MATHEMATICAL ANALYSIS OF NONLINEAR DIFFERENTIAL EQUATION ARISING IN MEMS

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Abstract. In this paper, we study nonlinear equation arising in MEMS modeling electrostatic actuation. We will prove the local and global existence of solutions of the generalized parabolic MEMS equation. We present that there exists a constant λ^* such that the associated stationary problem has a solution for any $\lambda < \lambda^*$ and no solution for any $\lambda > \lambda^*$. We show that when $\lambda < \lambda^*$ the global solution converges to its unique maximal steady-state as $t \to \infty$. We also obtain the condition for the existence of a touchdown time $T \leq \infty$ for the dynamical solution. Furthermore, there exists $p_0 > 1$, as a function of p, the pull-in voltage $\lambda^*(p)$ is strictly decreasing with respect to $1 < p < p_0$, and increasing with respect to $p > p_0$.

1. Introduction

Electrostatic actuation is crucial for micro-electromechanical systems (MEMS), which are based on an electrostatic-controlled tunable capacitor, and widely used in accelerometers for airbag deployment in automobiles, micro-resonators, switches, micro-mirrors, accelerometers, and so on. The simplicity and importance of this technique have led many applied mathematicians and engineers to study mathematical models of electrostatic-elastic interactions. A mathematical model of the physical phenomena, leading to a partial differential equation for the dimensionless dynamic deflection of the membrane, was derived and analyzed in [2] and [6]. In the damping-dominated limit, and using a narrow-gap asymptotic analysis, the dimensionless dynamic deflection $v = v(x,t)$ of the membrane on a bounded domain Ω in $\mathbb{R}^N (N \geq 2)$, is found to satisfy the following parabolic problem ([1, 5, 6, 13])

$$(1.1) \quad \begin{cases} -v_t + D\Delta v = \frac{\lambda f(x)}{(L+v)^p}, & x \in \Omega,\ t \in (0,T), \\ v(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ v(x,0) = 0, & x \in \Omega, \end{cases}$$

where $\lambda > 0$ is a parameter, $D > 0$, $p > 1$ are fixed constants. In particular the problem with $p = 2$ has been intensively studied ([5, 6]), which models a simple electrostatic MEMS device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at $-L$. Here u is modeled to describe dynamic deflection of the elastic membrane. The voltage V must be limited, or the elastic plate will be pulled-in.

The parameter $D > 0$ is the tension constant. The parameter λ in (1.1) characterizes the relative strength of the electro-static and mechanical forces in the system, and is proportional to the supply voltage squared V^2. We shall use here the parameter λ (resp., λ^*) to represent the applied voltage V (resp., pull-in voltage V^*). Referred to as the permittivity profile, $f(x)$ satisfy

(1.2) $$\begin{cases} f \in C^\alpha(\bar{\Omega}) \text{ for some } \alpha \in (0,1],\ 0 \le f \le 1 \\ f > 0 \text{ on a subset of } \Omega \text{ of positive measure.} \end{cases}$$

Recently, Pelesko, Ghoussoub and Guo, Wang and Ruan studied in [4, 11, 12] the steady-states of (1.1) with $p = 2$, $p > 0$, respectively, when $f(x)$ is assumed to be bounded away from zero. They established in above cases an upper bound $\bar{\lambda}_1$ for λ^*, respectively, and derived numerical results for the power-law permittivity profile, from which the larger pull-in voltage and thereby the larger pull-in distance, the existence and multiplicity of the steady-states were observed.

In [5, 6], authors studied the dynamic behavior of (1.1) with $p = 2$. They considered a more general class of profiles $f(x)$, i.e., $0 \le f(x) \le 1$ on Ω with $f(x) > 0$ on a subset of positive measure. They obtained larger pull-in voltage λ^* and larger pull-in distance for different classes of varying permittivity profiles. They established upper and lower bounds on first touchdown times, and analyzed their dependence on f, λ and Ω by applying various analytical and numerical techniques.

In [8], Hui studied the following nonlocal parabolic equation

(1.3) $$\begin{cases} u_t = \Delta u + \frac{\lambda}{(1-u)^2(1+\chi \int_\Omega \frac{dy}{1-u})^2}, & x \in \Omega,\ t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ u(x,0) = u_0, & x \in \Omega, \end{cases}$$

where $\chi > 0$ is a constant. He obtained upper bounds for the quenching time of the solution of the nonlocal parabolic MEMS equation in $\Omega \times (0,\infty)$, when λ is large. Furthermore, he studied the generalized parabolic MEMS equation in [7]. He obtained a series of sharp existence theorems for solution of the generalized MEMS equation.

