

**EXISTENCE OF  $n$  POSITIVE SOLUTIONS TO  
SECOND-ORDER MULTI-POINT BOUNDARY VALUE  
PROBLEM AT RESONANCE**

FENG WANG AND FANG ZHANG

ABSTRACT. The existence of  $n$  positive solutions is established for second-order multi-point boundary value problem at resonance where  $n$  is an arbitrary natural number. The proof is based on a theory of fixed point index for A-proper semilinear operators defined on cones due to Cremins.

**1. Introduction**

In this paper, we are concerned the multi-point boundary value problem (BVP for short)

$$(1.1) \quad \begin{cases} -x''(t) = f(t, x), & t \in (0, 1), \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), & x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

where  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $m > 2$ ,  $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ ,  $\alpha_i, \beta_i \geq 0$ ,  $i = 1, 2, \dots, m - 2$  with condition

$$(1.2) \quad \sum_{i=1}^{m-2} \alpha_i = \sum_{i=1}^{m-2} \beta_i = 1.$$

We note that (1.2) means that the multi-point boundary value problem (1.1) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{cases} -x''(t) = 0, & t \in (0, 1), \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), & x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) \end{cases}$$

has  $x(t) \equiv c$ ,  $t \in [0, 1]$ ,  $c \in \mathbb{R}$ , as a nontrivial solution.

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Received April 20, 2011; Revised March 22, 2012.

2010 *Mathematics Subject Classification.* 34B10, 34B15.

*Key words and phrases.* boundary value problem, positive solution, resonance, A-proper, fixed point index.

The project is supported financially by the National Natural Science Foundation of China (11171090, 10971179), the Program for New Century Excellent Talents in University (NCE10-0325) and the Natural Science Foundation of Changzhou University (JS201008).

Motivated by the work of Bitsadze and Samarskii on nonlocal linear elliptic boundary value problem [2, 3], Il'in and Moiseev studied a multi-point boundary value problems for linear second-order ordinary differential equations [13]. Since then, great efforts have been devoted to the multi-point boundary value problems for more general nonlinear ordinary differential equations due to its theoretical challenge and its great potential applications; see for example [1, 6, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 21, 22, 25, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39] and the references therein. Some theorems and methods of nonlinear functional analysis have been applied to research on this problem, such as the method of topological degree and Leray-Schauder fixed point theorem [15, 30, 31, 33, 34, 37], the upper and lower solutions method and monotone iterative technique [38], and Mawhin coincidence degree theory [21, 22]. Boundary value problems at resonance in the continuous setting have been studied by several authors, see [6, 8, 9, 19] for some references along this line. In most real problems, only the positive solution is significant. It is well known that the problem of existence of positive solutions to boundary value problem is very difficult when the resonant case is considered. To our best knowledge, only few papers deal with positive solutions to boundary value problems at resonance [1, 12, 14, 16, 25, 32, 35, 36, 39]. Recently, Yang and Shen [36] obtained the existence of a positive solution for problem (1.1) with resonant condition (1.2) or

$$(1.3) \quad \sum_{i=1}^{m-2} \alpha_i \neq 1, \quad \sum_{i=1}^{m-2} \beta_i \neq 1, \quad \sum_{i=1}^{m-2} \alpha_i \xi_i \left(1 - \sum_{i=1}^{m-2} \beta_i\right) + \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \left(1 - \sum_{i=1}^{m-2} \beta_i \xi_i\right) = 0.$$

The main method is the Leggett-Williams norm-type theorem due to O'Regan and Zima [24]. But there are no results concerning with the multiplicity of positive solutions for (1.1). Inspired by the above work, the aim of this paper is to establish the existence results of  $n$  positive solutions for boundary value problem (1.1), where  $n$  is an arbitrary natural number. This will be done by applying the theory of a fixed point index for A-proper semilinear operators defined on cones obtained by Cremins [4].

The remaining part of the paper is organized as follows. In Section 2, some preliminaries and lemmas will be given, and the main theorems are formulated and proved in Section 3. Finally, in Section 4, we give an example to illustrate our results.

## 2. Notation and preliminaries

We start by introducing some basic notation relative to theory of the fixed point index for A-proper semilinear operators defined on cones established by Cremins (see [4]).

