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L^p BOUNDS FOR THE PARABOLIC LITTLEWOOD-PALEY OPERATOR ASSOCIATED TO SURFACES OF REVOLUTION

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Abstract. In this paper the authors study the L^p boundedness for parabolic Littlewood-Paley operator

$$\mu_{\Phi,\Omega}(f)(x) = \left(\int_0^\infty |F_{\Phi,t}(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Phi,t}(x) = \int_{\rho(y)\le t} \frac{\Omega(y)}{\rho(y)^{\alpha-1}} f(x-\Phi(y))dy$$

and Ω satisfies a condition introduced by Grafakos and Stefanov in [6]. The result in the paper extends some known results.

1. Introduction

Let $\alpha_1, \ldots, \alpha_n$ be fixed real numbers, $\alpha_i \ge 1$. For fixed $x \in \mathbb{R}^n$, the function $F(x,\rho) = \sum_{i=1}^n \frac{{x_i}^2}{\rho^{2\alpha_i}}$ is a decreasing function in $\rho > 0$. We denote the unique solution of the equation $F(x,\rho) = 1$ by $\rho(x)$. In [5], Fabes and Rivière showed that $\rho(x)$ is a metric on $\mathbb{R}^n$, and $(\mathbb{R}^n, \rho)$ is called the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^n$.

For $\lambda > 0$, let $A_\lambda = \begin{pmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{pmatrix}$. Suppose that $\Omega(x)$ is a real valued and measurable function defined on $\mathbb{R}^n$. We say $\Omega(x)$ is homogeneous of degree zero with respect to A_λ, if for any $\lambda > 0$ and $x \in \mathbb{R}^n$

$$\Omega(A_\lambda x) = \Omega(x). \tag{1.1}$$

Moreover, $\Omega(x)$ satisfies the following condition

$$\int_{S^{n-1}} \Omega(x')J(x')d\sigma(x') = 0, \tag{1.2}$$

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where $J(x')$ is a function defined on the unit sphere S^{n-1} in $\mathbb{R}^n$, which will be defined in Section 2.

In 1966, Fabes and Rivière [5] proved that if $\Omega \in C^1(S^{n-1})$ satisfying (1.1) and (1.2), then the parabolic singular integral operator T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{\rho(y)^\alpha} f(x-y)\, dy \quad \text{ and } \quad \alpha = \sum_{i=1}^n \alpha_i.$$

In 1976, Nagel, Rivière and Wainger [7] improved the above result. They showed T_Ω is still bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if replacing $\Omega \in C^1(S^{n-1})$ by a weaker condition $\Omega \in L \log^+ L(S^{n-1})$.

Inspired by the works in [5] and [7], recently, Ding, Xue and Yabuta [10] defined the parabolic Littlewood-Paley operator by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{\rho(y)\le t} \frac{\Omega(y)}{\rho(y)^{\alpha-1}} f(x-y) dy.$$

The authors of [10] gave the $L^p\,(1 < p < \infty)$ boundedness of the parabolic Littlewood-Paley operator:

Theorem A. *If* $\Omega \in L^q(S^{n-1})(q > 1)$ *satisfies* (1.1) *and* (1.2)*, then*

$$\|\mu_\Omega(f)\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

Note that on S^{n-1},

$$L^q(S^{n-1})\,(q > 1) \subsetneq L \log^+ L(S^{n-1}) \subsetneq H^1(S^{n-1}).$$

Recently, Chen and Ding [2] improve Theorem A, the result is:

Theorem B. *If* $\Omega \in H^1(S^{n-1})$ *satisfies* (1.1) *and* (1.2)*, then*

$$\|\mu_\Omega(f)\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

For a suitable mapping $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^d$, we define the parabolic Littlewood-Paley operator $\mu_{\Phi,\Omega}$ along a mapping Φ on $\mathbb{R}^d$ by

$$\mu_{\Phi,\Omega}(f)(x) = \left(\int_0^\infty |F_{\Phi,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Phi,t}(x) = \int_{\rho(y)\le t} \frac{\Omega(y)}{\rho(y)^{\alpha-1}} f(x-\Phi(y)) dy.$$

If $d = n$ and $\Phi(y) = (y_1, y_2, \dots, y_n)$, then $\mu_{\Phi,\Omega}$ is the parabolic Littlewood-Paley operator μ_Ω.