It is often convenient to take the change of variable $u = -v$ in considering problem (1.1). Then we have

(1.4) $$\begin{cases} u_t - D\Delta u = \frac{\lambda f(x)}{(L-u)^p}, & x \in \Omega,\ t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ u(x,0) = 0, & x \in \Omega, \end{cases}$$

and the associated stationary problem

$$(1.5) \qquad \begin{cases} -D\Delta u = \frac{\lambda f(x)}{(L-u)^p}, & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega. \end{cases}$$

Let $Q = \{\lambda > 0 \,|\, (1.5) \text{ possesses at least one solution}\}$ and the number

$$(1.6) \qquad \lambda^* = \lambda^*(\Omega, f, p) = \sup_{\lambda \in Q} \lambda.$$

The constant λ^* is called the pull-in voltage of the equation (1.5) in the literature of MEMS.

In this paper, we deal with issues of global convergence in the dynamic problem (1.4). For $p > 1$, and $f(x)$ satisfies (1.2), we will prove the local and global existence and comparison theorems of solutions of the dynamic problem (1.4). We present that there exists a constant λ^* such that the associated stationary problem (1.5) has a solution for any $\lambda < \lambda^*$ and no solution for any $\lambda > \lambda^*$. We show that when $\lambda < \lambda^*$ the global solution converges to its unique minimal steady-state as $t \to \infty$. We also obtain the condition for the existence of a touchdown time $T \leq \infty$ for the dynamical solution. Furthermore, there exists $p_0 > 1$, as a function of p, the pull-in voltage $\lambda^*(p)$ is strictly decreasing with respect to $1 < p < p_0$, and increasing with respect to $p > p_0$. It is interesting that, we extend our method to study a more general model which is widely useful in applications ([1, 5, 6]).

2. Global regular solution for problem (1.4)

In this section, we will use upper-and-lower solution method to study the existence of solutions for the parabolic equation according to the problem (1.4).

We define upper and lower solutions, u^+ and u^-, if they satisfy, respectively,

$$(2.1) \qquad \begin{cases} u_t^+ - D\Delta u^+ \geq \frac{\lambda f(x)}{(L-u^+)^p}, & x \in \Omega, \ t \in (0,T), \\ u^+(x,t) \geq 0, & x \in \partial\Omega, \ t \in (0,T), \\ u^+(x,0) \geq 0, & x \in \Omega, \ t = 0, \end{cases}$$

and

$$(2.2) \qquad \begin{cases} u_t^- - D\Delta u^- \leq \frac{\lambda f(x)}{(L-u^-)^p}, & x \in \Omega, \ t \in (0,T), \\ u^-(x,t) \leq 0, & x \in \partial\Omega, \ t \in (0,T), \\ u^-(x,0) \leq 0, & x \in \Omega, \ t = 0. \end{cases}$$

We now invoke an iterative scheme

$$(2.3) \qquad \begin{cases} (u_n)_t - D\Delta u_n + Cu_n = \frac{\lambda f(x)}{(L-u_{n-1})^p} + Cu_{n-1}, & x \in \Omega, \ t \in (0,T), \\ u_n(x,t) = 0, & x \in \partial\Omega, \ t \in (0,T), \ n = 1,2,3,\dots \\ u_n(x,0) = 0, & x \in \Omega, \ t = 0, \end{cases}$$

where $C > 0$ is a suitable constant and $u_0 = 0$. Because $0 \leq u < L$, we can confirm that $u = 0$ is the lower solution u^-. To obtain a solution of the problem

(1.4), we assume that it has a upper solution u^+, then u_1 and u^+ satisfy:

$$(u^+ - u_1)_t - D\Delta(u^+ - u_1) + C(u^+ - u_1)$$

$$\geq \frac{\lambda f(x)}{(L - u^+)^p} - \frac{\lambda f(x)}{L^p} + C(u^+ - u_0)$$

$$= \lambda f(x)\frac{L^p - (L - u^+)^p}{L^p(L - u^+)^p} + Cu^+$$

$$> 0, \quad x \in \Omega, t \in (0, T).$$

Because of $u^+ - u_1 \geq 0$, $x \in \partial\Omega$, $t \in [0, T)$, we can use the maximum principle to obtain $u_1 \leq u^+$, $x \in \Omega$, $t \in (0, T)$.