Let  $X$  and  $Y$  be Banach spaces,  $D$  a linear subspace of  $X$ ,  $\{X_n\} \subset D$ , and  $\{Y_n\} \subset Y$  sequences of oriented finite dimensional subspaces such that  $Q_n y \rightarrow y$

in  $Y$  for every  $y$  and  $\text{dist}(x, X_n) \rightarrow 0$  for every  $x \in D$  where  $Q_n : Y \rightarrow Y_n$  and  $P_n : X \rightarrow X_n$  are sequences of continuous linear projections. The projection scheme  $\Gamma = \{X_n, Y_n, P_n, Q_n\}$  is then said to be admissible for maps from  $D \subset X$  to  $Y$ .

**Definition 2.1** ([4]). A map  $T : D \subset X \rightarrow Y$  is called approximation-proper (abbreviated A-proper) at a point  $y \in Y$  with respect to  $\Gamma$ , if  $T_n \equiv Q_n T|_{D \cap X_n}$  is continuous for each  $n \in \mathbb{N}$  and whenever  $\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$  is bounded with  $T_{n_j} x_{n_j} \rightarrow y$ , then there exists a subsequence  $\{x_{n_{j_k}}\}$  such that  $x_{n_{j_k}} \rightarrow x \in D$ , and  $Tx = y$ .  $T$  is said to be A-proper on a set  $D$  if it is A-proper at all points of  $D$ .

Let  $K$  be a cone in a finite dimensional Banach space  $X$  and  $\Omega \subset X$  be open and bounded with  $\Omega \cap K = \Omega_K \neq \emptyset$ . Let  $T : \bar{\Omega}_K \rightarrow K$  be continuous such that  $Tx \neq x$  on  $\partial\Omega_K = \partial\Omega \cap K$  where  $\bar{\Omega}_K$  and  $\partial\Omega_K$  denote the closure and boundary, respectively, of  $\Omega_K$  relative  $K$ . Let  $\rho : X \rightarrow K$  be an arbitrary retraction.

The following definition of finite dimensional index forms the basis of generalized index for A-proper maps  $I - T$ .

**Definition 2.2** ([4]). We define

$$i_K(T, \Omega) = \text{deg}_B(I - T\rho, \rho^{-1}(\Omega) \cap B_R, 0),$$

where the degree is the Brouwer degree and  $B_R$  is a ball containing  $\Omega_K$ .

Now let  $K$  be a cone in an infinite dimensional Banach space  $X$  with projection scheme  $\Gamma$  such that  $Q_n(K) \subseteq K$  for every  $n \in \mathbb{N}$ . Let  $\rho : X \rightarrow K$  be an arbitrary retraction and  $\Omega \subset X$  an open bounded set such that  $\Omega_K = \Omega \cap K \neq \emptyset$ . Let  $T : \bar{\Omega}_K \rightarrow K$  be such that  $I - T$  is A-proper at 0. Write  $K_n = K \cap X_n = Q_n K$  and  $\Omega_n = \Omega_K \cap X_n$ . Then  $Q_n \rho : X_n \rightarrow K_n$  is a finite dimensional retraction.

**Definition 2.3** ([4]). If  $Tx \neq x$  on  $\partial\Omega_K$ , then we define

$$\text{ind}_K(T, \Omega) = \{k \in \mathbb{Z} \cup \{\pm\infty\} : i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j}) \rightarrow k \text{ for some } n_j \rightarrow \infty\},$$

that is, the index is the set of limit points of  $i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j})$ , where the finite dimensional index is that defined above.

Let  $L : \text{dom}L \subset X \rightarrow Y$  be a Fredholm map of index zero and  $P : X \rightarrow X$ ,  $Q : Y \rightarrow Y$  be continuous projectors such that  $\text{Im}P = \text{Ker}L$ ,  $\text{Ker}Q = \text{Im}L$  and  $X = \text{Ker}L \oplus \text{Ker}P$ ,  $Y = \text{Im}L \oplus \text{Im}Q$ . The restriction of  $L$  to  $\text{dom}L \cap \text{Ker}P$ , denote  $L_1$ , is a bijection onto  $\text{Im}L$  with continuous inverse  $L_1^{-1} : \text{Im}L \rightarrow \text{dom}L \cap \text{Ker}P$ . Since  $\dim \text{Im}Q = \dim \text{Ker}L$ , there exists a continuous bijection  $J : \text{Im}Q \rightarrow \text{Ker}L$ . let  $K$  be a cone in an infinite dimensional Banach space  $X$  with projection scheme  $\Gamma$ . If we let  $H = L + J^{-1}P$ , then  $H : \text{dom}L \subset X \rightarrow Y$  is a linear bijection with bounded inverse. Thus  $K_1 = H(K \cap \text{dom}L)$  is a cone in the Banach space  $Y$ .