On the other hand, we note that if $\alpha_1 = \cdots = \alpha_n = 1$, then $\rho(x) = |x|$, $\alpha = n$ and $(\mathbb{R}^n, \rho) = (\mathbb{R}^n, |\cdot|)$. In this case, $\mu_{\Phi,\Omega}$ is just the classical Marcinkiewicz integral along surfaces of revolution, which was studied by [4]. Moveover, if $d = n$ and $\Phi(y) = (y_1, y_2, \ldots, y_n)$, then $\mu_{\Phi,\Omega}$ is just the classical Marcinkiewicz integral which was studied by many authors (See [8], [1] and [3]).

The purpose of this paper is to investigate the L^p boundedness of the parabolic Littlewood-Paley operator $\mu_{\Phi,\Omega}$ along surfaces of revolution when $\Omega \in F_\beta(S^{n-1})$. For a $\beta > 0$, $F_\beta(S^{n-1})$ denotes the set of all Ω which are integrable over S^{n-1} and satisfies

$$\sup_{\xi\in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)|(\ln \frac{1}{|\theta\cdot\xi|})^{1+\beta} d\theta < \infty. \tag{1.3}$$

Condition (1.3) was introduced by Grafakos and Stefanov in [6]. The examples in [6] show that there is the following relationship between $F_\beta(S^{n-1})$ and $H^1(S^{n-1})$:

$$\bigcap_{\beta>0} F_\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\beta>0} F_\beta(S^{n-1}).$$

We shall state our main results as follows:

Theorem 1. *Let $d = n + 1$, $m \in \mathbb{N}$, and $\Phi(y) = (y, \phi(\rho(y)))$, where ϕ is a polynomial of degree m and $\frac{d^{\alpha_i}\phi(t)}{dt}|_{t=0} = 0$, where $\alpha_i' s$ are the all positive integers which is less than m in $\{\alpha_1, \ldots, \alpha_n\}$. In addition, let $\Omega \in F_\beta(S^{n-1})$ for some $\beta > 0$ and satisfies* (1.1) *and* (1.2)*, then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $p \in (\frac{2+2\beta}{1+2\beta}, 2 + 2\beta)$.*

Corollary 1. *Let $d = n + 1$ and $\Phi(y) = (y, \phi(\rho(y)))$, where ϕ is a polynomial and $\frac{d^{\alpha_i}\phi(t)}{dt}|_{t=0} = 0$, where $\alpha_i' s$ are the all positive integers which is less than m in $\{\alpha_1, \ldots, \alpha_n\}$. In addition, let $\Omega \in \bigcap_{\beta>0} F_\beta(S^{n-1})$ and satisfies* (1.1) *and* (1.2)*, then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$.*

2. Lemmas

In this section, we give some lemmas which will be used in the proof of Theorem 1. For any $x \in \mathbb{R}^n$, set

$$\begin{aligned}
&x_1 = \rho^{\alpha_1} \cos\varphi_1 \cdots \cos\varphi_{n-2}\cos\varphi_{n-1}\\
&x_2 = \rho^{\alpha_2} \cos\varphi_1 \cdots \cos\varphi_{n-2}\sin\varphi_{n-1}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&x_{n-1} = \rho^{\alpha_{n-1}} \cos\varphi_1 \sin\varphi_2\\
&x_n = \rho^{\alpha_n} \sin\varphi_1.
\end{aligned}$$

Then $dx = \rho^{\alpha-1}J(\varphi_1, \ldots, \varphi_{n-1})d\rho d\sigma$, where $\alpha = \sum_{i=1}^n \alpha_i$, $d\sigma$ is the element of area of S^{n-1} and $\rho^{\alpha-1}J(\varphi_1, \ldots, \varphi_{n-1})$ is the Jacobian of the above transform. In [5], it was shown there exists a constant $L \geq 1$ such that $1 \leq J(\varphi_1, \ldots, \varphi_{n-1}) \leq L$ and $J(\varphi_1, \ldots, \varphi_{n-1}) \in C^\infty((0, 2\pi)^{n-2} \times (0, \pi))$. So, it is easy to see that J is also a C^∞ function in the variable $y' \in S^{n-1}$. For simplicity, we denote still it by $J(y')$.

