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WEAK AND STRONG CONVERGENCE FOR QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we first show that the iteration $\{x_n\}$ defined by $x_{n+1} = P((1-\alpha_n)x_n + \alpha_n TP[\beta_n Tx_n + (1-\beta_n)x_n])$ converges strongly to some fixed point of T when E is a real uniformly convex Banach space and T is a quasi-nonexpansive non-self mapping satisfying Condition **A**, which generalizes the result due to Shahzad [11]. Next, we show the strong convergence of the Mann iteration process with errors when E is a real uniformly convex Banach space and T is a quasi-nonexpansive self-mapping satisfying Condition **A**, which generalizes the result due to Senter-Dotson [10]. Finally, we show that the iteration $\{x_n\}$ defined by $x_{n+1} = \alpha_n Sx_n + \beta_n T[\alpha'_n Sx_n + \beta'_n Tx_n + \gamma'_n v_n] + \gamma_n u_n$ converges strongly to a common fixed point of T and S when E is a real uniformly convex Banach space and T, S are two quasi-nonexpansive self-mappings satisfying Condition **D**, which generalizes the result due to Ghosh-Debnath [3].

1. Introduction

Let E be a real uniformly convex Banach space and let C be a nonempty closed convex subset of E. Then a mapping T from C into E is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping T from C into E is also called *quasi-nonexpansive* if the set $F(T)$ of fixed points of T is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. For a mapping T of C into itself, we consider the following iteration scheme: $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[\beta_n Tx_n + (1-\beta_n)x_n] \tag{1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. Such an iteration scheme was introduced by Ishikawa [5]; see also Mann [7]. Let C be a nonexpansive retract of E. For a mapping T from C into E, we also consider the following iteration scheme (Shahzad [11]): $x_1 \in C$,

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n TP[\beta_n Tx_n + (1-\beta_n)x_n]) \tag{2}$$

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for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ and P is a nonexpansive retraction of E onto C. If T is a self-mapping, then (2) reduces to an iteration scheme (1). For two mappings S, T of C into itself, we also consider a more general iterative scheme of the type (cf., Ghosh-Debnath [3], Xu [13]) emphasizing the randomness of errors as follows:

$$
\begin{aligned}
&x_1 \in C,\\
&x_{n+1} = \alpha_n S x_n + \beta_n T y_n + \gamma_n u_n, \\
&y_n = \alpha'_n S x_n + \beta'_n T x_n + \gamma'_n v_n,
\end{aligned} \tag{3}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in [0,1] and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \geq 1$,
(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$.

If $S = I$, the identity mapping and $\gamma_n = \gamma'_n = 0$ for all $n \geq 1$, then (3) reduces to an iteration scheme (1), while setting $S = I$, $\beta'_n = 0$ and $\gamma'_n = 0$ for all $n \geq 1$ reduces to the Mann iteration process with errors which is a generalized case of the Mann iteration process. Recently, Shahzad [11] proved that if E is a real uniformly convex Banach space, and C is a nonempty closed convex subset of E which is also a nonexpansive retract of E, and $T : C \to E$ is a nonexpansive mapping with $F(T) \neq \emptyset$, and T satisfies Condition **A**, then for any x_1 in C, the sequence $\{x_n\}$ defined by (2) converges strongly to some fixed point of T under the assumption that $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$. On the other hand, Senter-Dotson [10] proved that if E is a real uniformly convex Banach space, and C is a nonempty closed convex subset of E, and $T : C \to C$ is a quasi-nonexpansive mapping satisfying Condition **A**, then for any $x_1 \in C$, the sequence $\{x_n\}$ defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$ converges strongly to some fixed point of T under the assumption that $\{\alpha_n\}$ in $[0,1]$ is chosen so that $\alpha_n \in [a,b]$ for all $n \geq 1$ and some $a, b \in (0,1)$. Ghosh-Debnath [3] proved that if E is a real uniformly convex Banach space and C is a nonempty closed convex subset of E and $T, S : C \to C$ are two quasi-nonexpansive mappings, and T, S satisfy Condition **C** with $F(T) \bigcap F(S) \neq \emptyset$, then for any x_1 in C, the sequence $\{x_n\}$ defined by $x_{n+1} = (1-\alpha_n)Sx_n + \alpha_n T[(1-\beta_n)Sx_n + \beta_n T x_n]$ converges strongly to a common fixed point of T and S under the assumption that $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $0 < a \leq \alpha_n \leq b < 1$, $0 \leq \beta_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$, which generalized the result due to M. Maiti and M. K. Ghosh [6].