Let u_1 and u^- satisfy:

$$(u_1 - u^-)_t - D\Delta(u_1 - u^-) + C(u_1 - u^-) \geq 0, \quad x \in \Omega, \ t \in (0, T).$$

By $u_1 - u^- \geq 0$, $x \in \partial\Omega$, $t \in [0, T)$, we can also use the maximum principle to obtain $u^- \leq u_1, x \in \Omega, t \in (0, T)$.

We can obtain a sequence as the same method:

$$(2.4) \qquad 0 = u^- \leq u_1 \leq \cdots \leq u_{n-1} \leq u_n \leq \cdots \leq u^+ < L.$$

We now show that when λ is small enough, the problem (1.4) has a upper solution. To see this, we use k_1 and U_1 to denote the first eigenvalue and the associated eigenfunction of the operate $-\Delta$ over a large bounded domain Ω_1 containing $\overline{\Omega}$, so that $U_1 = 0$ on $\partial\Omega_1$ and $U_1 > 0$ in Ω_1. Let $\varepsilon > 0$ be a small number so that $0 < \varepsilon U_1 < \frac{L}{2}$. Then $-D\Delta(\varepsilon U_1) = \varepsilon k_1 D U_1$. We need that

$$(2.5) \qquad \varepsilon k_1 D U_1 \geq \frac{\lambda f(x)}{(L - \varepsilon U_1)^p}, \quad x \in \Omega, \ t \in (0, T),$$

thus

$$(2.6) \qquad \lambda f(x) \leq \varepsilon k_1 D U_1 (L - \varepsilon U_1)^p,$$

i.e.,

$$(2.7) \qquad 0 < \lambda^* \leq k_1 D \frac{p^p}{(p+1)^{p+1}} \frac{L^{p+1}}{\inf_{x \in \Omega} f(x)}, \quad p > 1.$$

If $0 \leq \lambda < \lambda^*$, then εU_1 is the upper solution, i.e., $0 \leq u^- < \varepsilon U_1 = u^+ < L$.

So, we can obtain the existence of the solution of the problem (1.4) with the upper and lower solution.

Lemma 2.1. *Assume f is a function satisfying (1.2) on a bounded domain Ω in $\mathbb{R}^N$, and $p > 1$. If $0 \leq \lambda < \lambda^*$, then there exists a global regular solution $u(x, t)$ for (1.4), where the finite voltage $\lambda^* = \lambda^*(\Omega, f, p) > 0$ is as in (2.7).*

Corollary 2.2. *Let $0 \leq \lambda < \lambda^*$ and $0 \leq f \in C(\bar{\Omega})$. Suppose u_1, u_2 are nonnegative subsolution and supersolution of (1.4) in $\Omega \times (0, T)$, respectively. Then $u_1 \leq u_2$ in $\bar{\Omega} \times (0, T)$.*

Lemma 2.3. *Under the condition of the Lemma 2.1, then problem* (1.4) *has a unique global solution.*

Proof. Suppose u_1 and u_2 are solutions of (1.4) on the domain $\Omega \times [0, T]$ such that $\|u_i\|_{L^\infty(\Omega \times [0,T])} < L$ for $i = 1, 2$. Let $u = u_1 - u_2$, then we have

$$(2.8) \qquad u_t - D \triangle u = gu,$$

with initial data $u(x, 0) = 0$ and zero boundary condition. Here

$$(2.9) \quad g(x,t) = \lambda f(x) \frac{(L - u_2)^{p-1} + (L - u_2)^{p-2}(L - u_1) + \cdots + (L - u_1)^{p-1}}{(L - u_1)^p (L - u_2)^p}.$$

The assumption on u_1, u_2 implies that $g(x, t) \in L^\infty(\Omega \times [0, T])$. We now fix $T_1 \in [0, T]$ and consider the solution v of the problem

$$(2.10) \qquad \begin{cases} v_t + D\Delta v + gv = 0, & x \in \Omega, \ t \in (0, T_1), \\ v(x, T_1) = \xi(x) \in C_0(\Omega), \\ v(x, t) = 0, & x \in \partial\Omega. \end{cases}$$

