u\|_{L^\infty(E_i \cup E_j)} \\ &\leq Ch^{-\frac{(d-1)}{2}} h^{-\frac{1}{2}} (\|\nabla(\tilde{u} - P_h u)\|_{L^2(E_i \cup E_j)} + h\|\nabla^2(\tilde{u} - P_h u)\|_{L^2(E_i \cup E_j)}) \\ &\quad + C\|u\|_{1,\infty} \\ &\leq Ch^{-\frac{d}{2}} \frac{h^{\mu-1}}{r^{s-2}} (\|u_0\|_s + \|u\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^s)}) + C\|u\|_{1,\infty}. \end{aligned}$$

This proves (ii) since $\mu \geq d/2 + 1$. By the similar method, (iii) and (iv) can be proved. \square

Throughout this paper ε , ε_1 and ε_2 denote generic positive constants sufficiently small depending on C and the Sobolev norms of u but independent of h and r .

Remark. By Lemma 3.2, $\|u(x, 0) - U(x, 0)\|_{L^\infty} = \|u_0(x) - \tilde{u}(x, 0)\|_{L^\infty} = \|u_0(x) - P_h(u_0(x))\|_{L^\infty} < K^*$ holds.

Lemma 4.3. For a sufficiently small $\varepsilon > 0$ and a generic constant $C > 0$ the following estimation holds:

$$\begin{aligned} &\left| \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) + b(x, U)) \nabla \xi \cdot \mathbf{n}_k\} [\xi_t] ds \right. \\ &\quad \left. + \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) + b(x, U)) \nabla \xi_t \cdot \mathbf{n}_k\} [\xi] ds \right| \\ &\leq \varepsilon \|\xi_t\|_1^2 + C((a(x, U) + b(x, U)) \nabla \xi, \nabla \xi) + J_\beta^\sigma(\xi, \xi). \end{aligned}$$

Proof. Since $a(x, y)$ and $b(x, y)$ satisfy Condition (A), we have

$$\begin{aligned}
& \left| \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) + b(x, U)) \nabla \xi \cdot \mathbf{n}_k\} [\xi_t] ds \right. \\
& \quad \left. + \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) + b(x, U)) \nabla \xi_t \cdot \mathbf{n}_k\} [\xi] ds \right| \\
& \leq C \sum_{k=1}^{L_h} (\|\nabla \xi\|_{L^2(e_k)} \|\xi_t\|_{L^2(e_k)} + \|\nabla \xi_t\|_{L^2(e_k)} \|\xi\|_{L^2(e_k)}) \\
& \leq C \left(h^{(d-1)\beta/2} \sum_{k=1}^{L_h} \|\nabla \xi\|_{L^2(e_k)} \frac{1}{|e_k|^{\beta/2}} \|\xi_t\|_{L^2(e_k)} \right. \\
& \quad \left. + h^{(d-1)\beta/2} \sum_{k=1}^{L_h} \|\nabla \xi_t\|_{L^2(e_k)} \frac{1}{|e_k|^{\beta/2}} \|\xi\|_{L^2(e_k)} \right) \\
& \leq Ch \sum_{k=1}^{L_h} \|\nabla \xi\|_{L^2(e_k)}^2 + \varepsilon_1 \|\xi_t\|_1^2 + \varepsilon_1 h \sum_{k=1}^{L_h} \|\nabla \xi_t\|_{L^2(e_k)}^2 + CJ_\beta^\sigma(\xi, \xi) \\
& < \varepsilon \|\xi_t\|_1^2 + C((a(x, U) + b(x, U)) \nabla \xi, \nabla \xi) + J_\beta^\sigma(\xi, \xi),
\end{aligned}$$

where we apply Lemma 3.1 and Condition (A). \square

Lemma 4.4. *For a sufficiently small $\varepsilon > 0$, if $u_0 \in H^s$, $u(t) \in H^s$, $u(t) \in W^{1,\infty}$, $u_t \in L^2(H^s)$, $u_t(t) \in W^{1,\infty}$ we have the following estimations:*