In this paper, we first prove that the iteration $\{x_n\}$ defined by (2) converges strongly to some fixed point of T when E is a real uniformly convex Banach space and T is a quasi-nonexpansive non-self mapping satisfying Condition **A**, which generalizes the result due to Shahzad [11]. Next, we prove the strong convergence of the Mann iteration process with errors when E is a real uniformly convex Banach space and T is a quasi-nonexpansive self-mapping satisfying Condition **A**, which generalizes the result due to Senter-Dotson [10].

Finally, we prove that the iteration $\{x_n\}$ defined by (3) converges strongly to a common fixed point of T and S when E is a real uniformly convex Banach space and T, S are two quasi-nonexpansive self-mappings satisfying Condition **D**, which generalizes the result due to Ghosh-Debnath [3].

2. Preliminaries

Throughout this paper, we denote by E a real Banach space. Let C be a nonempty closed convex subset of E and let T be a mapping from C into E. Then we denote by $F(T)$ the set of all fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. A subset C of E is said to be a *retract* of E if there exists a continuous mapping $P : E \to C$ such that $Px = x$ for all $x \in C$. A mapping $P : E \to E$ is said to be a *retraction* if $P^2 = P$. A Banach space E is said to be *uniformly convex* if the modulus of convexity $\delta_E = \delta_E(\epsilon)$, $0 < \epsilon \leq 2$, of E defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

satisfies the inequality $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x. A mapping $T : C \to E$ is said to be demiclosed with respect to $y \in E$ [1] if for any sequence $\{x_n\}$ in C, it follows from $x_n \rightharpoonup x$ and $Tx_n \to y$ that $x \in C$ and $T(x) = y$. If $I - T$ is *demiclosed* at zero, i.e., for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply $x - Tx = 0$. A Banach space E is said to satisfy *Opial's condition* [8] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. All Hilbert spaces and $l^p (1 < p < \infty)$ satisfy Opial's condition, while L^p with $1 < p \neq 2 < \infty$ do not.

Condition 1 ([10])**.** A mapping $T : C \to E$ with $F(T) \neq \emptyset$ is said to satisfy Condition **A** if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$, where $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$.

Condition 2 ([6])**.** A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy Condition **B** if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Ty\| \geq f(d(x, F(T)))$$

for all $x \in C$ with $y = (1-t)x + tTx$, where $0 \leq t \leq \beta < 1$ and $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$.

Condition 3 ([3])**.** Two mappings $T, S : C \to C$ are said to satisfy Condition **C** if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\|Sx - Ty\| \geq f(d(x, \mathbf{F}))$$

for all $x, y \in C$ with $y = (1-t)Sx + tTx$, where $0 \leq t \leq \beta < 1$, $\mathbf{F} = F(T) \bigcap F(S) \neq \emptyset$ and $d(x, \mathbf{F}) = \inf_{z\in\mathbf{F}} \|x - z\|$.

Condition 4. Two mappings $T, S : C \to C$ are said to satisfy Condition **D** if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\|Sx - Ty\| \geq f(d(x, \mathbf{F}))$$

for all $x, y \in C$ with $y = \alpha Sx + \beta Tx + \gamma v$ for all $v \in C$, where $0 < a \leq \alpha \leq 1$, $0 \leq \beta, \gamma \leq b < 1$ with $\alpha + \beta + \gamma = 1$, $\mathbf{F} = F(T) \bigcap F(S) \neq \emptyset$ and $d(x, \mathbf{F}) = \inf_{z\in\mathbf{F}} \|x - z\|$.

If we set $\gamma = 0$, then Condition **D** becomes identical with Condition **C**, while setting $S = I$ and $\gamma = 0$ becomes identical with Condition **B**.

3. Weak and strong convergence theorems

We first begin with the following lemma.

Lemma 1 ([9])**.** *Let E be a uniformly convex Banach space, $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$ and some $b, c \in (0,1)$ and some $a \geq 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$. Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2 ([4])**.** *Let E be a uniformly convex Banach space. Let $x, y \in E$. If $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon > 0$, then $\|\lambda x + (1-\lambda)y\| \leq 1 - 2\lambda(1-\lambda)\delta(\epsilon)$ for λ with $0 \leq \lambda \leq 1$.*

Lemma 3 ([12])**.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and*

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

Our Theorem 1 carries over Theorem 3.3 of Shahzad [11] to a quasi-nonexpansive mapping.