Let  $\Omega \subset X$  be open and bounded with  $\Omega_K \cap \text{dom}L \neq \emptyset$ ,  $L : \text{dom}L \subset X \rightarrow Y$  a bounded Fredholm operator of index zero,  $N : \overline{\Omega}_K \cap \text{dom}L \rightarrow Y$  a bounded continuous nonlinear operator such that  $L - N$  is A-proper at 0.

We can now extend the definition of the index to A-proper maps of the form  $L - N$  acting on cones.

**Definition 2.4** ([4]). Let  $\rho_1$  be a retraction from  $Y$  to  $K_1$  and assume  $Q_n K_1 \subset K_1$ ,  $P + JQN + L_1^{-1}(I - Q)N$  maps  $K \cap \text{dom}L$  to  $K \cap \text{dom}L$  and  $Lx \neq Nx$  on  $\partial\Omega_K$ . We define the fixed point index of  $L - N$  over  $\Omega_K$  as

$$\text{ind}_K([L, N], \Omega) = \text{ind}_{K_1}(T, U),$$

where  $U = H(\Omega_K)$ ,  $T : Y \rightarrow Y$  is defined as  $Ty = (N + J^{-1}P)H^{-1}y$  for each  $y \in Y$ , and the index on the right is that of Definition 2.3.

For convenience, we recall some properties of  $\text{ind}_K$ .

**Proposition 2.1** ([4]). Let  $L : \text{dom}L \rightarrow Y$  be a Fredholm operator of index zero,  $\Omega \subset X$  be open and bounded. Assume that  $P + JQN + L_1^{-1}(I - Q)N$  maps  $K$  to  $K$ , and  $Lx \neq Nx$  on  $\partial\Omega_K$ . Then we have

(P<sub>1</sub>) (Existence property) if  $\text{ind}_K([L, N], \Omega) \neq \{0\}$ , then there exists  $x \in \Omega_K$  such that  $Lx = Nx$ .

(P<sub>2</sub>) (Normality property) if  $x_0 \in \Omega_K$ , then  $\text{ind}_K([L, -J^{-1}P + \hat{y}_0], \Omega) = \{1\}$ , where  $\hat{y}_0 = Hx_0$  and  $\hat{y}_0(y) = y_0$  for every  $y \in H(\Omega_K)$ .

(P<sub>3</sub>) (Additivity property) if  $Lx \neq Nx$  for  $x \in \overline{\Omega}_K \setminus (\Omega_1 \cup \Omega_2)$ , where  $\Omega_1$  and  $\Omega_2$  are disjoint relatively open subsets of  $\Omega_K$ , then

$$\text{ind}_K([L, N], \Omega) \subseteq \text{ind}_K([L, N], \Omega_1) + \text{ind}_K([L, N], \Omega_2)$$

with equality if either of indices on the right is a singleton.

(P<sub>4</sub>) (Homotopy invariance property) if  $L - N(\lambda, x)$  is an A-proper homotopy on  $\Omega_K$  for  $\lambda \in [0, 1]$  and  $(N(\lambda, x) + J^{-1}P)H^{-1} : K_1 \rightarrow K_1$  and  $\theta \notin (L - N(\lambda, x))(\partial\Omega_K)$  for  $\lambda \in [0, 1]$ , then  $\text{ind}_K([L, N(\lambda, x)], \Omega) = \text{ind}_{K_1}(T_\lambda, U)$  is independent of  $\lambda \in [0, 1]$ , where  $T_\lambda = (N(\lambda, x) + J^{-1}P)H^{-1}$ .

The following two lemmas will be used in this paper.

**Lemma 2.1** ([29]). If  $L : \text{dom}L \rightarrow Y$  is a Fredholm operator of index zero,  $\Omega$  is an open bounded set, and  $\Omega_K \cap \text{dom}L \neq \emptyset$ , and let  $L - \lambda N$  be A-proper for  $\lambda \in [0, 1]$ . Assume that  $N$  is bounded and  $P + JQN + L_1^{-1}(I - Q)N$  maps  $K$  to  $K$ . If there exists  $e \in K_1 \setminus \{\theta\}$  such that

$$(2.1) \quad Lx - Nx \neq \mu e$$

for every  $x \in \partial\Omega_K$  and all  $\mu \geq 0$ , then  $\text{ind}_K([L, N], \Omega) = \{0\}$ .

