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STRONG COHOMOLOGICAL RIGIDITY OF A PRODUCT OF PROJECTIVE SPACES

Suyoung Choi and Dong Youp Suh

Abstract. We prove that for a toric manifold (respectively, a quasitoric manifold) M, any graded ring isomorphism $H^*(M) \to H^*(\prod_{i=1}^m \mathbb{C}P^{n_i})$ can be realized by a diffeomorphism (respectively, a homeomorphism) $\prod_{i=1}^m \mathbb{C}P^{n_i} \to M$.

1. Introduction

The *cohomological rigidity problem* for toric manifolds asks whether the integral cohomology ring of a toric manifold determines its topological type or not. So far, there is no negative answer to the question but some positive results. In [2], the authors with M. Masuda show that if M is a toric manifold whose cohomology ring is isomorphic to that of $\prod_{i=1}^m \mathbb{C}P^{n_i}$, a product of complex projective spaces, then M is actually diffeomorphic to $\prod_{i=1}^m \mathbb{C}P^{n_i}$, which gives a positive result to the cohomological rigidity problem.

On the other hand, one might ask a stronger question as follows. Throughout this paper, $H^*(X)$ denotes the integral cohomology ring of a topological space X.

Problem 1.1. *Let M and N be toric manifolds, and $\varphi\colon H^*(N) \to H^*(M)$ a graded ring isomorphism. Then, does there exist a homeomorphism or a diffeomorphism $f\colon M \to N$ such that $f^* = \varphi$?*

We call this the *strong cohomological rigidity problem* for toric manifolds. Problem 1.1 for homeomorphism is still open. However, the answer to this question for diffeomorphism is negative in general; for instance, it is shown

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by Friedman and Morgan [5] that some cohomology ring automorphism of $\mathbb{C}P^2\#10\overline{\mathbb{C}P^2}$ is not induced by its self-diffeomorphism while it is a toric manifold. Nevertheless, we do conjecture that the strong cohomological rigidity (even for diffeomorphism) holds for some specific subclass of toric manifolds such as generalized Bott manifolds [3, Section 5].

A *generalized Bott tower* of height m is a sequence of $\mathbb{C}P^{n_i}$-bundles with $n_i \geq 1$:

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{1}$$

where each $\pi_i \colon B_i \to B_{i-1}$ for $i = 1, \ldots, m$ is the projectivization of a Whitney sum of $n_i + 1$ complex line bundles over B_{i-1}. We call B_i an *i-stage generalized Bott manifold* or a *generalized Bott manifold of height i*. If all fibration in (1) are trivial, B_m is $\prod_{i=1}^m \mathbb{C}P^{n_i}$.

In this article, we would like to answer to Problem 1.1 for the case when $N = \prod_{i=1}^m \mathbb{C}P^{n_i}$. Namely, we prove the following theorem.

Theorem 1.2. *Let M be a toric manifold. If there is a graded ring isomorphism $\varphi : H^*(M) \to H^*(\prod_{i=1}^m \mathbb{C}P^{n_i})$, then there is a diffeomorphism $f : \prod_{i=1}^m \mathbb{C}P^{n_i} \to M$ such that $f^* = \varphi$.*

Combining Theorem 1.2 with [1, Theorem 8.1], we obtain the following corollary which generalizes [6, Theorem 5.1] treating the case where $n_i = 1$ for any i. Note that a quasitoric manifold is a topological analogue of toric manifold, which is introduced by Davis and Januszkiewicz in [4].

Corollary 1.3. *Let M be a quasitoric manifold. If there is a graded ring isomorphism $\varphi : H^*(M) \to H^*(\prod_{i=1}^m \mathbb{C}P^{n_i})$, then there is a homeomorphism $f : \prod_{i=1}^m \mathbb{C}P^{n_i} \to M$ such that $f^* = \varphi$.*

2. Proof of Theorem 1.2

Let $R = \mathbb{Z}[x_1, \ldots, x_m]/\langle x_i^{n_i+1} : i = 1, \ldots, m\rangle \cong H^*(\prod_{i=1}^m \mathbb{C}P^{n_i})$.

Lemma 2.1. *Let $y = \sum_{j=1}^m a_j x_j \in R$ such that $a_i \neq 0$ for some i. Then $y^{n_i} \neq 0$ in R.*

Proof. Suppose $y^{n_i} = 0$ on the contrary. Then y^{n_i} must lie in the ideal generated by the polynomials $x_j^{n_j+1}$ for $j = 1, \ldots, n$. Since $a_i \neq 0$, $y^{n_i} = (\sum_{j=1}^m a_j x_j)^{n_i}$ must contain the nonzero monomial term of $x_i^{n_i}$. However if a nonzero multiple of a power of x_i appear in the ideal generated by $x_j^{n_j+1}$ for $j = 1, \ldots, m$, then the exponent must be at least $n_i + 1$, which is a contradiction. □

Lemma 2.2. *If ψ is a graded ring automorphism on R, then there exists a permutation σ on $\{1, \ldots, m\}$ such that $n_i = n_{\sigma(i)}$ for all $i = 1, \ldots, m$ and $\psi(x_i) = \pm x_{\sigma(i)}$.*

Proof. Let $\psi(x_i) = \sum_{j=1}^m b_{ij}x_j$ for $i = 1, \dots, m$. Since ψ is an automorphism, $\det B = \pm 1$, where $B = (b_{ij})$. Note that the positive integers $n_1, \dots, n_m$ need not be distinct. Let $S = \{N_1, \dots, N_k \mid N_1 > \cdots > N_k\}$ be the set of distinct numbers from $n_1, \dots, n_m$, and let $\mu\colon \{1, \dots, m\} \to S$ be the function defined by $\mu(i) = n_i$. Let $J_\ell = \mu^{-1}(N_\ell)$ for $\ell = 1, \dots, k$.