Theorem 1. *Let E be a uniformly convex Banach space, and let C be a nonempty closed convex subset of E which is also a nonexpansive retract of E, and let $T : C \to E$ be a quasi-nonexpansive mapping. Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by (2), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$. Then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. For any $z \in F(T)$, since

$$
\begin{aligned}
\|x_{n+1} - z\| &= \|P((1-\alpha_n)x_n + \alpha_n TP[\beta_n Tx_n + (1-\beta_n)x_n]) - Pz\| \\
&\le \|(1-\alpha_n)x_n + \alpha_n TP[\beta_n Tx_n + (1-\beta_n)x_n] - z\| \\
&\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|TP[\beta_n Tx_n + (1-\beta_n)x_n] - z\| \\
&\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|P[\beta_n Tx_n + (1-\beta_n)x_n] - Pz\| \\
&\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|\beta_n Tx_n + (1-\beta_n)x_n - z\| \\
&\le (1-\alpha_n)\|x_n - z\| + \alpha_n[\beta_n\|Tx_n - z\| + (1-\beta_n)\|x_n - z\|] \\
&\le (1-\alpha_n)\|x_n - z\| + \alpha_n[\beta_n\|x_n - z\| + (1-\beta_n)\|x_n - z\|] \\
&\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|x_n - z\| \\
&= \|x_n - z\|,
\end{aligned} \tag{4}
$$

and thus the sequence $\{\|x_n - z\|\}$ is nonincreasing and bounded below. Hence we see that

$$\lim_{n\to\infty} \|x_n - z\| (\equiv c) \tag{5}$$

exists. If $c = 0$, then the conclusion is obvious. So, we assume $c > 0$. Put $y_n = P[\beta_n Tx_n + (1-\beta_n)x_n]$ for all $n \ge 1$. Then

$$
\begin{aligned}
\|y_n - z\| &= \|P[\beta_n Tx_n + (1-\beta_n)x_n] - Pz\| \\
&\le \|\beta_n Tx_n + (1-\beta_n)x_n - z\| \\
&\le \beta_n\|Tx_n - z\| + (1-\beta_n)\|x_n - z\| \\
&\le \beta_n\|x_n - z\| + (1-\beta_n)\|x_n - z\| \\
&= \|x_n - z\|.
\end{aligned} \tag{6}
$$

So, we have

$$\limsup_{n\to\infty} \|y_n - z\| \le c. \tag{7}$$

By using (6), we obtain

$$\|Ty_n - z\| \le \|y_n - z\| \le \|x_n - z\|.$$

By using Lemma 2, we have

$$
\begin{aligned}
\|x_{n+1} - z\| &= \|P((1-\alpha_n)x_n + \alpha_n Ty_n) - Pz\| \\
&\le \|(1-\alpha_n)x_n + \alpha_n Ty_n - z\| \\
&= \|\alpha_n(Ty_n - z) + (1-\alpha_n)(x_n - z)\| \\
&\le \Big(\|x_n - z\|\Big)\Big[1 - 2\alpha_n(1-\alpha_n)\delta_E\Big(\frac{\|Ty_n - x_n\|}{\|x_n - z\|}\Big)\Big].
\end{aligned}
$$

Hence we obtain

$$2\alpha_n(1-\alpha_n)\Big(\|x_n - z\|\Big)\delta_E\Big(\frac{\|Ty_n - x_n\|}{\|x_n - z\|}\Big) \le \|x_n - z\| - \|x_{n+1} - z\|.$$

Since

$$2a(1-b)\sum_{n=1}^{\infty}\Big(\|x_n - z\|\Big)\delta_E\Big(\frac{\|Ty_n - x_n\|}{\|x_n - z\|}\Big) < \infty,$$

and δ_E is strictly increasing and continuous, we obtain

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0. \tag{8}$$

Since

$$\begin{aligned}\|x_n - z\| &\le \|x_n - Ty_n\| + \|Ty_n - z\| \\ &\le \|x_n - Ty_n\| + \|y_n - z\|,\end{aligned}$$

and by (8), we obtain

$$c \le \liminf_{n\to\infty} \|y_n - z\| \tag{9}$$

By using (7) and (9), we obtain

$$\begin{aligned} c &= \lim_{n\to\infty} \|y_n - z\| \\ &= \lim_{n\to\infty} \|P[\beta_n Tx_n + (1-\beta_n)x_n] - Pz\| \\ &\le \lim_{n\to\infty} \|\beta_n Tx_n + (1-\beta_n)x_n - z\| \\ &= \lim_{n\to\infty} \|\beta_n(Tx_n - z) + (1-\beta_n)(x_n - z)\| \\ &\le \lim_{n\to\infty} \{\beta_n\|Tx_n - z\| + (1-\beta_n)\|x_n - z\|\} \\ &\le \lim_{n\to\infty} \{\beta_n\|x_n - z\| + (1-\beta_n)\|x_n - z\|\} \\ &= \lim_{n\to\infty} \|x_n - z\| = c. \end{aligned}$$