The standard linear theory [5, 9] gives that the solution of (2.10) is a unique and bounded. Now multiplying (2.8) by v, and integrating it on $\Omega \times [0, T_1]$, together with (2.10), yield that

$$(2.11) \qquad \int_\Omega u(x, T_1)\xi(x)dx = 0,$$

for arbitrary T_1 and $\xi(x)$, which implies that $u = 0$, $u_1 = u_2$. $\qquad\square$

Lemma 2.4. *Let $\lambda > 0, p > 1$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain such that ball $B_R = B_R(0) \subset \Omega$ for some constant $R > 0$. Let $0 < f \in C(\bar{\Omega})$, $0 \le u < L$ be a solution of* (1.4) *in $\Omega \times (0, T)$ and $0 < \delta \le \inf_{x\in\Omega} f(x)$. Suppose $\lambda > \frac{2nD}{\delta R^2}$. Then T satisfies*

$$(2.12) \qquad T \le \frac{1}{\lambda\delta}\left(1 - \frac{2nD}{\lambda\delta R^2}\right)^{-1}.$$

Proof. Let $w = L - u$ and $w_0 = L$. Then w satisfies

$$(2.13) \qquad \begin{cases} w_t - D\Delta w + \lambda f(x)w^{-p} = 0, & x \in \Omega, \ t \in (0, T), \\ w(x, t) = L, & x \in \partial\Omega, \ t \in (0, T), \\ w(x, 0) = L, & x \in \Omega. \end{cases}$$

Let

$$\eta(x, t) = 1 - \lambda\delta\gamma t\left(1 - \frac{|x|^2}{R^2}\right), \quad x \in B_R, \ t \in (0, T),$$

where $\gamma = 1 - \frac{2nD}{\lambda\delta R^2} > 0$. Then

$$(2.14) \qquad \frac{2nD}{R^2} \cdot \frac{1}{\lambda\delta\gamma} = \gamma^{-1} - 1.$$

By (2.14), we get

$$\eta_t - D\Delta\eta + \lambda\delta\eta^{-p} \ge \lambda\delta\gamma\left(\gamma^{-1} - 1 - \frac{2nD}{R^2}t\right) \ge 0, \quad x \in B_R, \ t \in (0, T_1),$$

where $T_1 = \min\{T, \frac{1}{\lambda\delta\gamma}\}$. Hence η is a supersolution of

$$(2.15) \qquad \begin{cases} \eta_t - D\Delta\eta + \lambda\delta\eta^{-p} = 0, & (x,t) \in B_R \times (0,T_1), \\ \eta(x,t) = L, & (x,t) \in \partial B_R \times (0,T_1), \\ \eta(x,0) = L, & x \in \bar{B}_R. \end{cases}$$

Suppose $T > \frac{1}{\lambda\delta\gamma}$. Since w is a subsolution of (2.15), by Corollary 2.2, we have

$$0 < w \leq \eta, \quad (x,t) \in \bar{B}_R \times (0,T_1).$$

So $0 < w(0, \frac{1}{\lambda\delta\gamma}) \leq \eta(0, \frac{1}{\lambda\delta\gamma}) = 0$. Contradiction arise and (2.12) follows. $\square$

Theorem 2.5. *Assume f is a function satisfying (1.2) on a bounded domain Ω in $\mathbb{R}^N$, and $p > 1$. Then for any $\lambda > 0$ there exists $T > 0$ such that (1.4) has a unique solution $0 \leq u < L$ in $\Omega \times (0,T)$ which satisfies*

$$(2.16) \quad u(x,t) = \lambda \int_0^t \int_\Omega G(x,y,t-s) \frac{f(y)}{(L - u(y,s))^p} dyds, \quad (x,t) \in \bar{\Omega} \times (0,T).$$

Where $G(x,y,t)$ is the Dirichlet Green function for the heat equation in $\Omega \times (0,T)$.