- (i) $|A_\lambda(u; \tilde{u}, \xi + \xi_t) - A_\lambda(U; \tilde{u}, \xi + \xi_t)|$
 $\leq C(\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\xi\|^2) + 3\varepsilon \|\xi\|_1^2 + 3\varepsilon \|\xi_t\|_1^2;$
- (ii) $|B_\lambda(u; \tilde{u}_t, \xi + \xi_t) - B_\lambda(U; \tilde{u}_t, \xi + \xi_t)|$
 $\leq C(\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\xi\|^2) + 3\varepsilon \|\xi\|_1^2 + 3\varepsilon \|\xi_t\|_1^2.$

Proof. By the definitions of A_λ , $A_\lambda(U; \tilde{u}, \xi + \xi_t) - A_\lambda(u; \tilde{u}, \xi + \xi_t)$ can be separated as follows,

$$\begin{aligned}
& A_\lambda(U; \tilde{u}, \xi + \xi_t) - A_\lambda(u; \tilde{u}, \xi + \xi_t) \\
& = ((a(x, U) - a(x, u)) \nabla \tilde{u}, \nabla(\xi + \xi_t)) \\
& \quad - \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) - a(x, u)) \nabla \tilde{u} \cdot \mathbf{n}_k\} [\xi + \xi_t] ds \\
& \quad - \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) - a(x, u)) \nabla(\xi + \xi_t) \cdot \mathbf{n}_k\} [\tilde{u}] ds \\
& = E_1 + E_2 + E_3.
\end{aligned}$$

Now by applying Lemma 4.2, we get

$$|E_1| = |((a(x, U) - a(x, u)) \nabla \tilde{u}, \nabla(\xi + \xi_t))|$$

$$\begin{aligned}
&\leq C \|\nabla \tilde{u}\|_{L^\infty} \|u - U\| \|\nabla \xi + \nabla \xi_t\| \\
&\leq C (\|\eta\| + \|\xi\|) (\|\nabla \xi\| + \|\nabla \xi_t\|) \\
&\leq C (\|\eta\|^2 + \|\xi\|^2) + \varepsilon \|\xi\|_1^2 + \varepsilon \|\xi_t\|_1^2.
\end{aligned}$$

Applying Lemma 4.2 and the trace inequalities yields

$$\begin{aligned}
|E_2| &= \left| \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) - a(x, u)) \nabla \tilde{u} \cdot \mathbf{n}_k\} [\xi + \xi_t] ds \right| \\
&\leq C \sum_{k=1}^{L_h} \|\nabla \tilde{u}\|_{L^\infty(e_k)} \|u - U\|_{L^2(e_k)} (\|\xi\|_{L^2(e_k)} + \|\xi_t\|_{L^2(e_k)}) \\
&\leq C \sum_{k=1}^{L_h} \|\nabla \tilde{u}\|_{L^\infty(e_k)} (\|\eta\|_{L^2(e_k)} + \|\xi\|_{L^2(e_k)}) h^{\frac{(d-1)\beta}{2}} \left(\frac{1}{|e_k|^{\beta/2}} \|\xi\|_{L^2(e_k)} \right. \\
&\quad \left. + \frac{1}{|e_k|^{\beta/2}} \|\xi_t\|_{L^2(e_k)} \right) \\
&\leq C (\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\xi\|^2) + \varepsilon \|\xi\|_1^2 + \varepsilon \|\xi_t\|_1^2.
\end{aligned}$$