We assume that there is a continuous bilinear form  $[y, x]$  on  $Y \times X$  such that  $y \in \text{Im}L$  if and only if  $[y, x] = 0$  for each  $x \in \text{Ker}L$ . This condition implies that if  $\{x_1, x_2, \dots, x_n\}$  is a basis in  $\text{Ker}L$ , then the linear map  $J : \text{Im}Q \rightarrow \text{Ker}L$  defined by  $Jy = \beta \sum_{i=1}^n [y, x_i]x_i$ ,  $\beta \in \mathbb{R}^+$  is an isomorphism and that

if  $y = \sum_{i=1}^n y_i x_i$ , then  $[J^{-1}y, x_i] = \frac{y_i}{\beta}$  for  $1 \leq i \leq n$  and  $[J^{-1}x_0, x_0] > 0$  for  $x_0 \in \text{Ker}L$ .

Cremins [4] extended a continuation theorem related to that of Mawhin [20] and Petryshyn [26] for semilinear equations to cones refer to [4, Corollary 1] for the details. By Lemma 2.1 and [4, Corollary 1], we obtain that following existence theorem of positive solutions to a semilinear equation in cones.

**Lemma 2.2.** *If  $L : \text{dom}L \rightarrow Y$  is a Fredholm operator of index zero,  $K \subset X$  is a cone,  $\Omega_1$  and  $\Omega_2$  are open bounded sets such that  $\theta \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$  and  $\Omega_2 \cap K \cap \text{dom}L \neq \emptyset$ , where  $\theta$  is the zero element of  $X$ . Suppose that  $L - \lambda N$  is  $A$ -proper for  $\lambda \in [0, 1]$  with  $N : \Omega_2 \cap \bar{K} \rightarrow Y$  is bounded. Assume that*

- (C<sub>1</sub>)  $(P + JQN)(K) \subset K$  and  $(P + JQN + L_1^{-1}(I - Q)N)(K) \subset K$ ,
- (C<sub>2</sub>)  $Lx \neq \lambda Nx$  for  $x \in \partial\Omega_2 \cap K$ ,  $\lambda \in (0, 1]$ ,
- (C<sub>3</sub>)  $QNx \neq 0$  for  $x \in \partial\Omega_2 \cap K \cap \text{Ker}L$ ,
- (C<sub>4</sub>)  $[QNx, x] \leq 0$  for all  $x \in \partial\Omega_2 \cap K \cap \text{Ker}L$ ,
- (C<sub>5</sub>) there exists  $e \in K_1 \setminus \{\theta\}$  such that

$$Lx - Nx \neq \mu e \text{ for every } \mu \geq 0, x \in \partial\Omega_1 \cap K.$$

Then there exists  $x \in \text{dom}L \cap K \cap (\bar{\Omega}_2 \setminus \Omega_1)$  such that  $Lx = Nx$ .

*Remark 2.1.* It is worth mentioning that the positive or nonnegative solutions of an operator equation  $Lx = Nx$  was also discussed by a recent paper of O'Regan and Zima [24] and the earlier papers [5, 7, 23, 27, 28].

### 3. Main results

The goal of this section is to apply Lemma 2.2 to discuss the existence and multiplicity of positive solutions for the BVP (1.1). For simplicity of notation, we denote  $\xi_0 = 0, \xi_{m-1} = 1, \alpha_0 = \alpha_{m-1} = \beta_0 = \beta_{m-1} = 0$  and the function  $l(s), s \in [0, 1]$  as follow

$$l(s) = 1 - s + \frac{\sum_{i=0}^{m-1} \beta_i \xi_i - 1}{\sum_{i=0}^{m-1} \alpha_i \xi_i} \sum_{i=k}^{m-1} \alpha_i (\xi_i - s) - \sum_{i=k}^{m-1} \beta_i (\xi_i - s),$$

$$\xi_{k-1} \leq s \leq \xi_k, k = 1, 2, \dots, m - 1.$$

It is easy to check that  $l(s) \geq 0, s \in [0, 1]$ . Denote the function  $G(t, s)$  as follow:

$$G(t, s) = \begin{cases} \frac{(1-s)^2}{2} + \frac{3t^2 + 5}{6 \int_0^1 l(s) ds} l(s), & 0 \leq t \leq s \leq 1, \\ \frac{(1-s)^2}{2} + s - t + \frac{3t^2 + 5}{6 \int_0^1 l(s) ds} l(s), & 0 \leq s \leq t \leq 1. \end{cases}$$

Note that  $G(t, s) \geq 0$ ,  $t, s \in [0, 1]$ . We also set

$$\kappa = \min \left\{ \min_{s \in [0, 1]} \frac{\int_0^1 l(s) ds}{l(s)}, \frac{1}{\max_{t, s \in [0, 1]} G(t, s)} \right\}.$$

Note that  $\kappa < 1$ .