Proof. Let

$$(2.17) \qquad T = \frac{L+a}{2\lambda\|f\|_{L^\infty}} \left(\frac{L-a}{2}\right)^p, \quad p > 1, \ 0 < a < 1,$$

and

$$(2.18) \qquad u_1(x,t) = \lambda \int_0^t \int_\Omega G(x,y,t-s) \frac{f(y)}{L^p} dyds, \quad (x,t) \in \bar{\Omega} \times (0,T).$$

Let $T_1 = \sup\{0 < t_1 < T : u_1 < \frac{L+a}{2}, (x,t) \in \bar{\Omega} \times (0,t_1)\}$. Suppose $T_1 < T$. $\forall (x,t) \in \bar{\Omega} \times (0,T_1]$, then $u_1 \leq \lambda\|f\|_{L^\infty} \cdot \frac{t}{L^p} < \frac{L+a}{2}$. By continuity of u_1 there exists $0 < \delta < \frac{T-T_1}{2}$ such that $u_1 < \frac{L+a}{2}$ holds for all $(x,t) \in \bar{\Omega} \times (0,T_1+\delta]$. This contradicts the maximality of T_1. Hence $T_1 = T$ and (2.18) holds for all $(x,t) \in \bar{\Omega} \times (0,T]$.

We define

$$(2.19) \quad u_{n+1}(x,t) = \lambda \int_0^t \int_\Omega G(x,y,t-s) \frac{f(y)}{(L-u_n)^p} dyds, \quad (x,t) \in \bar{\Omega} \times (0,T).$$

Let $T_n = \sup\{0 < t_1 < T : u_n < \frac{L+a}{2}, (x,t) \in \bar{\Omega} \times (0,t_1)\}$. By induction, we obtain $T_n = T$ for all $n \in \mathbb{Z}^+$. Then

$$(2.20) \qquad u_n < \frac{L+a}{2}, \quad \forall (x,t) \in \bar{\Omega} \times (0,T).$$

By (2.18), (2.19), (2.20) and standard parabolic theory [3], we have $u_n \in C^{2,1}(\bar{\Omega} \times (0,T])$ for all $n \geq 2$. Then u_n satisfies

$$(2.21) \qquad \begin{cases} \frac{\partial u_n}{\partial t} - D\Delta u_n = \frac{\lambda f(x)}{(L-u_n)^p}, & x \in \Omega, \ t \in (0,T), \\ u_n(x,t) = 0, & x \in \partial\Omega, \ t \in (0,T), \\ u_n(x,0) = 0, & x \in \Omega. \end{cases}$$

By (2.20), (2.21) and the Schauder estimates for the heat equation ([3, 7]) $\{u_n\}_{n=2}^{\infty}$ are uniformly bounded in $C^{2+\beta,1+\frac{\beta}{2}}(\Omega')$ ([7]) for any compact subset $\Omega' \subset \bar{\Omega} \times (0,T]$, where $0 < \beta < 1$ is some constant. By the Ascoli theorem and a diagonalization argument $\{u_n\}_{n=2}^{\infty}$ has a subsequence which converges uniformly in $C^{2+\beta,1+\frac{\beta}{2}}(\Omega')$ to some function u for any compact subset $\Omega' \subset \bar{\Omega} \times (0,T]$ as $n \to \infty$. Then u satisfies (2.16),

$$(2.22) \qquad u(x,t) \leq \frac{L+a}{2} < L, \qquad \forall (x,t) \in \bar{\Omega} \times (0,T),$$

and (1.4). Hence u is a solution of (1.4) in $\Omega \times (0,T)$.   □

By an argument similar to the proof of Theorem 2.5, we have the following theorem.

Theorem 2.6. *Let $p > 1$ and*

$$0 < f \in C^{\alpha}(\mathbb{R}^N) \text{ for some constant } 0 < \alpha < 1.$$

Then for any $\lambda > 0$ there exists a constant $T > 0$ such that Cauchy problem

$$(2.23) \qquad \begin{cases} u_t - D\Delta u = \frac{\lambda f(x)}{(L-u)^p}, & x \in \mathbb{R}^N, \ t \in (0,T), \\ u(x,0) = 0, & x \in \mathbb{R}^N, \end{cases}$$

has a solution $0 \leq u < L$ in $\mathbb{R}^N \times (0,T)$ which satisfies

$$(2.24) \qquad u(x,t) = \lambda \int_0^t \int_{\mathbb{R}^N} Z(x,y,t-s)\frac{f(y)}{(L-u(y,s))^p}dyds,$$

where $Z(x,y,t) = (4D\pi t)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4Dt}}$.

3. Global convergence at infinite time

In this section we will prove the convergence of solution of (1.4) for any $0 < \lambda < \lambda^*$ as $t \to \infty$. We also obtain the conditions for the solution of (1.4) to have finite touchdown time or infinite time.