We shall begin by establishing some notations. For a family of measures $\tau = \{\tau_{k,t} : k \in \mathbb{N},\, t \in \mathbb{R}\}$ on $\mathbb{R}^d$, we define the operators Δ_τ and τ_k^* by

$$\Delta_\tau(f)(x) = \sum_{k=1}^{\infty} \bigg(\int_{\mathbb{R}} \big|(\tau_{k,t} * f)(x)\big|^2 dt \bigg)^{1/2}, \tau_k^*(f)(x) = \sup_{t \in \mathbb{R}} \Big(\big|\tau_{k,t}\big| * |f| \Big)(x).$$

In order to prove our theorems, we need the following lemmas:

Lemma 2.1 ([9]). *Let $k \in \mathbb{N}$. Suppose that $\gamma(t) : \mathbb{R}^+ \mapsto \mathbb{R}^k$ satisfies $\gamma'(t) = M(\frac{\gamma(t)}{t})$ for a fixed matrix M, and assume $\gamma(t)$ doesn't lie in an affine hyperplane. Then*

$$\int_1^2 e^{i\gamma(t)\cdot\eta} dt \le C|\eta|^{1/k}.$$

Lemma 2.2 ([9]). *Suppose that λ_j's and α_j's are fixed real numbers, $\phi(t)$ is a polynomial and $\Gamma(t) = (\lambda_1 t^{\alpha_1}, \ldots, \lambda_n t^{\alpha_n}, \phi(t))$ is a function from $\mathbb{R}_+$ to $\mathbb{R}^{n+1}$. For suitable f, the maximal function associated to the homogeneous curve Γ is defined by*

$$M_\Gamma(f)(x) = \sup_h \frac{1}{h} \int_0^h |f(x - \Gamma(t))| dt, h > 0. \tag{2.1}$$

Then for $1 < p \le \infty$, there is a constant $C > 0$, independent of λ_j's, the coefficient of $\phi(t)$ and f, such that

$$\|M_\Gamma(f)\|_{L^p} \le C\|f\|_{L^p}. \tag{2.2}$$

Lemma 2.3. *Let $L : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a linear transformation. Suppose that there are constants $C_0, C_p, \beta, \gamma > 0$ such that the following hold for $k \in \mathbb{N}$, $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n+1}$:*

$$\|\tau_{k,t}\| \le C_0 2^{-k}, \tag{2.3}$$

$$|\widehat{\tau_{k,t}}(\xi)| \le C_0 2^{-k} |A_{2^{\gamma(t-k)}} L\xi|, \tag{2.4}$$

$$|\widehat{\tau_{k,t}}(\xi)| \le C_0 2^{-k} (\ln |A_{2^{\gamma(t-k)}} L\xi|)^{-(1+\beta)} \quad \text{if } |A_{2^{\gamma(t-k)}} L\xi| > 2, \tag{2.5}$$

$$\|\tau_k^*(f)\|_{L^p(\mathbb{R}^{n+1})} \le C_p 2^{-k} \|f\|_{L^p(\mathbb{R}^{n+1})} \quad \text{for } 1 < p < \infty. \tag{2.6}$$

Then, for

$$p \in (\tfrac{2+2\beta}{1+2\beta}, 2+\beta)$$

there exists a constant $A_p > 0$ such that

$$\|\Delta_\tau(f)\|_{L^p(\mathbb{R}^{n+1})} \le A_p \|f\|_{L^p(\mathbb{R}^{n+1})} \tag{2.7}$$

for all $f \in L^p(\mathbb{R}^{n+1})$. The constant A_p may depend on C_0, C_p, β, γ and n, but it is independent of the linear transformation L.

Proof. In the proof of Lemma 2.3, we use some idea from [4]. We may assume that $L\xi = (\xi_1, \dots, \xi_n) = \zeta$ for $\xi = (\xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$. Choose a C^∞ function $\psi : \mathbb{R} \to [0, 1]$ such that $\text{supp}(\psi) \subset [\frac{1}{4}, 4]$ and

$$\int_0^\infty \frac{\psi(r)}{r} dr = 2. \tag{2.8}$$

Define the Schwartz functions Ψ, $\Psi_t : \mathbb{R}^n \to C$ by

$$\widehat{\Psi}(\xi_1, \dots, \xi_n) = \psi(\rho^2(\zeta))$$

and $\Psi_t(u) = t^{-\alpha}\Psi(A_{t^{-1}}u)$ for $t > 0$ and $u \in \mathbb{R}^n$. If we let δ_1 represent the Dirac delta on $\mathbb{R}$, then by (2.8), for any Schwartz function f,

$$f(x) = \int_0^\infty (\Psi_t \otimes \delta_1) * f(x) \frac{dt}{t} = (r \ln 2) \int_{\mathbb{R}} (\Psi_{2^{rs}} \otimes \delta_1) * f(x) ds. \tag{2.9}$$