Consider the Banach spaces

$$X = Y = C[0, 1]$$

with

$$\|x\| = \max_{t \in [0, 1]} |x(t)|.$$

Let  $L : \text{dom}L \rightarrow Y$  and  $N : X \rightarrow Y$  with

$$\text{dom}L = \left\{ x \in X : x'' \in C[0, 1], x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i) \right\}$$

be given by  $(Lx)(t) = -x''(t)$  and  $(Nx)(t) = f(t, x(t))$  for  $t \in [0, 1]$ .

It is easy to check that

$$\begin{aligned} \text{Ker}L &= \{x \in \text{dom}L : x(t) \equiv c \text{ on } [0, 1], c \in \mathbb{R}\}, \\ \text{Im}L &= \left\{ y \in Y : \int_0^1 l(s)y(s)ds = 0 \right\}, \\ \dim \text{Ker}L &= \text{codim Im}L = 1, \end{aligned}$$

so that  $L$  is a Fredholm operator of index zero.

Next, define the projections  $P : X \rightarrow X$  by

$$Px = \int_0^1 x(s)ds,$$

and  $Q : Y \rightarrow Y$  by

$$Qy = \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)y(s)ds.$$

Furthermore, we define the isomorphism  $J : \text{Im}Q \rightarrow \text{Im}P$  as  $Jy = \beta y$ , where  $\beta = 1$ . We are easy to verify that the inverse operator  $L_1^{-1} : \text{Im}L \rightarrow \text{dom}L \cap \text{Ker}P$  of  $L|_{\text{dom}L \cap \text{Ker}P} : \text{dom}L \cap \text{Ker}P \rightarrow \text{Im}L$  is  $(L_1^{-1}y)(t) = \int_0^1 k(t, s)y(s)ds$ , where

$$k(t, s) = \begin{cases} \frac{(1-s)^2}{2}, & 0 \leq t \leq s \leq 1, \\ \frac{(1-s)^2}{2} + s - t, & 0 \leq s \leq t \leq 1. \end{cases}$$

Now we can state and prove our main results.

**Theorem 3.1.** *Assume that there exist two positive numbers  $a, b$  such that*

*(H<sub>1</sub>)  $f(t, x) \geq -\kappa x$  for all  $t \in [0, 1]$ ,  $x \geq 0$ .*

*(H<sub>2</sub>) If one of the two conditions*

$$(i) \min_{t \in [0,1]} f(t, a) > 0, \max_{t \in [0,1]} f(t, b) < 0$$

and

$$(ii) \max_{t \in [0,1]} f(t, a) < 0, \min_{t \in [0,1]} f(t, b) > 0$$

is satisfied, then the BVP (1.1) has at least one positive solution  $x^* \in X$  satisfying  $\min\{a, b\} \leq \|x^*\|_X \leq \max\{a, b\}$ .

*Proof.* Consider the cone

$$K = \{x \in X : x(t) \geq 0, t \in [0, 1]\}.$$

It is easy to see  $a \neq b$ . Without loss of generality, let  $a < b$ .

First, we note that  $L$ , as so defined, is a Fredholm operator of index zero,  $L_1^{-1}$  is compact by Arzela-Ascoli theorem and thus  $L - \lambda N$  is A-proper for  $\lambda \in [0, 1]$  by (a) of Lemma 2 in [26].

For each  $x \in K$ , then by condition  $(H_1)$  that

$$\begin{aligned} & Px + JQNx \\ &= \int_0^1 x(s)ds + \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)f(s, x(s))ds \\ &\geq \int_0^1 \left(1 - \frac{\kappa}{\int_0^1 l(s)ds} l(s)\right)x(s)ds \\ &\geq 0, \end{aligned}$$

$$\begin{aligned} & Px + JQNx + L_1^{-1}(I - Q)Nx \\ &= \int_0^1 x(s)ds + \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)f(s, x(s))ds \\ &\quad + \int_0^1 k(t, s) \left[ f(s, x(s)) - \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(\tau)f(\tau, x(\tau))d\tau \right] ds \\ &= \int_0^1 x(s)ds + \int_0^1 G(t, s)f(s, x(s))ds \\ &\geq \int_0^1 (1 - \kappa G(t, s))x(s)ds \geq 0. \end{aligned}$$

This implies that condition  $(C_1)$  of Lemma 2.2 is satisfied. To apply Lemma 2.2, we should define two open bounded subsets  $\Omega_1, \Omega_2$  of  $X$  so that  $(C_2)$ - $(C_5)$  of Lemma 2.2 hold.