By using $\limsup_{n\to\infty} \|Tx_n - z\| \le c$ and Lemma 1, we obtain $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. □

Theorem 2. *Let E be a uniformly convex Banach space satisfying Opial's condition, and let C be a nonempty closed convex subset of E which is also a nonexpansive retract of E, and let $T : C \to E$ be a quasi-nonexpansive mapping with $I - T$ demiclosed at zero. Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by (2), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \le \alpha_n, \beta_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0,1)$. Then $\{x_n\}$ converges weakly to some fixed point of T.*

Proof. For any $z \in F(T)$, by (5) in the proof of Theorem 1, $\{x_n\}$ is bounded. Let z_1 and z_2 be two weak subsequential limits of the sequence $\{x_n\}$. Then we claim that the conditions $x_{n_p} \rightharpoonup z_1$ and $x_{n_q} \rightharpoonup z_2$ imply $z_1 = z_2 \in F(T)$. In fact, since $I - T$ demiclosed at zero and by using Theorem 1, we have $z_1, z_2 \in F(T)$. Next, we show $z_1 = z_2$. If not, by Opial's condition and (5) in the proof of Theorem 1,

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{p\to\infty}\|x_{n_p} - z_1\| \\ &< \lim_{p\to\infty}\|x_{n_p} - z_2\| \\ &= \lim_{n\to\infty}\|x_n - z_2\|\end{aligned}$$

and by using similar method, we have

$$\lim_{n\to\infty}\|x_n - z_2\| < \lim_{n\to\infty}\|x_n - z_1\|.$$

This is a contradiction. Hence we have $z_1 = z_2$. Therefore $\{x_n\}$ converges weakly to some fixed point of T. □

Our Theorem 3 carries over Theorem 3.6 of Shahzad [11] to a quasi-non-expansive mapping.

Theorem 3. *Let E be a uniformly convex Banach space, and let C be a nonempty closed convex subset of E which is also a nonexpansive retract of E, and let $T : C \to E$ be a quasi-nonexpansive mapping, and T satisfies Condition* **A**. *Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by (2), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \le \alpha_n, \beta_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. By using Condition **A**, we obtain

$$f(d(x_n, F(T))) \le \|Tx_n - x_n\|$$

for all $n \ge 1$. By using (4) in the proof of Theorem 1, we obtain

$$\inf_{z\in F(T)}\|x_{n+1} - z\| \le \inf_{z\in F(T)}\|x_n - z\|.$$

Thus $\lim_{n\to\infty} d(x_n, F(T))(\equiv k)$ exists. We first claim that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. In fact, assume that $k = \lim_{n\to\infty} d(x_n, F(T)) > 0$. Then we can choose $n_0 \in N$ such that $0 < \frac{k}{2} < d(x_n, F(T))$ for all $n \ge n_0$. By using Condition **A** and Theorem 1, we obtain

$$0 < f(\tfrac{k}{2}) \le f(d(x_n, F(T))) \le \|Tx_n - x_n\| \to 0$$

as $n \to \infty$. This is a contradiction. So, we obtain $k = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists $n_0 \in N$ such that for all $n \ge n_0$, we obtain

$$d(x_n, F(T)) < \frac{\epsilon}{2}. \tag{10}$$

Let $n, m \ge n_0$ and $p \in F(T)$. Then, from (4) in the proof of Theorem 1, we obtain

$$\begin{aligned}\|x_n - x_m\| &\le \|x_n - p\| + \|x_m - p\| \\ &\le 2[\|x_{n_0} - p\|].\end{aligned}$$

Taking the infimum over all $p \in F(T)$ on both sides and by (10), we obtain

$$\|x_n - x_m\| \le 2[d(x_{n_0}, F(T))] < \epsilon$$

for all $n, m \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} x_n = q$. Then $d(q, F(T)) = 0$. Since $F(T)$ is closed, we obtain $q \in F(T)$. Hence $\{x_n\}$ converges strongly to some fixed point of T. □