Definition 3.1. If u_* is a solution of (1.5) such that for any other solution v of (1.5), one has

$$(3.1) \qquad\qquad u_* \leq v, \ \text{ a.e. } \ x \in \Omega,$$

we say that u_* is a unique minimal solution of (1.5).

In [10, 12], Lin and Yang, Wang and Ruan derived the equation (1.5) with $p = 2$, $p > 0$ in the study of charged plates in electrostatic actuators, respectively. Here $\lambda = kV^2$ where V is the electric voltage and k is a positive constant. For the boundary condition $u|_{\partial\Omega} = 0$, they found that there exists $0 < \lambda^* < \infty$ such that for $\lambda \in (0, \lambda^*)$, (1.5) has a minimal and regular solution u_*. For $\lambda > \lambda^*$, (1.5) does not have any regular solution.

Similarly to the method in [4, 6, 7, 10, 11, 12], by the definition λ^* (1.6) and the proof of Lemma 2.1 we can get:

Lemma 3.2. *Suppose* $\lambda^* = \lambda^*(\Omega, f, p) > 0$ *is as in Lemma 2.1. Then the following hold:*
  1. *If* $0 \le \lambda < \lambda^*$, *there exists the regular solution of* (1.5).
  2. *If* $\lambda > \lambda^*$, *there exists no solution of* (1.5).

Proposition 3.3. *Let* $\Omega_1 \subset \Omega_2$ *and let* f_1, f_2 *satisfy* (1.2) *in* Ω_1, Ω_2, *respectively for some constant* $0 < \alpha < 1$ *such that* $f_1 \le f_2$ *in* Ω_1. *Then* $\lambda^*(\Omega_1, f_1, p) \ge \lambda^*(\Omega_2, f_2, p)$.
  If $0 < \lambda < \lambda^*(\Omega_2, f_2, p)$ *and* v_1, v_2 *are the minimal solutions of* (1.5) *with* $\Omega = \Omega_1, \Omega_2$, $f = f_1, f_2$, *respectively, then* $v_1 \le v_2$ *in* Ω_1. *If moreover* $\Omega = \Omega_1 = \Omega_2$ *and* $f_1 \not\equiv f_2$, *then* $v_1 < v_2$ *in* Ω.

Proof. For any $\lambda < \lambda^*(\Omega_2, f_2, p)$, let v_2 be the minimal solution of (1.5) with $\Omega = \Omega_2$, $f = f_2$. Then v_2 is a supersolution of (1.5) with $\Omega = \Omega_1$, $f = f_1$, the minimal solution v_1 of (1.5) satisfies $0 \le v_1 \le v_2$ in Ω_1. Hence $\lambda^*(\Omega_1, f_1, p) \ge \lambda^*(\Omega_2, f_2, p)$.

Suppose that $\Omega = \Omega_1 = \Omega_2$, and $G(x, y)$ be the Green function for Δ in Ω. Then

$$(3.2) \qquad v_i(x) = \lambda \int_\Omega G(x, y) \frac{f_i(y)}{(L - v_i(y))^p} dy, \quad \forall i = 1, 2.$$

If $f_1 \not\equiv f_2$, there exists a set $A \subset \Omega$ of positive measure such that $\frac{f_1}{(L - v_1)^p} < \frac{f_2}{(L - v_2)^p}$ in A. Then by (3.2), $v_1 < v_2$ in Ω and the proposition follows. $\qquad \square$

Theorem 3.4. *Assume* f *satisfies* (1.2), $p > 1$, *then for every* $0 < \lambda < \lambda^*$, *there exists a unique global solution* $u(x, t)$ *of* (1.4) *which monotonically converges as* $t \longrightarrow +\infty$ *to the unique minimal solution* u_* *of* (1.5). *Furthermore, there exists* $p_0 > 1$, *as a function of* p, *the pull-in voltage* $\lambda^*(p)$ *is smooth, strictly decreasing with respect to* $1 < p < p_0$, *and increasing with respect to* $p > p_0$.

Proof. Fix $0 < \lambda < \lambda^*$, using the method of [4, 10] we can obtain the existence of an unique minimal solution u_* of (1.5). It is clear that the pair $u^- = 0$ and $u^+ = u_*$ are sub- and sup-solutions of (1.4). This implies that the unique global solution $u(x, t)$ of (1.4) satisfies $L > u_* \ge u \ge 0$ in $\Omega \times (0, \infty)$.