We prove only Case  $(H_2)$ (i). In the same way, we can prove the case  $(H_2)$ (ii). Let

$$\Omega_1 = \{x \in X : \|x\|_X < a\}, \Omega_2 = \{x \in X : \|x\|_X < b\}.$$

Clearly,  $\Omega_1$  and  $\Omega_2$  are bounded and open sets and

$$\theta \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2.$$

Next we show that  $(H_2)(i)$  implies  $(C_2)$ . That is, suppose that there exist  $x_1 \in K \cap \partial\Omega_2 \cap \text{dom}L$  and  $\lambda_1 \in (0, 1]$  such that  $Lx_1 = \lambda_1 Nx_1$  then  $x_1''(t) = -\lambda_1 f(t, x_1(t))$  for all  $t \in [0, 1]$ . Let  $t_1 \in [0, 1]$ , such that  $x_1(t_1) = \max_{t \in [0, 1]} x_1(t) = b$ . From boundary conditions, we have  $t_1 \in (0, 1)$ . In this case,  $x_1'(t_1) = 0$ ,  $x_1''(t_1) \leq 0$ . This gives

$$0 \geq x_1''(t_1) = -\lambda_1 f(t_1, x(t_1)) = -\lambda_1 f(t_1, b) > 0,$$

which contradicts  $(H_2)(i)$ . So for each  $x \in \partial\Omega_2 \cap K \cap \text{dom}L$  and  $\lambda \in (0, 1]$ , we have  $Lx \neq \lambda Nx$ . Thus  $(C_2)$  of Lemma 2.2 is satisfied.

To prove  $(C_4)$  of Lemma 2.2, we define the bilinear form  $[\cdot, \cdot] : Y \times X \rightarrow \mathbb{R}$  as

$$[y, x] = \int_0^1 l(s)y(s)x(s)ds.$$

It is clear that  $[\cdot, \cdot]$  is continuous and satisfies  $[y, x] = 0$  for every  $x \in \text{Ker}L$ ,  $y \in \text{Im}L$ . In fact, for any  $x \in \text{Ker}L$  and  $y \in \text{Im}L$ , we have  $x \equiv c$ , a constant, and there exists  $x \in X$  such that  $y(s) = -x''(s)$  for each  $s \in [0, 1]$ . By  $x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$ ,  $x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)$ , we get

$$[y, x] = \int_0^1 l(s)y(s)x(s)ds = -c \int_0^1 l(s)x''(s)ds = 0.$$

Let  $x \in \text{Ker}L \cap \partial\Omega_2 \cap K$ . Then  $x(t) \equiv b$ , so we have by condition  $(H_2)(i)$

$$QNx = \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)f(s, b)ds \neq 0,$$

$$\begin{aligned} [QNx, x] &= \int_0^1 l(s) \left( \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)f(s, b)ds \cdot b \right) ds \\ &= b \cdot \int_0^1 l(s)ds \cdot \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)f(s, b)ds \\ &= b \cdot \int_0^1 l(s)f(s, b)ds \\ &< 0. \end{aligned}$$

Thus  $(C_3)$  and  $(C_4)$  of Lemma 2.2 are verified.

Finally, we prove  $(C_5)$  of Lemma 2.2 is satisfied. We may suppose that  $Lx \neq Nx$ ,  $\forall x \in \partial\Omega_1 \cap K \cap \text{dom}L$ . Otherwise, the proof is completed. Let  $e_* \equiv 1 \in K_1 \setminus \{\theta\}$ . We claim that

$$(3.2) \quad Lx - Nx \neq \mu e_*, \quad \forall x \in \partial\Omega_1 \cap K \cap \text{dom}L, \quad \mu \geq 0.$$

In fact, if not, there exist  $x_2 \in \partial\Omega_1 \cap K \cap \text{dom}L$ ,  $\mu_1 > 0$ , such that

$$Lx_2 - Nx_2 = \mu_1.$$

Since  $QL = \theta$ , operating on both sides of the latter equation by  $Q$ , we obtain

$$QNx_2 + Q\mu_1 = 0,$$



that is

$$(3.3) \quad \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)(f(s, x_2(s)) + \mu_1)ds = 0.$$

For any  $x_2 \in \partial\Omega_1 \cap K \cap \text{dom}L$ , we have  $\|x_2\|_X = a$ . There exists  $t_2 \in [0, 1]$ , such that  $x_2(t_2) = a$ . By condition  $(H_2)(i)$  and  $\mu_1 > 0$ ,

$$\frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)(f(s, x_2(s)) + \mu_1)ds = \frac{1}{\int_0^1 l(s)ds} \int_0^1 l(s)(f(s, a) + \mu_1)ds > 0,$$

in contradiction to (3.3). So (3.2) holds, that is  $(C_5)$  of Lemma 2.2 is verified.