Theorem 4. *Let E be a uniformly convex Banach space, and let C be a nonempty closed convex subset of E which is also a nonexpansive retract of E, and let $T : C \to E$ be a continuous quasi-nonexpansive mapping, and let $T(C)$ be contained in a compact subset of E. Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by (2), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. Since $\{x_n\}$ is well-defined and the closure of $T(C)$ is compact, there exists a subsequence $\{Tx_{n_i}\}$ of the sequence $\{Tx_n\}$ such that $\{Tx_{n_i}\} \to z$. By Theorem 1, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to z. Thus, by using Theorem 1 and the continuity of T, we obtain $z \in F(T)$. From (5) in the proof of Theorem 1, we obtain $\lim_{n\to\infty} \|x_n - z\| = 0$. □

Theorem 5. *Let E be a uniformly convex Banach space, and let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a quasi-nonexpansive mapping. Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n Tx_n + \gamma_n u_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ with $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$, and $\{u_n\}$ is a bounded sequence in C such that (i) *$\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$,* (ii) *$\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. For a fixed $z \in F(T)$, since $\{u_n\}$ is bounded in C, let

$$M := \sup_{n\geq 1} \|u_n - z\| < \infty.$$

From

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n Tx_n + \gamma_n u_n - z\| \\ &\leq \alpha_n\|x_n - z\| + \beta_n\|Tx_n - z\| + \gamma_n\|u_n - z\| \\ &\leq \alpha_n\|x_n - z\| + \beta_n\|x_n - z\| + \gamma_n\|u_n - z\| \\ &= (1 - \gamma_n)\|x_n - z\| + \gamma_n\|u_n - z\| \\ &\leq \|x_n - z\| + \gamma_n M \end{aligned}$$

and Lemma 3, we readily see that

$$\lim_{n\to\infty} \|x_n - z\| (\equiv c) \tag{11}$$

exists. If $c = 0$, then the conclusion is obvious. So, we assume $c > 0$. Since

$$\begin{aligned} \|Tx_n - z + \gamma_n(u_n - x_n)\| &\leq \|Tx_n - z\| + \gamma_n\|u_n - x_n\| \\ &\leq \|x_n - z\| + \gamma_n M', \end{aligned}$$

where $M' = \sup_{n\geq 1} \|u_n - x_n\| < \infty$ and

$$\|x_n - z + \gamma_n(u_n - x_n)\| \leq \|x_n - z\| + \gamma_n M',$$

and by Lemma 2, we have

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&= \|\alpha_n x_n + \beta_n Tx_n + \gamma_n u_n - z\| \\
&= \|\alpha_n(x_n - z) + \beta_n(Tx_n - z) + \gamma_n(u_n - z)\| \\
&= \|\beta_n(Tx_n - z) + \alpha_n(x_n - z) + \gamma_n(u_n - x_n + x_n - z) \\
&\quad + \beta_n\gamma_n(u_n - x_n) - \beta_n\gamma_n(u_n - x_n)\| \\
&= \|\beta_n(Tx_n - z) + (1 - \beta_n)(x_n - z) + \gamma_n(u_n - x_n) \\
&\quad + \beta_n\gamma_n(u_n - x_n) - \beta_n\gamma_n(u_n - x_n)\| \\
&= \|\beta_n(Tx_n - z) + \beta_n\gamma_n(u_n - x_n) + (1 - \beta_n)(x_n - z) \\
&\quad + (1 - \beta_n)\gamma_n(u_n - x_n)\| \\
&= \|\beta_n(Tx_n - z + \gamma_n(u_n - x_n)) + (1 - \beta_n)(x_n - z + \gamma_n(u_n - x_n))\| \\
&\leq \Big(\|x_n - z\| + \gamma_n M'\Big)\Big[1 - 2\beta_n(1 - \beta_n)\delta_E\Big(\frac{\|Tx_n - x_n\|}{\|x_n - z\| + \gamma_n M'}\Big)\Big].
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&2\beta_n(1 - \beta_n)\Big(\|x_n - z\| + \gamma_n M'\Big)\delta_E\Big(\frac{\|Tx_n - x_n\|}{\|x_n - z\| + \gamma_n M'}\Big) \\
&\leq \|x_n - z\| - \|x_{n+1} - z\| + \gamma_n M'.
\end{aligned}$$