By applying Lemma 3.1 and Theorem 3.1

$$\begin{aligned}
|E_3| &= \left| \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) - a(x, u)) \nabla (\xi + \xi_t) \cdot \mathbf{n}_k\} [\tilde{u}] ds \right| \\
&\leq \sum_{k=1}^{L_h} \int_{e_k} \{|(a(x, U) - a(x, u)) \nabla (\xi + \xi_t) \cdot \mathbf{n}_k\} [\eta] | ds \\
&\leq C \sum_{k=1}^{L_h} (\|\nabla \xi\|_{L^\infty(e_k)} + \|\nabla \xi_t\|_{L^\infty(e_k)}) \|\{u - U\}\|_{L^2(e_k)} \|\eta\|_{L^2(e_k)} \\
&\leq C \sum_{j=1}^{N_h} h^{-\frac{1}{2}} (\|\nabla \xi\|_{L^2(E_j)} + \|\nabla \xi_t\|_{L^2(E_j)}) \left(h^{-1/2} \|\eta\|_{L^2(E_j)} + h^{1/2} \|\nabla \eta\|_{L^2(E_j)} \right. \\
&\quad \left. + h^{-1/2} \|\xi\|_{L^2(E_j)} \right) \cdot h^{-\frac{1}{2}} (\|\eta\|_{L^2(E_j)} + h \|\nabla \eta\|_{L^2(E_j)}) \\
&\leq C (\|\nabla \xi\| + \|\nabla \xi_t\|) (\|\eta\| + h \|\nabla \eta\| + \|\xi\|) (\|u_t\|_{L^2(H^s)} + \|u_0\|_s) \\
&\leq C (\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\xi\|^2) + \varepsilon \|\xi\|_1^2 + \varepsilon \|\xi_t\|_1^2,
\end{aligned}$$

where C depends on $\|u_t\|_{L^2(H^s)}$ and $\|u_0\|_s$. Combining the estimations of E_1 , E_2 and E_3 we have

$$\begin{aligned}
&|A_\lambda(U; \tilde{u}, \xi + \xi_t) - A_\lambda(u; \tilde{u}, \xi + \xi_t)| \\
&\leq C (\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\xi\|^2) + 3\varepsilon \|\xi\|_1^2 + 3\varepsilon \|\xi_t\|_1^2,
\end{aligned}$$

which completes the proof of (i). Similarly, by applying Lemma 4.2(iii), (iv) we can obtain the estimation (ii) as follows

$$\begin{aligned} & |B_\lambda(u; \tilde{u}_t, \xi + \xi_t) - B_\lambda(U; \tilde{u}_t, \xi + \xi_t)| \\ & \leq C (\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\xi\|^2) + 3\varepsilon \|\xi\|_1^2 + 3\varepsilon \|\xi_t\|_1^2, \end{aligned}$$

where C depends on $\|u_0\|_s, \|u_t\|_s, \|u_t\|_{1,\infty}$ and $\|u_t\|_{L^2(H^s)}$. \square

Theorem 4.2. *For a $\lambda > 1$ and $\beta = \frac{1}{d-1}$, if $u_0 \in H^s$, $u(t) \in H^s$, $u(t) \in W^{1,\infty}$, $u_t \in L^2(H^s)$, $u_t(t) \in W^{1,\infty}$ and $u_{tt} \in L^2(H^s)$ and f is locally Lipschitz continuous at u , i.e., there exists $c(u, K^*)$ such that if $|u(x, t) - v| \leq 2K^*$, then $|f(x, u(x, t)) - f(x, v)| \leq c(u, K^*)|u(x, t) - v|$, $\forall (x, t) \in \Omega \times [0, T]$, $\forall v \in \mathbb{R}$, then there exists a constant $C > 0$ independent of h and r such that*

$$\|u(\cdot, t) - U(\cdot, t)\| \leq C \frac{h^\mu}{r^{s-2}} (\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}),$$

where $\mu = \min(r+1, s)$, $r \geq \frac{d}{2}$ and $s \geq 1 + \frac{d}{2}$.