Claim: *B is conjugate to a block upper triangular matrix by a permutation matrix.*

Since $x_i^{n_i+1} = 0$ in R, $0 = \psi(x_i^{n_i+1}) = (\sum_{j=1}^m b_{ij}x_j)^{n_i+1}$. Therefore, by Lemma 2.1, $b_{ij} = 0$ if $n_i < n_j$. Hence by an appropriate permutation of the index set $\{1, \dots, m\}$, we may assume that $n_1 \geq n_2 \geq \cdots \geq n_m$ and B is an upper triangular matrix of the form

$$\begin{pmatrix} C_{J_1} & & & * \\ & C_{J_2} & & \\ & & \ddots & \\ 0 & & & C_{J_k} \end{pmatrix},$$

where C_{J_ℓ} is the matrix formed from b_{ij} with $i, j \in J_\ell$. This proves the claim.

Now let $J_{<\ell} = \bigcup_{\{N \in S \mid N < N_\ell\}} \mu^{-1}(N)$. By the previous claim, if $k \in J_\ell$, then we may write $\psi(x_k) = \sum_{j \in J_\ell} b_{kj}x_j + \sum_{j \in J_{<\ell}} b_{kj}x_j$. Let us denote $z_\ell = \sum_{j \in J_\ell} b_{kj}x_j$ and $w_\ell = \sum_{j \in J_{<\ell}} b_{kj}x_j$ for simplicity. Then $\psi(x_k) = z_\ell + w_\ell$. Therefore,

$$0 = \psi(x_k^{N_\ell+1}) = z_\ell^{N_\ell+1} + \binom{N_\ell+1}{1} w_\ell z_\ell^{N_\ell} + \binom{N_\ell+1}{2} w_\ell^2 z_\ell^{N_\ell-1} + \cdots.$$

We note that $z_\ell \neq 0$ since $\det B = \pm 1$. On the other hand, there is no way to get the polynomial equation

$$-\binom{N_\ell+1}{1} w_\ell z_\ell^{N_\ell} = z_\ell^{N_\ell+1} + \binom{N_\ell+1}{2} w_\ell^2 z_\ell^{N_\ell-1} + \cdots$$

in the ring R unless $w_\ell = 0$. Hence, $z_\ell^{N_\ell+1} = 0$. But then there is a unique nonzero b_{ij} for $i \in J_\ell$, and, hence, $b_{ij} = \pm 1$.

Therefore, we have shown that B is conjugate to a diagonal matrix all of whose diagonal entries are ± 1. Therefore if $k \in J_\ell$, then ψ sends x_k to $\pm x_i$ for some $i \in J_\ell$. □

Corollary 2.3. *Any graded ring automorphism ψ on $H^*(\prod_{i=1}^m \mathbb{C}P^{n_i})$ is induced by a self-diffeomorphism g on $\prod_{i=1}^m \mathbb{C}P^{n_i}$, i.e., $g^* = \psi$.*

Proof. Note that any automorphism ψ on

$$H^*(\prod_{i=1}^m \mathbb{C}P^{n_i}) = \mathbb{Z}[x_1, \dots, x_m]/\langle x_i^{n_i+1} \colon i = 1, \dots, m\rangle$$

of the form $\psi(x_i) = \pm x_{\sigma(i)}$ for some permutation σ satisfying $n_i = n_{\sigma(i)}$ for all i is realized by composition of permutating factors and reversing orientations of factors of $\prod_{i=1}^m \mathbb{C}P^{n_i}$, appropriately; namely, ψ is induced by a self-diffeomorphism on $\prod_{i=1}^m \mathbb{C}P^{n_i}$. Hence the corollary follows from Lemma 2.2. □

We are now ready to prove Theorem 1.2. Let $\varphi : H^*(M) \to H^*(\prod_{i=1}^m \mathbb{C}P^{n_i})$ be a graded ring isomorphism. As mentioned in the introduction, by [2, Theorem 1.1], there is a diffeomorphism $h : M \to \prod_{i=1}^m \mathbb{C}P^{n_i}$. Then we have

$$\begin{array}{ccc} H^*(M) & \xrightarrow{\varphi} & H^*(\prod_{i=1}^m \mathbb{C}P^{n_i}) . \\ \uparrow{\scriptstyle h^*} & \nearrow{\scriptstyle \varphi\circ h^*} & \\ H^*(\prod_{i=1}^m \mathbb{C}P^{n_i}) & & \end{array}$$

By Corollary 2.3, there is a diffeomorphism

$$g : \prod_{i=1}^m \mathbb{C}P^{n_i} \to \prod_{i=1}^m \mathbb{C}P^{n_i}$$

such that $g^* = \varphi \circ h^*$. Then $\varphi = (\varphi \circ h^*) \circ (h^*)^{-1} = g^* \circ (h^{-1})^* = (h^{-1} \circ g)^*$. Therefore, the diffeomorphism

$$f := h^{-1} \circ g : \prod_{i=1}^m \mathbb{C}P^{n_i} \to M$$

induces φ, which proves the theorem.

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Suyoung Choi
Department of Mathematics
Ajou University
Suwon 443-749, Korea
E-mail address: schoi@ajou.ac.kr

Dong Youp Suh
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Daejeon 305-701, Korea
E-mail address: dysuh@math.kaist.ac.kr