Since

$$2\sum_{n=1}^{\infty} \beta_n(1 - \beta_n)\Big(\|x_n - z\| + \gamma_n M'\Big)\delta_E\Big(\frac{\|Tx_n - x_n\|}{\|x_n - z\| + \gamma_n M'}\Big) < \infty,$$

and δ_E is strictly increasing and continuous, we obtain

$$\liminf_{n\to\infty} \|Tx_n - x_n\| = 0.$$

□

Theorem 6. *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T : C \to C$ be a quasi-nonexpansive mapping satisfying Condition* **A**. *Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n Tx_n + \gamma_n u_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ with $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$, and $\{u_n\}$ is a bounded sequence in C such that (i) *$\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$,* (ii) *$\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. By Theorem 5, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0. \tag{12}$$

By Condition **A**, we obtain

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\|$$

for all $n \geq 1$. As in the proof of Theorem 5, we obtain

$$\|x_{n+1} - z\| \leq \|x_n - z\| + \gamma_n M. \tag{13}$$

Thus

$$\inf_{z \in \mathbf{F}} \|x_{n+1} - z\| \leq \inf_{z \in \mathbf{F}} \|x_n - z\| + \gamma_n M.$$

By Lemma 3, we see that $\lim_{n\to\infty} d(x_n, F(T))(\equiv r)$ exists. We first claim that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. In fact, assume that $r = \lim_{n\to\infty} d(x_n, F(T)) > 0$. Then we can choose $n_0 \in N$ such that $0 < \frac{r}{2} < d(x_n, F(T))$ for all $n \geq n_0$. By using Condition **A** and (12), we obtain

$$0 < f(\tfrac{r}{2}) \leq f(d(x_{n_k}, F(T))) \leq \|Tx_{n_k} - x_{n_k}\| \to 0$$

as $k \to \infty$. This is a contradiction. So, we obtain $r = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, there exists $n_0 \in N$ such that for all $n \geq n_0$, we obtain

$$d(x_n, F(T)) < \frac{\epsilon}{4} \quad \text{and} \quad \sum_{i=n_0}^{\infty} \gamma_i < \frac{\epsilon}{4(M+1)}. \tag{14}$$

Let $n, m \geq n_0$ and $p \in F(T)$. Then, by using (13), we obtain

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\
&\leq \|x_{n_0} - p\| + \sum_{i=n_0}^{n-1} \gamma_i M + \|x_{n_0} - p\| + \sum_{i=n_0}^{m-1} \gamma_i M \\
&\leq 2[\|x_{n_0} - p\| + \sum_{i=n_0}^{\infty} \gamma_i (M+1)].
\end{aligned}$$

Taking the infimum over all $p \in F(T)$ on both sides and by using (14), we obtain

$$\begin{aligned}
\|x_n - x_m\| &\leq 2\left[d(x_{n_0}, F(T)) + \sum_{i=n_0}^{\infty} \gamma_i (M+1)\right] \\
&< 2\left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) = \epsilon
\end{aligned}$$

for all $n, m \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} x_n = q$. Then $d(q, F(T)) = 0$. Since $F(T)$ is closed, we obtain $q \in F(T)$. Hence $\{x_n\}$ converges strongly to some fixed point of T. □

As a direct consequence, taking $\gamma_n = 0$ for all $n \geq 1$ in Theorem 6, we obtain the following result, which improves Theorem 2 of Senter-Dotson [10] under much less restriction on the iterative parameter $\{\alpha_n\}$.

Theorem 7. *Let E be a uniformly convex Banach space, and let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a quasi-nonexpansive mapping satisfying Condition* **A**. *Suppose that for any $x_1 \in C$, the sequence $\{x_n\}$ is given by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$$

for all $n \geq 1$, where $\{\alpha_n\}$ in $[0, 1]$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to some fixed point of T.

Corollary 1 ([10])**.** *Let E be a uniformly convex Banach space, and let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a quasi-nonexpansive mapping satisfying Condition* **A**. *Suppose that for any $x_1 \in C$, the sequence $\{x_n\}$ is given by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$$

for all $n \geq 1$, where $\{\alpha_n\}$ in $[0, 1]$ is chosen so that $\alpha_n \in [a, b]$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to some fixed point of T.