Define the g-function $g(f)$ by

$$g(f)(x) = \left(\int_{\mathbb{R}} |(\Psi_{2^{\gamma s}} \otimes \delta_1) * f(x)|^2 ds \right)^{1/2}.$$

By $\int_{\mathbb{R}^n} \Psi_t(z)dz = \psi(0) = 0$ and Littlewood-Paley theory, we have

$$\|g(f)\|_{L^p(\mathbb{R}^{n+1})} \le C\|f\|_{L^p(\mathbb{R}^{n+1})} \text{ for } 1 < p < \infty. \tag{2.10}$$

For $s \in \mathbb{R}, k \in \mathbb{N}$ and Schwartz function f, let

$$H_{s,k}(f)(x) = \left(\int_{\mathbb{R}} |(\Psi_{2^{\gamma(s+t)}} \otimes \delta_1) * \tau_{k,t} * f(x)|^2 dt \right)^{1/2} \tag{2.11}$$

and

$$H_s(f) = \sum_{k=1}^\infty H_{s,k}(f).$$

It follows from (2.9) and Minkowski's inequality that:

$$\Delta_\tau(f)(x) \le (\gamma \ln 2) \int_{\mathbb{R}} H_s(f)(x) ds. \tag{2.12}$$

Hence if we can prove that, for

$$p \in (\tfrac{2+2\beta}{1+2\beta}, 2 + 2\beta)$$

there exist $\theta_p > 0$ and $\theta_p' > 1$ such that

$$\|H_s\|_{p,p} \le \begin{cases} C_p 2^{-s\theta_p} & \text{if } s > 0, \\ C_p |s|^{-\theta_p'} & \text{if } s < -N, \\ C_p & \text{if } -N \le s \le 0, \end{cases} \tag{2.13}$$

where $N > 0$ depended only α and γ, then (2.7) follows from (2.12) and (2.13). We shall first establish (2.13) for $p = 2$. When $s > 0$, by (2.4) we have

(2.14)

$$\begin{aligned}
&\int_{\mathbb{R}} |\psi(\rho^2(A_{2^{\gamma(s+t)}}\zeta))\widehat{\tau_{k,t}}(\xi)|^2 dt \\
&\le C2^{-2k} \int_{(2^{\gamma s+1}\rho(\zeta))^{-1} \le 2^{\gamma t} \le 2(2^{\gamma s}\rho(\zeta))^{-1}} |A_{2^{\gamma(t-k)}}\zeta|^2 dt \\
&\le C2^{-2k} \int_{(2^{\gamma s+1}\rho(\zeta))^{-1} \le 2^{\gamma t} \le 2(2^{\gamma s}\rho(\zeta))^{-1}} \left(2^{2\alpha_1\gamma(t-k)}\xi_1^2 + \cdots + 2^{2\alpha_n\gamma(t-k)}\xi_n^2\right) dt \\
&\le C2^{-2k}2^{-2\min\{\alpha_i\}\gamma k} \int_{(2^{\gamma s+1}\rho(\zeta))^{-1} \le 2^{\gamma t} \le 2(2^{\gamma s}\rho(\zeta))^{-1}} \\
&\quad \left(2^{-2\alpha_1\gamma s}2^{2\alpha_1\gamma t}\rho(\zeta)^{2\alpha_1}({\zeta_1}')^2 + \cdots + 2^{-2\alpha_n\gamma s}2^{2\alpha_n\gamma t}\rho(\zeta)^{2\alpha_n}({\zeta_n}')^2\right) dt \\
&\le C2^{-2k}2^{-2\min\{\alpha_i\}\gamma(k+s)} \int_{(2^{\gamma s+1}\rho(\zeta))^{-1} \le 2^{\gamma t} \le 2(2^{\gamma s}\rho(\zeta))^{-1}} \left(({\zeta_1}')^2 + \cdots + ({\zeta_n}')^2\right) dt \\
&\le C(2^{k(\gamma+1)+\gamma s})^{-2},
\end{aligned}$$

where $\xi_i = \rho(\zeta)^{\alpha_i}\zeta_i'$, $1 \le i \le n$, $\zeta' = (\zeta_1', \ldots, \zeta_n') \in S^{n-1}$.