Proof. From (3.1), (3.3) and the definitions of A_λ and B_λ we have

$$\begin{aligned} & (u_{tt} - \tilde{u}_{tt} + \tilde{u}_{tt} - U_{tt}, v) + A_\lambda(u; u - \tilde{u}, v) + A_\lambda(u; \tilde{u}, v) \\ & + A_\lambda(U; \tilde{u} - U, v) - A_\lambda(U; \tilde{u}, v) + B_\lambda(u; (u_t - \tilde{u}_t), v) + B_\lambda(u; \tilde{u}_t, v) \\ (4.3) \quad & + B_\lambda(U; \tilde{u}_t - U_t, v) - B_\lambda(U; \tilde{u}_t, v) \\ & = (f(x, u) - f(x, U), v) + \lambda(u - U, v) + \lambda(u_t - U_t, v), \quad \forall v \in D_r(\mathcal{E}_h). \end{aligned}$$

Now, we choose $v = \xi + \xi_t$ as a test function in (4.3) and we apply (3.2) to get the following

$$\begin{aligned} & (\eta_{tt} + \xi_{tt}, \xi + \xi_t) + A_\lambda(u; \tilde{u}, \xi + \xi_t) + A_\lambda(U; \xi, \xi + \xi_t) - A_\lambda(U; \tilde{u}, \xi + \xi_t) \\ & + B_\lambda(u; \tilde{u}_t, \xi + \xi_t) + B_\lambda(U; \xi_t, \xi + \xi_t) - B_\lambda(U; \tilde{u}_t, \xi + \xi_t) \\ & = (f(x, u) - f(x, U), \xi + \xi_t) + \lambda(u - U, \xi + \xi_t) + \lambda(u_t - U_t, \xi + \xi_t). \end{aligned}$$

By simple computation we get

$$\begin{aligned} (4.4) \quad & (\xi_{tt}, \xi_t) + (\xi_{tt}, \xi) + A_\lambda(U; \xi, \xi) + B_\lambda(U; \xi_t, \xi_t) + A_\lambda(U; \xi, \xi_t) + B_\lambda(U; \xi_t, \xi) \\ & = -A_\lambda(u; \tilde{u}, \xi + \xi_t) + A_\lambda(U; \tilde{u}, \xi + \xi_t) - B_\lambda(u; \tilde{u}_t, \xi + \xi_t) + B_\lambda(U; \tilde{u}_t, \xi + \xi_t) \\ & + (f(x, u) - f(x, U), \xi + \xi_t) + \lambda(u - U, \xi + \xi_t) + \lambda(u_t - U_t, \xi + \xi_t) \\ & - (\eta_{tt}, \xi + \xi_t). \end{aligned}$$

Applying the definitions of $A_\lambda(U; \xi, \xi_t)$ and $B_\lambda(U; \xi_t, \xi)$ in (4.4) we obtain

$$\begin{aligned} & (\xi_{tt}, \xi_t) + A_\lambda(U; \xi, \xi) + B_\lambda(U; \xi_t, \xi_t) + (\xi_{tt}, \xi) \\ & + \frac{1}{2} \frac{d}{dt} ((a(x, U) \nabla \xi, \nabla \xi) + \lambda(\xi, \xi) + J_\beta^\sigma(\xi, \xi)) \\ & + \frac{1}{2} \frac{d}{dt} ((b(x, U) \nabla \xi, \nabla \xi) + \lambda(\xi, \xi) + J_\beta^\sigma(\xi, \xi)) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{\partial}{\partial t} (a(x, U) + b(x, U)) \nabla \xi, \nabla \xi \right) \\
&\quad + \sum_{k=1}^{L_h} \int_{e_k} \{ (a(x, U) + b(x, U)) \nabla \xi \cdot \mathbf{n}_k \} [\xi_t] \, ds \\
&\quad + \sum_{k=1}^{L_h} \int_{e_k} \{ (a(x, U) + b(x, U)) \nabla \xi_t \cdot \mathbf{n}_k \} [\xi] \, ds \\
&\quad - A_\lambda(u; \tilde{u}, \xi + \xi_t) + A_\lambda(U; \tilde{u}, \xi + \xi_t) - B_\lambda(u; \tilde{u}_t, \xi + \xi_t) + B_\lambda(U; \tilde{u}_t, \xi + \xi_t) \\
&\quad + (f(x, u) - f(x, U), \xi + \xi_t) + \lambda(u - U, \xi + \xi_t) + \lambda(u_t - U_t, \xi + \xi_t) \\
&\quad - (\eta_{tt}, \xi + \xi_t).
\end{aligned}$$