Theorem 8. *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T, S : C \to C$ be two quasi-nonexpansive mappings satisfying Condition* **D** *with $\mathbf{F} = F(T) \bigcap F(S) \neq \emptyset$. Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by (3), where $\{\alpha_n\}, \{\beta_n\}$ in $[0, 1]$ with the restriction that $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T and S.*

Proof. For a fixed $z \in \mathbf{F}$, since $\{u_n\}$ and $\{v_n\}$ are bounded in C, let

$$M := \sup_{n \geq 1} \|u_n - z\| \vee \sup_{n \geq 1} \|v_n - z\| < \infty.$$

From

$$\begin{aligned} \|Ty_n - z\| &\leq \|y_n - z\| \\ &= \|\alpha'_n S x_n + \beta'_n T x_n + \gamma'_n v_n - z\| \\ &\leq \alpha'_n \|Sx_n - z\| + \beta'_n \|Tx_n - z\| + \gamma'_n \|v_n - z\| \\ &\leq \alpha'_n \|x_n - z\| + \beta'_n \|x_n - z\| + \gamma'_n \|v_n - z\| \\ &= (1 - \gamma'_n)\|x_n - z\| + \gamma'_n \|v_n - z\| \\ &\leq \|x_n - z\| + \gamma'_n M, \end{aligned} \tag{15}$$

we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n S x_n + \beta_n T y_n + \gamma_n u_n - z\| \\ &\leq \alpha_n \|Sx_n - z\| + \beta_n \|Ty_n - z\| + \gamma_n \|u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \{\|x_n - z\| + \gamma'_n M\} + \gamma_n \|u_n - z\| \\ &\leq (1 - \gamma_n)\|x_n - z\| + \gamma'_n M + \gamma_n M \\ &\leq \|x_n - z\| + (\gamma'_n + \gamma_n) M. \end{aligned} \tag{16}$$

By using Lemma 3, we readily see that

$$\lim_{n\to\infty} \|x_n - z\| (\equiv d)$$

exists. Without loss of generality, we assume $d > 0$. By using (15), we obtain

$$\begin{aligned}\|Ty_n - z + \gamma_n(u_n - Sx_n)\| &\le \|Ty_n - z\| + \gamma_n\|u_n - Sx_n\| \\ &\le \|x_n - z\| + \gamma_n' M + \gamma_n M'',\end{aligned}$$

where $M'' = \sup_{n\ge 1} \|u_n - Sx_n\| < \infty$ and

$$\begin{aligned}\|Sx_n - z + \gamma_n(u_n - Sx_n)\| &\le \|Sx_n - z\| + \gamma_n\|u_n - Sx_n\| \\ &\le \|x_n - z\| + \gamma_n M'' \\ &\le \|x_n - z\| + \gamma_n' M + \gamma_n M''.\end{aligned}$$

Thus by Lemma 2, we have

$$\begin{aligned}&\|x_{n+1} - z\| \\ &= \|\alpha_n Sx_n + \beta_n Ty_n + \gamma_n u_n - z\| \\ &= \|\alpha_n(Sx_n - z) + \beta_n(Ty_n - z) + \gamma_n(u_n - z)\| \\ &= \|\beta_n(Ty_n - z) + \alpha_n(Sx_n - z) + \gamma_n(u_n - Sx_n + Sx_n - z) \\ &\quad + \beta_n\gamma_n(u_n - Sx_n) - \beta_n\gamma_n(u_n - Sx_n)\| \\ &= \|\beta_n(Ty_n - z) + (1-\beta_n)(Sx_n - z) + \gamma_n(u_n - Sx_n) \\ &\quad + \beta_n\gamma_n(u_n - Sx_n) - \beta_n\gamma_n(u_n - Sx_n)\| \\ &= \|\beta_n(Ty_n - z) + \beta_n\gamma_n(u_n - Sx_n) + (1-\beta_n)(Sx_n - z) \\ &\quad + (1-\beta_n)\gamma_n(u_n - Sx_n)\| \\ &= \|\beta_n(Ty_n - z + \gamma_n(u_n - Sx_n)) + (1-\beta_n)(Sx_n - z + \gamma_n(u_n - Sx_n))\| \\ &\le \Big(\|x_n - z\| + \gamma_n' M + \gamma_n M''\Big)\Big[1 - 2\beta_n(1-\beta_n)\delta_E\Big(\frac{\|Ty_n - Sx_n\|}{\|x_n - z\| + \gamma_n' M + \gamma_n M''}\Big)\Big].\end{aligned}$$

Hence we obtain

$$\begin{aligned}&2\beta_n(1-\beta_n)\Big(\|x_n - z\| + \gamma_n' M + \gamma_n M''\Big)\delta_E\Big(\frac{\|Ty_n - Sx_n\|}{\|x_n - z\| + \gamma_n' M + \gamma_n M''}\Big) \\ &\le \|x_n - z\| - \|x_{n+1} - z\| + \gamma_n' M + \gamma_n M''.\end{aligned}$$