Thus all conditions of Lemma 2.2 are satisfied and there exists  $x^* \in K \cap (\bar{\Omega}_2 \setminus \Omega_1) \cap \text{dom}L$  such that  $Lx^* = Nx^*$  and the assertion follows. Thus  $x^* \in K$  and  $a \leq \|x^*\|_X \leq b$ .  $\square$

Let  $[c]$  be the integer part of  $c$ . The following result concerns the existence results of  $n$  positive solutions.

**Theorem 3.2.** *Assume that there exist  $n+1$  positive numbers  $a_1 < a_2 < \dots < a_{n+1}$  such that*

$$(H_1)' \quad f(t, x) \geq -\kappa x \text{ for all } t \in [0, 1], \quad x \geq 0.$$

$(H_2)'$  *If one of the two conditions*

$$(i) \quad \min_{t \in [0, 1]} f(t, a_{2i-1}) > 0, \quad i = 1, 2, \dots, [\frac{n+2}{2}],$$

$$\max_{t \in [0, 1]} f(t, a_{2i}) < 0, \quad i = 1, 2, \dots, [\frac{n+1}{2}]$$

and

$$(ii) \quad \max_{t \in [0, 1]} f(t, a_{2i-1}) < 0, \quad i = 1, 2, \dots, [\frac{n+2}{2}],$$

$$\min_{t \in [0, 1]} f(t, a_{2i}) > 0, \quad i = 1, 2, \dots, [\frac{n+1}{2}]$$

is satisfied, then the BVP (1.1) has at least one positive solutions  $x_i^* \in K$ ,  $i = 1, 2, \dots, n$  satisfying  $a_i < \|x_i^*\|_X < a_{i+1}$ .

*Proof.* Modeling the proof of Theorem 3.1, we can prove that if there exist two positive numbers  $a, b$  such that  $\min_{t \in [0, 1]} f(t, a) > 0$  and  $\max_{t \in [0, 1]} f(t, b) < 0$ , then BVP (1.1) has at least one positive solution  $x^* \in K$  satisfying  $\min\{a, b\} < \|x^*\|_X < \max\{a, b\}$ .

By the claim, for every pair of positive numbers  $\{a_i, a_{i+1}\}, i = 1, 2, \dots, n$ , (1.1) has at least one positive solutions  $x_i^* \in K$  satisfying  $a_i < \|x_i^*\|_X < a_{i+1}$ .  $\square$

We have the following existence result for two positive solutions.

**Corollary 3.1.** *Assume that there exist three positive numbers  $a_1 < a_2 < a_3$  such that*

$$(H_1)'' \quad f(t, x) \geq -\kappa x \text{ for all } t \in [0, 1], \quad x \geq 0.$$

$(H_2)''$  *If one of the two conditions*

$$(i) \min_{t \in [0,1]} f(t, a_1) > 0, \max_{t \in [0,1]} f(t, a_2) < 0, \min_{t \in [0,1]} f(t, a_3) > 0$$

and

$$(ii) \max_{t \in [0,1]} f(t, a_1) < 0, \min_{t \in [0,1]} f(t, a_2) > 0, \max_{t \in [0,1]} f(t, x) < 0$$

is satisfied, then the BVP (1.1) has at least two positive solutions  $x_1^*, x_2^* \in K$  satisfying  $a_1 \leq \|x_1^*\|_X < a_2 < \|x_2^*\|_X \leq a_3$ .

We also have the following existence result for three positive solutions.

**Corollary 3.2.** Assume that there exist four positive numbers  $a_1 < a_2 < a_3 < a_4$  such that

$$(H_1)''' f(t, x) \geq -\kappa x \text{ for all } t \in [0, 1], x \geq 0.$$

(H<sub>2</sub>)''' If one of the two conditions

$$(i) \min_{t \in [0,1]} f(t, a_1) > 0, \max_{t \in [0,1]} f(t, a_2) < 0, \min_{t \in [0,1]} f(t, a_3) > 0, \max_{t \in [0,1]} f(t, a_4) < 0$$

and

$$(ii) \max_{t \in [0,1]} f(t, a_1) < 0, \min_{t \in [0,1]} f(t, a_2) > 0, \max_{t \in [0,1]} f(t, a_3) < 0, \min_{t \in [0,1]} f(t, a_4) > 0$$

is satisfied, then the BVP (1.1) has at least three positive solutions  $x_1^*, x_2^*, x_3^* \in K$  satisfying  $a_1 \leq \|x_1^*\|_X < a_2 < \|x_2^*\|_X < a_3 < \|x_3^*\|_X \leq a_4$ .