By applying Lemma 3.4 to the above equation, we have

$$\begin{aligned}
(4.5) \quad &\frac{1}{2} \frac{d}{dt} \|\xi_t\|^2 + c \|\xi\|_1^2 + c \|\xi_t\|_1^2 \\
&\quad + \frac{d}{dt} \left(\frac{1}{2} ((a(x, U) + b(x, U)) \nabla \xi, \nabla \xi) + \lambda(\xi, \xi) + J_\beta^\sigma(\xi, \xi) \right) \\
&\leq -(\xi_{tt}, \xi) - (\eta_{tt}, \xi + \xi_t) + \frac{1}{2} \left(\frac{\partial}{\partial t} (a(x, U) + b(x, U)) \nabla \xi, \nabla \xi \right) \\
&\quad + \left(\sum_{k=1}^{L_h} \int_{e_k} \{ (a(x, U) + b(x, U)) \nabla \xi \cdot \mathbf{n}_k \} [\xi_t] \, ds \right. \\
&\quad \left. + \sum_{k=1}^{L_h} \int_{e_k} \{ (a(x, U) + b(x, U)) \nabla \xi_t \cdot \mathbf{n}_k \} [\xi] \, ds \right) \\
&\quad + (-A_\lambda(u; \tilde{u}, \xi + \xi_t) + A_\lambda(U; \tilde{u}, \xi + \xi_t)) \\
&\quad + (-B_\lambda(u; \tilde{u}_t, \xi + \xi_t) + B_\lambda(U; \tilde{u}_t, \xi + \xi_t)) \\
&\quad + (f(x, u) - f(x, U), \xi + \xi_t) + \lambda(u - U, \xi + \xi_t) \\
&\quad + \lambda(u_t - U_t, \xi + \xi_t).
\end{aligned}$$

To continue the proof we temporarily assume that there exists $0 < h^* < 1$ such that

$$(4.6) \quad \|\xi_t(t)\| < \tilde{C} \frac{h^{\frac{d}{2} + \delta_0}}{r^{s-2}}, \quad 0 < \delta_0 < 1$$

and

$$(4.7) \quad \|\xi(t)\|_{L^\infty} < K^*$$

hold for $h < h^*$, $\forall t \in [0, T]$ and $\tilde{C} > 0$. Later we verify that these hypotheses hold. By the hypothesis (4.6) and the inverse inequality we have

$$(4.8) \quad \|\xi_t\|_{L^\infty} \leq Ch^{-\frac{d}{2}} \|\xi_t\| \leq Ch^{\delta_0} < C^*, \quad \forall t \in [0, T].$$

By the hypothesis (4.6), the inverse inequality, Lemma 3.1 and Lemma 4.1 we have

$$\begin{aligned} \|(u - U)(t)\|_{L^\infty} &\leq \|(u - P_h u)(t)\|_{L^\infty} + \|(P_h u - \tilde{u})(t)\|_{L^\infty} + \|(\tilde{u} - U)(t)\|_{L^\infty} \\ &\leq C\left(h + \frac{h^{\mu - \frac{d}{2}}}{r^{s-2}}\right)(\|u\|_{W^{1,\infty}} + \|u_0\|_s + \|u\|_s + \|u_t\|_s \\ &\quad + \|u_t\|_{L^2(H^s)}) + K^* \\ &\leq 2K^*. \end{aligned}$$

Now we estimate the terms in the right hand side of (4.5). For sufficiently small $\varepsilon > 0$, we have the following estimation for $(\eta_{tt}, \xi + \xi_t)$

$$|(\eta_{tt}, \xi + \xi_t)| \leq \|\eta_{tt}\|(\|\xi\| + \|\xi_t\|) \leq C\|\xi\|^2 + \varepsilon\|\xi_t\|^2 + C\|\eta_{tt}\|^2.$$