Since

$$2\sum_{n=1}^{\infty} \beta_n(1-\beta_n)\Big(\|x_n - z\| + \gamma_n' M + \gamma_n M''\Big)\delta_E\Big(\frac{\|Ty_n - Sx_n\|}{\|x_n - z\| + \gamma_n' M + \gamma_n M''}\Big) < \infty,$$

and δ_E is strictly increasing and continuous, we obtain

$$\liminf_{n\to\infty} \|Ty_n - Sx_n\| = 0. \tag{17}$$

By using (16) and Lemma 3, we see that

$$\lim_{n\to\infty} d(x_n, \mathbf{F}) \tag{18}$$

exists. By using Condition **D**, (17) and taking lim inf on both sides, we obtain

$$\liminf_{n\to\infty} f(d(x_n, \mathbf{F})) \le \liminf_{n\to\infty} \|Sx_n - Ty_n\| \to 0 \tag{19}$$

as $n \to \infty$. From the Condition **D**, (18) and (19), we obtain $\lim_{n\to\infty} d(x_n, \mathbf{F}) = 0$. By using similar method in the proof of Theorem 6, $\{x_n\}$ converges strongly to a common fixed point of T and S. □

As a direct consequence, taking $\gamma_n = 0$ for all $n \ge 1$ in Theorem 8, we obtain the following result, which improves Theorem 1 of Ghosh-Debnath [3] under much less restriction on the iterative parameter $\{\alpha_n\}$.

Theorem 9. *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T, S : C \to C$ be two quasi-nonexpansive mappings satisfying Condition* **C** *with $\mathbf{F} = F(T) \bigcap F(S) \neq \emptyset$. Suppose that for any x_1 in C, the sequences $\{x_n\}$ and $\{y_n\}$ are defined by*

$$x_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n, \quad y_n = (1 - \beta_n)Sx_n + \beta_n Tx_n,$$

where $\{\alpha_n\}, \{\beta_n\}$ in $[0, 1]$ with the restriction that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T and S.

Corollary 2 ([3]). *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T, S : C \to C$ be two quasi-nonexpansive mappings satisfying Condition* **C** *with $\mathbf{F} = F(T) \bigcap F(S) \neq \emptyset$. Suppose that for any x_1 in C, the sequences $\{x_n\}$ and $\{y_n\}$ are defined by*

$$x_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n, \quad y_n = (1 - \beta_n)Sx_n + \beta_n Tx_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \le \alpha_n \le b < 1$, $0 \le \beta_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to a common fixed point of T and S.

Corollary 3 ([6]). *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T : C \to C$ be a quasi-nonexpansive mappings satisfying Condition* **B**. *Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by (1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \le \alpha_n \le b < 1$, $0 \le \beta_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 1. If $\{\alpha_n\}$ is bounded away from both 0 and 1, i.e., $a \le \alpha_n \le b$ for all $n \ge 1$ and some $a, b \in (0, 1)$, then $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ holds. However, the converse is not true. For example, consider $\alpha_n = \frac{1}{n}$.

Remark 2. The concept of quasi-nonexpansive mapping is more general than that of nonexpansive mapping.

We give two examples of quasi-nonexpansive mappings which are not nonexpansive mappings.

Example 1. Let $E = [-\pi, \pi]$ and let T be defined by

$$Tx = x \cos x$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = |x||\cos x| \leq |x| = \|x - z\|.$$

But it is not a nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$\|Tx - Ty\| = \left\| \frac{\pi}{2} \cos \frac{\pi}{2} - \pi \cos \pi \right\| = \pi,$$

whereas,

$$\|x - y\| = \left\| \frac{\pi}{2} - \pi \right\| = \frac{\pi}{2}.$$

Example 2 (cf. [2])**.** Let $E = \mathbb{R}$ and let T be defined by

$$\begin{aligned} Tx &= \tfrac{x}{2} \cos \tfrac{1}{x}, \quad & x \neq 0, \\ &= 0, & x = 0. \end{aligned}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2} \cos \frac{1}{x}$. Thus $2 = \cos \frac{1}{x}$. This is not hold. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = \left| \frac{x}{2} \right| \left| \cos \frac{1}{x} \right| \leq \frac{|x|}{2} < |x| = \|x - z\|.$$

But it is not a nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$\|Tx - Ty\| = \left\| \frac{1}{3\pi} \cos \frac{3\pi}{2} - \frac{1}{2\pi} \cos \pi \right\| = \frac{1}{2\pi},$$

whereas,

$$\|x - y\| = \left\| \frac{2}{3\pi} - \frac{1}{\pi} \right\| = \frac{1}{3\pi}.$$

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