It then follows from Plancherel's Theorem and (2.14) that

$$\|H_s\|_{2,2} \le C2^{-\gamma s}. \tag{2.15}$$

Now let us consider the case $s < 0$. For given $\beta > 0$ and $\gamma > 0$, take

$$-s > \max\left\{1 + \frac{8}{\gamma}, \frac{\gamma(1+\beta)}{\ln 2}\right\}.$$

Then for $1 \le k < -s - (4/\gamma)$, similar to the proof (2.14), by (2.5) we have

$$\begin{aligned}
&\int_{\mathbb{R}} |\psi(\rho^2(A_{2^{\gamma(s+t)}}\zeta))\widehat{\tau_{k,t}}(\xi)|^2 dt \\
&\le C2^{-2k} \int_{(2^{\gamma s+1}\rho(\zeta))^{-1} \le 2^{\gamma t} \le 2(2^{\gamma s}\rho(\zeta))^{-1}} (\ln |A_{2^{\gamma(t-k)}}\zeta|)^{-2(1+\beta)} dt \\
&\le C2^{-2k}(1 + \gamma|s+k|)^{-2(1+\beta)}.
\end{aligned} \tag{2.16}$$

On the other hand, for s chosen above and $k \ge -s - (4/\gamma)$, by (2.4) we have

$$\int_{\mathbb{R}} |\psi(\rho^2(A_{2^{\gamma(s+t)}}\zeta))\widehat{\tau_{k,t}}(\xi)|^2 dt \le C2^{-2k}2^{-2\gamma(s+k)}. \tag{2.17}$$

Apply Plancherel's Theorem again, by (2.16) and (2.17), for s chosen above we have

$$\begin{aligned}
&\|H_{s,k}(f)\|_{L^2(\mathbb{R}^{n+1})} \\
&\le \begin{cases} C2^{-k}(1+\gamma|s+k|)^{-(1+\beta)}\|f\|_{L^2(\mathbb{R}^{n+1})} & \text{if } 1 \le k < -s - (4/\gamma), \\ C2^{-k}2^{-\gamma(s+k)}\|f\|_{L^2(\mathbb{R}^{n+1})} & \text{if } k \ge -s - (4/\gamma), \end{cases}
\end{aligned} \tag{2.18}$$

Similar to the proof of (2.20) in [4], by (2.18), there exists $N > \max\{1 + \frac{8}{\gamma}, \frac{\gamma(1+\beta)}{\log 2}\}$, we get

(2.19)
$$\|H_s\|_{2,2} \le C\Big\{ \sum_{1\le k< -s-(4/\gamma)} 2^{-k}(1+\gamma|s+k|)^{-(1+\beta)} + \sum_{k\ge -s-(4/\gamma)} 2^{-k}2^{-\gamma(s+k)} \Big\}$$
$$\le C|s|^{-(1+\beta)} \quad \text{for} \quad s < -N.$$

Similar to the proof of (2.21) in [4], we can prove that, for every $1 < p < \infty$, there exists $C_p > 0$ such that for every $s \in \mathbb{R}$,

$$\|H_s\|_{p,p} \le C_p. \tag{2.20}$$

Finally, by interpolating (2.15) and (2.20), (2.19) and (2.20), respectively, we obtain (2.13) for every p in

$$p \in (\tfrac{2+2\beta}{1+2\beta}, 2+2\beta)$$

with $\theta_p > 0$ and $\theta'_p > 1$. Lemma 2.3 is proved. □

3. Proof of Theorem 1

The idea of proving Theorem 1 is taken from [4] and [10]. Let Ω satisfies (1.1), (1.2) and (1.3) for some $\beta > 0$. $\Phi(y) = (y, \phi(\rho(y)))$, where $\phi(t) = \sum_{j=0}^{m} a_j t^j$, $m \in \mathbb{N}$. Let $D_j = \{y \in \mathbb{R}^n : 2^j < \rho(y) \le 2^{j+1}\}$ and define the family of measures $\tau = \{\tau_{k,t} : k \in \mathbb{N}, t \in \mathbb{R}\}$ on $\mathbb{R}^{n+1}$ by

$$\int_{\mathbb{R}^{n+1}} f(y, y_{n+1}) d\tau_{k,t} = 2^{-t} \int_{D_{t-k}} f(y, \phi(\rho(y))) \frac{\Omega(y)}{\rho(y)^{\alpha-1}}\, dy.$$