By applying Condition (B) on the functions a and b and (4.8), we get the following estimation.

$$\left| \frac{1}{2} \left(\frac{\partial}{\partial t} (a(x, U) + b(x, U)) \nabla \xi, \nabla \xi \right) \right| \leq C \|U_t\|_{L^\infty} \|\nabla \xi\|^2 \leq C(a(x, U) \nabla \xi, \nabla \xi),$$

where C depends on $\|u\|_{L^\infty(L^\infty)}$ and C^* . By applying Lemma 4.3 we obtain

$$\begin{aligned} &\left| \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) + b(x, U)) \nabla \xi \cdot \mathbf{n}_k\} [\xi_t] ds \right. \\ &\quad \left. + \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, U) + b(x, U)) \nabla \xi_t \cdot \mathbf{n}_k\} [\xi] ds \right| \\ &\leq \varepsilon \|\xi_t\|_1^2 + C((a(x, U) + b(x, U)) \nabla \xi, \nabla \xi) + J_\beta^\sigma(\xi, \xi). \end{aligned}$$

By applying (4.8) and the local Lipschitz property of f at $u(x, t)$, we obtain the estimations in the following,

$$\begin{aligned} |(f(x, u) - f(x, U), \xi + \xi_t)| &\leq c(u, K^*)(\|\eta\| + \|\xi\|)(\|\xi\| + \|\xi_t\|) \\ &\leq C(\|\eta\|^2 + \|\xi\|^2) + \varepsilon\|\xi_t\|^2, \\ |\lambda(u - U, \xi + \xi_t)| &\leq \lambda(\|\eta\| + \|\xi\|)(\|\xi\| + \|\xi_t\|) \leq C(\|\eta\|^2 + \|\xi\|^2) + \varepsilon\|\xi_t\|^2, \\ |\lambda(u_t - U_t, \xi + \xi_t)| &\leq \lambda(\|\eta_t\| + \|\xi_t\|)(\|\xi\| + \|\xi_t\|) \\ &\leq C(\|\eta_t\|^2 + \|\xi\|^2) + (\lambda + \varepsilon)\|\xi_t\|^2. \end{aligned}$$

Now substituting the above estimations and applying Lemma 4.3 and Lemma 4.4 in (4.5) we get

$$\begin{aligned} (4.9) \quad &\frac{d}{dt} \left(\frac{1}{2} \|\xi_t\|^2 + \frac{1}{2} ((a(x, U) + b(x, U)) \nabla \xi, \nabla \xi) + \lambda(\xi, \xi) + J_\beta^\sigma(\xi, \xi) \right) + \frac{c}{2} \|\xi\|_1^2 \\ &\quad + \frac{c}{2} \|\xi_t\|_1^2 \\ &\leq -(\xi_{tt}, \xi) + C(\|\eta_{tt}\|^2 + ((a(x, U) + b(x, U)) \nabla \xi, \nabla \xi) + J_\beta^\sigma(\xi, \xi)) \end{aligned}$$

$$+ \|\eta\|^2 + \|\eta_t\|^2 + h^2 \|\nabla \eta\|^2 + \|\xi\|^2 + \|\xi_t\|^2).$$

Applying the integration by parts and Cauchy Schwarz's inequality we have

$$(4.10) \quad - \int_0^t (\xi_{tt}, \xi) d\tau \leq \frac{1}{4} \|\xi_t(t)\|^2 + \|\xi(t)\|^2 + \frac{1}{2} \|\xi_t(0)\|^2 + \frac{1}{2} \|\xi(0)\|^2 \\ + \int_0^t \|\xi_t\|^2 d\tau.$$