By differentiating in time and setting $v = u_t$, we get for any fixed $T_0 > 0$

$$(3.3) \qquad \begin{cases} v_t = D\Delta v + \frac{p\lambda f(x)}{(L - u)^{p+1}} v, & (x, t) \in \Omega \times (0, T_0), \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T_0), \\ v(x, 0) \ge 0, & x \in \Omega. \end{cases}$$

Here $\frac{p\lambda f(x)}{(L - u)^{p+1}}$ is a locally bounded non-negative function, and by the strong maximum principle, we get that $u_t = v > 0$ for $(x, t) \in \Omega \times (0, T_0)$. Otherwise $u_t = 0$, i.e., $u(x, t) = u_*$ for any $t > 0$. It follows that $u_t > 0$ holds, and since u is bounded, this monotonicity property implies that the unique global solution

$u(x, t)$ converges to some function u^* as $t \longrightarrow +\infty$. Hence,

$$(3.4) \qquad\qquad 0 \leq u^* \leq u_* < L, \quad x \in \Omega.$$

The linear stationary boundary problem

$$(3.5) \qquad\qquad \begin{cases} D\Delta v + \frac{\lambda f(x)}{(L-u^*)^p} = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

Let $\omega(x, t) = u(x, t) - v(x)$. Then ω satisfies

$$(3.6) \qquad\qquad \begin{cases} \omega_t - D\Delta\omega = \lambda f(x)[\frac{1}{(L-u)^p} - \frac{1}{(L-u^*)^p}], & (x, t) \in \Omega \times (0, \infty), \\ \omega(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ \omega(x, 0) = -v(x), & x \in \Omega. \end{cases}$$

By a standard eigenfunction expansion, we imply that the solution ω of (3.6) also converges to zero in $L^p(\Omega)$ as $t \longrightarrow +\infty$. This shows that $u \to v$ in $L^p(\Omega)$ as $t \longrightarrow +\infty$. But since $u \to u^*$ pointwise in Ω as $t \longrightarrow +\infty$, we deduce that $v(x) \equiv u^*(x)$ in $L^p(\Omega)$, which implies that $u^*(x)$ is also a solution for (1.5). The minimal property of $u_*(x)$ then yields that $u_*(x) \equiv u^*(x)$ on Ω, from which it follows that for every $x \in \Omega$, we have $u(x, t) \downarrow u_*$ as $t \longrightarrow +\infty$.

Following Section 2, we define the pull-in voltage λ^*, that is as in Lemma 2.1. We use $\lambda^*(p)$ to denote the function

$$(3.7) \qquad\qquad \lambda^*(p) = C\frac{(Lp)^p}{(p+1)^{p+1}},$$

where $C = \frac{k_1 DL}{\inf_{x \in \Omega} f(x)}$. However, since

$$(3.8) \qquad\qquad \lambda_p^* = \lambda^*(p) \ln(\frac{Lp}{p+1}).$$

Fix $1 < L < 2$, there exists $p_0 = \frac{1}{L-1} > 1$, we see that there holds

$$(3.9) \qquad\qquad \lambda_p^* < 0, \quad \text{if} \quad 1 < p < p_0,$$

and

$$(3.10) \qquad\qquad \lambda_p^* > 0, \quad \text{if} \quad p > p_0.$$

Which imply that $\lambda^*(p)$ is strictly decreasing with respect to $1 < p < p_0$, and increasing with respect to $p > p_0$. The proof of the theorem is complete.   □

We know from Lemma 3.2 that there is no solution for (1.5) as soon as $\lambda > \lambda^*$. Since the solution $u(x, t)$ of (1.4) is strictly increasing in time t, then there must be $T \leq \infty$ such that $u(x, t)$ reaches L at some point of $\bar{\Omega}$ as $t \to T^-$. Otherwise, a proof similar to Theorem 3.4 would imply that $u(x, t)$ will converge to its steady-state which is then the unique minimal solution u_* of (1.5), contrary to the hypothesis that $\lambda > \lambda^*$. We have:

Corollary 3.5. *Assume f satisfies* (1.2), $p > 1$, *if $\lambda > \lambda^*$, then the solution $u(x, t)$ of* (1.4) *must touchdown at a finite time or infinite time.*

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