*Remark 3.1.* In this paper, we give some multiplicity results for positive solutions. Up to now, there were fewer papers that considered the existence of multiple positive solutions for multi-point boundary value problem at resonance. Therefore, the results of this paper is new.

#### 4. Example

In this section, we give an example to illustrate the main results of the paper. Consider the following second-order three-point boundary value problem (BVP)

$$(4.1) \quad \begin{cases} -x''(t) = -\frac{433(5t+2)}{9200} \sin x, & 0 < t < 1, \\ x(0) = \frac{1}{2}x(\frac{1}{4}) + \frac{1}{2}x(\frac{1}{2}), & x(1) = \frac{1}{3}x(\frac{1}{4}) + \frac{2}{3}x(\frac{1}{2}), \end{cases}$$

Corresponding to problem (1.1), we have  $\alpha_1 = \alpha_2 = \frac{1}{2}, \beta_1 = \frac{1}{3}, \beta_2 = \frac{2}{3}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}$ . By calculating directly, we have  $\kappa = \frac{433}{920}$ . It is easy to check that  $-\frac{433(5t+2)}{9200} \sin x \geq -\kappa x, x \geq 0, t \in [0, 1]$ .

(1) We take  $a = \frac{\pi}{2}, b = \frac{3\pi}{2}$ , then all the conditions of Theorem 3.1 are satisfied. Thus BVP (4.1) has at least one positive solution  $x^*$  satisfying  $\frac{\pi}{2} \leq \|x^*\|_X \leq \frac{3\pi}{2}$ .

(2) We take  $a_1 = \frac{\pi}{2}, a_2 = \frac{3\pi}{2}, a_3 = \frac{5\pi}{2}$ , then all the conditions of Corollary 3.1 are satisfied. Thus BVP (4.1) has at least two positive solutions  $x_1^*, x_2^*$  satisfying  $\frac{\pi}{2} \leq \|x_1^*\|_X < \frac{3\pi}{2} < \|x_2^*\|_X \leq \frac{5\pi}{2}$ .

(3) We take  $a_1 = \frac{\pi}{2}, a_2 = \frac{3\pi}{2}, a_3 = \frac{5\pi}{2}, a_4 = \frac{7\pi}{2}$ , then all the conditions of Corollary 3.2 are satisfied. Thus BVP (4.1) has at least three positive solutions  $x_1^*, x_2^*, x_3^*$  satisfying  $\frac{\pi}{2} \leq \|x_1^*\|_X < \frac{3\pi}{2} < \|x_2^*\|_X < \frac{5\pi}{2} < \|x_3^*\|_X \leq \frac{7\pi}{2}$ .

(4) We take  $a_{2i-1} = \frac{(4i-3)\pi}{2}, i = 1, 2, \dots, [\frac{n+2}{2}]; a_{2i} = \frac{(4i-1)\pi}{2}, i = 1, 2, \dots, [\frac{n+1}{2}]$ , then all the conditions of Theorem 3.2 are satisfied. Thus BVP (4.1) has

at least  $n$  positive solutions  $x_i^*$ ,  $i = 1, 2, \dots, n$  satisfying  $a_i < \|x_i^*\|_X < a_{i+1}$ ,  $i = 1, 2, \dots, n$ .

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FENG WANG  
 SCHOOL OF MATHEMATICS AND PHYSICS  
 CHANGZHOU UNIVERSITY  
 CHANGZHOU, JIANGSU 213164, P. R. CHINA  
 AND  
 DEPARTMENT OF MATHEMATICS  
 COLLEGE OF SCIENCE  
 HOHAI UNIVERSITY  
 NANJING, JIANGSU 210098, P. R. CHINA  
 E-mail address: fengwang188@163.com

FANG ZHANG  
SCHOOL OF MATHEMATICS AND PHYSICS  
CHANGZHOU UNIVERSITY  
CHANGZHOU, JIANGSU 213164, P. R. CHINA  
*E-mail address:* [fangzhang188@163.com](mailto:fangzhang188@163.com)