By integrating both sides of (4.9) from 0 to t and applying (4.10), we have

$$\begin{aligned} & \frac{1}{2} \|\xi_t(t)\|^2 + \frac{1}{2} ((a(x, U) + b(x, U)) \nabla \xi(t), \nabla \xi(t)) + \lambda \|\xi(t)\|^2 + J_\beta^\sigma(\xi(t), \xi(t)) \\ & + \frac{c}{2} \left(\int_0^t \|\xi\|_1^2 + \|\xi_t\|_1^2 d\tau \right) \\ \leq & \frac{1}{2} \|\xi_t(0)\|^2 + \frac{1}{2} ((a(x, U) + b(x, U)) \nabla \xi(0), \nabla \xi(0)) + \lambda \|\xi(0)\|^2 + J_\beta^\sigma(\xi(0), \xi(0)) \\ & + \frac{1}{4} \|\xi_t(t)\|^2 + \|\xi(t)\|^2 + \frac{1}{2} \|\xi_t(0)\|^2 + \frac{1}{2} \|\xi(0)\|^2 \\ & + C \left(\int_0^t (\|\eta_{tt}\|^2 + \|\eta_t\|^2 + \|\eta\|^2 + h^2 \|\nabla \eta\|^2) d\tau \right) \\ & + C \left(\int_0^t ((a(U) + b(U)) \nabla \xi, \nabla \xi) + J_\beta^\sigma(\xi, \xi) + \|\xi\|^2 + \|\xi_t\|^2 d\tau \right). \end{aligned}$$

Adopting the initial approximations from (3.3) to the inequality above, we obtain

$$\begin{aligned} & \frac{1}{4} \|\xi_t(t)\|^2 + \frac{1}{2} ((a(x, U) + b(x, U)) \nabla \xi(t), \nabla \xi(t)) + (\lambda - 1) \|\xi(t)\|^2 \\ & + J_\beta^\sigma(\xi(t), \xi(t)) + \frac{c}{2} \int_0^t (\|\xi\|_1^2 + \|\xi_t\|_1^2) d\tau \\ \leq & C \int_0^t \|\xi_t\|^2 d\tau + C \left(\int_0^t (\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2) d\tau \right) \\ & + C \left(\int_0^t ((a(x, U) + b(x, U)) \nabla \xi, \nabla \xi) + J_\beta^\sigma(\xi, \xi) + \|\xi\|^2 d\tau \right) \end{aligned}$$

for $\lambda > 1$. By applying the Gronwall's Lemma, we have

$$\begin{aligned} & \|\xi_t(t)\|^2 + ((a(x, U) + b(x, U)) \nabla \xi(t), \nabla \xi(t)) + \|\xi(t)\|^2 + J_\beta^\sigma(\xi(t), \xi(t)) \\ \leq & C \int_0^t (\|\eta\|^2 + h^2 \|\nabla \eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2) d\tau. \end{aligned}$$

Applying the estimations from Theorems 3.1 and 3.2 to the above inequality we get

$$(4.11) \quad \|\xi_t(t)\|^2 + \|\xi(t)\|^2 + J_\beta^\sigma(\xi(t), \xi(t)) \leq C \left(\frac{h^\mu}{r^{s-2}} \right)^2 (\|u_0\|_s^2 + \|u_t\|_{L^2(H^s)}^2 + \|u_{tt}\|_{L^2(H^s)}^2).$$

Combining (4.11) and Theorem 3.1 we conclude that the following optimal error estimate holds

$$\|u(\cdot, t) - U(\cdot, t)\| \leq C \frac{h^\mu}{r^{s-2}} (\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^2(H^s)}),$$

where $\mu = \min(r + 1, s)$.