Then by the Minkowski inequality, we get

$$\mu_{\Phi,\Omega}(f) \le \sqrt{\ln 2}\,\Delta_\tau(f)(x). \tag{3.1}$$

It is easy to see that

(3.2)
$$\|\tau_{k,t}\| = \int_{\mathbb{R}^{n+1}} |d\tau_{k,t}| = 2^{-t} \int_{D_{t-k}} \frac{|\Omega(y')|}{\rho(y)^{\alpha-1}} d\sigma(y')$$
$$= 2^{-t} \int_{S_{n-1}} \int_{2^{t-k}}^{2^{t-k+1}} \frac{|\Omega(y')|J(y')}{\rho^{\alpha-1}} \rho^{\alpha-1} d\rho d\sigma(y') < C_0 2^{-k}.$$

In light of (3.1) and Lemma 2.3, it suffices to show that (2.4), (2.5) and (2.6) also hold when we choose $\gamma = 1$.

For $(\xi, \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, $y' \in S^{n-1}$, and $\lambda \in \mathbb{Z}$. Let

$$I_\lambda(\xi, \xi_{n+1}, y') = \int_1^2 e^{i[A_{\lambda\rho}\xi\cdot y' + \xi_{n+1}\phi(\lambda\rho)]} d\rho.$$

Set $\Lambda = \big\{\alpha_i : \alpha_i$ is the positive integers which is less than m in$\{\alpha_1, \dots, \alpha_n\}\big\}$, and $\overline{\Lambda} = \{1, 2, \dots, m\}\backslash\Lambda$. Then $\frac{d^{\alpha_i}\phi(t)}{dt}|_{t=0} = 0$, where $\alpha_i \in \Lambda$, and $\overline{\Lambda}$ is not a subset of $\{\alpha_1, \ldots, \alpha_n\}$. Therefore, we get

$$A_{\lambda\rho}\xi \cdot y' + \xi_{n+1}\phi(\lambda\rho) = \rho^{\alpha_1}\lambda^{\alpha_1}\xi_1 y_1' + \cdots + \rho^{\alpha_n}\lambda^{\alpha_n}\xi_n y_n' + \xi_{n+1}\sum_{j\in\overline{\Lambda}} a_j(\lambda\rho)^j.$$

Without loss of generality, we may assume Λ consists of r distinct numbers and let $\overline{\Lambda} = \{i_1, i_2, \ldots, i_{m-r}\}$. If $\alpha_j' s$ are all distinct, we get immediately

$$\begin{aligned} &|I_\lambda(\xi, \xi_{n+1}, y')| \\ &\leq \Big(|\lambda^{\alpha_1}\xi_1 y_1'| + \cdots + |\lambda^{\alpha_n}\xi_n y_n'| + (m-r)|\lambda\xi_{n+1}|\Big)^{-1/(n+m-r)} \\ &\leq (|\lambda^{\alpha_1}\xi_1 y_1' + \cdots + \lambda^{\alpha_n}\xi_n y_n'|)^{-1/(n+m-r)} = |A_\lambda\xi \cdot y'|^{-1/(n+m-r)}. \end{aligned} \tag{3.3}$$

If $\{\alpha_j\}$ only consists of s distinct numbers, we suppose that $\alpha_1 = \alpha_2 = \cdots = \alpha_{l_1}$, $\alpha_{l_1+1} = \cdots = \alpha_{l_1+l_2}$, $\ldots$, $\alpha_{l_1+\cdots+l_{s-1}+1} = \cdots = \alpha_n$, where s is a positive integer with $1 \leq s \leq n$, $l_1, l_2, \ldots, l_s$ are positive integers such that $l_1 + l_2 + \cdots + l_s = n$ and $\alpha_1, \alpha_{l_1+l_2}, \ldots, \alpha_{l_1+\cdots+l_{s-1}}$, α_n are distinct. Obviously,

$$\gamma(t) = (t^{\alpha_1}, t^{\alpha_{l_1+l_2}}, \ldots, t^{\alpha_{l_1+\cdots+l_{s-1}}}, t^{\alpha_n}, t^{i_1}, t^{i_2}, \ldots, t^{i_{m-r}})$$

doesn't lie in an affine hyperplane in $\mathbb{R}^{s+m-r}$. Then using Lemma 2.1 again, there exists $C > 0$ such that for any vector $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$,

$$\begin{aligned} &\int_1^2 e^{2i(\eta_1+\cdots+\eta_{l_1})t^{\alpha_{l_1}} + (\eta_{l_1+1}+\cdots+\eta_{l_1+l_2})t^{\alpha_{l_1+l_2}} + \cdots + (\eta_{l_1+\cdots+l_{s-1}+1}+\cdots+\eta_n)t^{\alpha_n} + \lambda\xi_{n+1}\sum_{j\in\overline{\Lambda}} t^j}\, dt \\ &\leq C\Big(|\eta_1 + \cdots + \eta_{l_1}|^2 + |\eta_{l_1+1} + \cdots + \eta_{l_1+l_2}|^2 + \cdots \\ &\qquad + |\eta_{l_1+\cdots+l_{s-1}+1} + \cdots + \eta_n|^2 + (m-r)|\lambda\xi_{n+1}|^2\Big)^{-1/2(s+m-r)} \\ &\leq C\Big(|\eta_1 + \cdots + \eta_{l_1}| + |\eta_{l_1+1} + \cdots + \eta_{l_1+l_2}| + \cdots \\ &\qquad + |\eta_{l_1+\cdots+l_{s-1}+1} + \cdots + \eta_n|\Big)^{-1/(s+m-r)} \\ &\leq C\Big|\sum_{j=1}^n \eta_j\Big|^{-1/(s+m-r)}. \end{aligned}$$

Let $\eta_j = \lambda^{\alpha_j}\xi_j y_j'$, we have

$$\begin{aligned} |I_\lambda(\xi, \xi_{n+1}, y)| &\leq \Big(|\lambda^{\alpha_1}\xi_1 y_1'| + \cdots + |\lambda^{\alpha_n}\xi_n y_n'|\Big)^{-1/(s+m-r)} \\ &\leq (|\lambda^{\alpha_1}\xi_1 y_1' + \cdots + \lambda^{\alpha_n}\xi_n y_n'|)^{-1/(s+m-r)} \\ &= |A_\lambda\xi \cdot y'|^{-1/(s+m-r)}. \end{aligned} \tag{3.3$'$}$$

On the other hand, it is easy to see that

$$|I_\lambda(\xi, \xi_{n+1}, y')| \leq 1. \tag{3.4}$$

From (3.3), (3.3′) and (3.4)

$$|I_\lambda(\xi,\xi_{n+1},y')| \le \frac{C[\ln(1/|\eta'\cdot y'|)]^{1+\beta}}{(\ln|A_\lambda\xi|)^{1+\beta}} \text{ for } |A_\lambda\xi|\ge 2,$$

where $\eta' = \frac{A_\lambda\xi}{|A_\lambda\xi|}$. Thus, by (1.3), we get

$$\int_{S^{n-1}} |I_\lambda(\xi,\xi_{n+1},y')\Omega(y')|d\sigma(y') \le C(\ln|A_\lambda\xi|)^{-(1+\beta)}.$$

Therefore,

$$\begin{aligned}
&|\widehat{\tau_{k,t}}(\xi,\xi_{n+1})|\\
&= \Big|2^{-t}\int_{D_{t-k}} e^{i\left(\xi\cdot y+\xi_{n+1}\phi(\rho(y))\right)}\frac{\Omega(y)}{\rho(y)^{\alpha-1}}dy\Big|\\
&= \Big|2^{-t}\int_{S^{n-1}}\int_{2^{t-k}}^{2^{t-k+1}} e^{i\left(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho)\right)}\Omega(y')J(y')d\rho d\sigma(y')\Big|\\
&= \Big|2^{-k}\int_{S^{n-1}}\int_1^2 e^{i\left(\xi\cdot A_{2^{t-k}\rho}y'+\xi_{n+1}\phi(2^{t-k}\rho)\right)}\Omega(y')J(y')d\rho d\sigma(y')\Big|\\
&\le C2^{-k}\int_{S^{n-1}}\big|I_{2^{t-k}}(\xi,\xi_{n+1},y')\big|\big|\Omega(y')\big|d\sigma(y')\\
&\le C2^{-k}\big(\ln\big|A_{2^{t-k}}\xi\big|\big)^{-(1+\beta)}.
\end{aligned} \tag{3.5}$$