Now we show that the hypotheses (4.6) hold. From (3.3), (4.6) obviously holds for $t = 0$. Now we assume that there exists $0 < t^* \leq T$ such that (4.6) holds for $0 \leq t < t^*$ but

$$(4.12) \quad \|\xi_t(t^*)\| \geq \tilde{C} \frac{h^{\frac{d}{2} + \delta_0}}{r^{s-2}}.$$

Now we take a sequence of $\{t_n\} \subset [0, t^*)$ such that $\lim_{n \rightarrow \infty} t_n = t^*$. For $t = t_n$ the hypothesis (4.6) is satisfied and so (4.11) can be obtained for $t = t_n$ by the aforementioned process in this proof. Therefore $\|\xi_t(t_n)\| \leq C \frac{h^\mu}{r^{s-2}} \leq \frac{\tilde{C}}{2} \frac{h^{\frac{d}{2} + \delta_0}}{r^{s-2}}$. By the continuity property of $\|\xi_t\|$ with respect to t proved in Theorem 4.1 we get $\|\xi_t(t^*)\| \leq \frac{\tilde{C}}{2} \frac{h^{\frac{d}{2} + \delta_0}}{r^{s-2}}$ which contradicts to (4.12). Therefore the hypothesis (4.6) holds. Now we will prove that the hypothesis (4.7) holds. Obviously (4.7) holds for $t = 0$. Now we assume that there exists $\tau \in (0, T]$ such that (4.7) holds for $0 \leq t < \tau$ but

$$(4.13) \quad \|\xi(\tau)\|_{L^\infty} \geq K^*.$$

Now again we take a sequence of $\{t_n\} \subset [0, \tau)$ such that $\lim_{n \rightarrow \infty} t_n = \tau$. For $t = t_n$, by applying Lemma 3.2 and Lemma 4.1 we have

$$\begin{aligned} & \|(u - U)(t_n)\|_{L^\infty} \\ & \leq \|(u - P_h u)(t_n)\|_{L^\infty} + \|(P_h u - \tilde{u})(t_n)\|_{L^\infty} + \|\tilde{u}(t_n) - U(t_n)\|_{L^\infty} \\ & \leq Ch \|u\|_{1,\infty} + \|\theta(t_n)\|_{L^\infty} + \|\xi(t_n)\|_{L^\infty} \\ & \leq C \left(h + \frac{h^{\mu - \frac{d}{2} - 1}}{r^{s-2}} \right) (\|u_0\|_s + \|u\|_{1,\infty} + \|u\|_s + \|u_t\|_s + \|u_t\|_{L^2(H^2)}) + K^*, \end{aligned}$$

where C depends on the sobolev norms of u . Therefore for sufficiently small h , we have $\|(u - U)(t_n)\|_{L^\infty} \leq 2K^*$. For $t = t_n$ we conclude that the following property, needed to obtain the result (4.11), $|f(x, u(x, v)) - f(x, U(x, t_n))| \leq c(u, K^*)|u(x, t_n) - U(x, t_n)|$ holds. Therefore we have (4.11) holds for $t = t_n$

so that $\|\xi(t_n)\|_{L^\infty} \leq C \frac{h^{\mu - \frac{d}{2}}}{r^{s-2}} (\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^2(H^s)}) \leq \frac{K^*}{2}$ holds for sufficiently small h . In Theorem 4.1 we prove the continuity of $U(t)$ with respect to t so we have the continuity of $\|\xi(t)\|_{L^\infty}$ hence we have $\|\xi(\tau)\|_{L^\infty} \leq \frac{K^*}{2}$ which contradicts to (4.13), so that the hypothesis (4.7) holds. This completes the proof of Theorem 4.2. \square

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MI RAY OHM
DIVISION OF INFORMATION SYSTEMS ENGINEERING
DONGSEO UNIVERSITY
PUSAN 617-716, KOREA
E-mail address: mrohmdongseo.ac.kr

HYUN YOUNG LEE
DEPARTMENT OF MATHEMATICS
KYUNGSUNG UNIVERSITY
PUSAN 608-736, KOREA
E-mail address: hylee@ks.ac.kr

JUN YONG SHIN
DEPARTMENT OF APPLIED MATHEMATICS
PUKYONG NATIONAL UNIVERSITY
PUSAN 608-737, KOREA
E-mail address: jyshin@pknu.ac.kr