On the other hand, by (1.1), we can obtain

$$\begin{aligned}
&|\widehat{\tau_{k,t}}(\xi,\xi_{n+1})|\\
&= \Big|2^{-t}\int_{D_{t-k}} e^{i(\xi\cdot y+\xi_{n+1}\phi(\rho(y)))}\frac{\Omega(y)}{\rho(y)^{\alpha-1}}dy\Big|\\
&= \Big|2^{-t}\int_{2^{t-k}}^{2^{t-k+1}}\int_{S^{n-1}} e^{i(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho))}\Omega(y')J(y')d\sigma(y')d\rho\Big|\\
&= \Big|2^{-t}\int_{2^{t-k}}^{2^{t-k+1}}\int_{S^{n-1}} \Big(e^{i(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho))}-e^{i\xi_{n+1}\phi(\rho)}\Big)\Omega(y')J(y')d\sigma(y')d\rho\Big|\\
&\le C2^{-t}\int_{2^{t-k}}^{2^{t-k+1}}\int_{S^{n-1}} \big|e^{i(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho))}-e^{i\xi_{n+1}\phi(\rho)}\big|\big|\Omega(y')\big|\big|J(y')\big|d\sigma(y')d\rho\\
&\le C2^{-t}\int_{2^{t-k}}^{2^{t-k+1}}\int_{S^{n-1}} \big|\xi\cdot A_\rho y'\big|\big|\Omega(y')\big|\big|J(y')\big|d\sigma(y')d\rho\\
&\le C2^{-t}\int_{2^{t-k}}^{2^{t-k+1}}\int_{S^{n-1}} \big|A_{2^{t-k+1}}\xi\cdot y'\big|\big|\Omega(y')\big|\big|J(y')\big|d\sigma(y')d\rho\\
&\le C2^{-t}\cdot|A_{2^{t-k}}\xi|\int_{2^{t-k}}^{2^{t-k+1}}\int_{S^{n-1}} \big|\Omega(y')\big|\big|J(y')\big|\left|\frac{A_{2^{t-k+1}}\xi}{|A_{2^{t-k+1}}\xi|}\cdot y'\right|d\sigma(y')d\rho\\
&\le C2^{-k}|A_{2^{t-k}}\xi|.
\end{aligned} \tag{3.6}$$

Clearly, (3.5) and (3.6) imply (2.4) and (2.5) hold. Finally we shall show that (2.6) holds.

$$\begin{aligned}
&\tau_k^*(f)(x)\\
&=\sup_{t\in\mathbb{R}}(|\tau_{k,t}|*|f|)(x)\\
&=\sup_{t\in\mathbb{R}}2^{-t}\int_{D_{t-k}}|f(x-\Phi(y))|\frac{|\Omega(y)|}{\rho(y)^{\alpha-1}}dy\\
&=\sup_{t\in\mathbb{R}}2^{-t}\int_{S^{n-1}}\int_{2^{t-k}}^{2^{t-k+1}}|f(x-\Phi(A_\rho y'))||\Omega(y')|d\rho d\sigma(y')\\
&=\sup_{t\in\mathbb{R}}2^{-k+1}2^{-t}\int_{S^{n-1}}\int_{2^{t-1}}^{2^t}|f(x-\Phi(A_{2^{-k+1}\rho}y'))||\Omega(y')|d\rho d\sigma(y')\\
&\le 2^{-k+1}\int_{S^{n-1}}|\Omega(y')|\big(\sup_{t\in\mathbb{R}}2^{-t}\int_0^{2^t}|f(x-\Phi(A_{2^{-k+1}\rho}y'))|d\rho\big)d\sigma(y')\\
&\le C2^{-k}\int_{S^{n-1}}|\Omega(y')|M_\Phi(f)(x)d\sigma(y').
\end{aligned}$$

By Lemma 2.2, we obtain $\|M_\Phi(f)\|_p\le C\|f\|_p$, where $C>0$ is independent of k, the coefficient of $\phi(t)$ and f, since Ω is integrable on S^{n-1}, thus $\|\tau_k^*(f)\|_p\le C2^{-k}\|f\|_p$. This shows (2.7) holds. This completes the proof of Theorem 1.

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