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DIVISOR FUNCTIONS AND WEIERSTRASS FUNCTIONS ARISING FROM q-SERIES

Daeyeoul Kim and Min-Soo Kim

Abstract. We consider Weierstrass functions and divisor functions arising from q-series. Using these we can obtain new identities for divisor functions. Farkas [3] provided a relation between the sums of divisors satisfying congruence conditions and the sums of numbers of divisors satisfying congruence conditions. In the proof he took logarithmic derivative to theta functions and used the heat equation. In this note, however, we obtain a similar result by differentiating further. For any $n \geq 1$, we have

$$k\cdot\tau_{2;k,l}(n) = 2n\cdot E_{\frac{k-l}{2}}(n;k)+l\cdot\tau_{1;k,l}(n)+2k\cdot\sum_{j=1}^{n-1}E_{\frac{k-l}{2}}(j;k)\tau_{1;k,l}(n-j).$$

Finally, we shall give a table for $E_1(N;3), \sigma(N), \tau_{1;3,1}(N)$ and $\tau_{2;3,1}(N)$ $(1 \leq N \leq 50)$ and state simulation results for them.

1. Introduction

Basic hypergeometric series plays a very important role in many fields, such as affine systems, Lie algebras and groups, number theory, orthogonal polynomials, and physics. Throughout this paper, we use the standard notation

$$(a;q)_\infty := \prod_{n\geq 0}(1-aq^n).$$

If there is no confusion, we briefly write $(a)_\infty$ instead of $(a;q)_\infty$. In general, q will denote a fixed complex number of absolute value less than 1, so we may write $q = e^{2\pi i\tau}$ with $\text{Im } \tau > 0$.

For $N, m, r, s, w, k, l \in \mathbb{Z}$, we define some divisor functions necessary for later use, which appear in many areas of number theory:

$$E_r(N;m) = \sum_{\substack{d|N \\ d\equiv r \bmod m}} 1 - \sum_{\substack{d|N \\ d\equiv -r \bmod m}} 1,$$

$$E_{r,\ldots,s}(N;m) = E_r(N;m) + \cdots + E_s(N;m),$$

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$$\sigma_s^{\{l\}}(N;m) = \sum_{\substack{d|N \\ d\equiv s \bmod m}} d^l, \quad \sigma(N) = \sum_{d|N} d,$$

$$\sigma_s(N;m) = \sigma_s^{\{1\}}(N;m) = \sum_{\substack{d|N \\ d\equiv s \bmod m}} d,$$

$$\tau_{1;k,l}(N) = \sum_{\substack{d|N \\ d\equiv \frac{k-l}{2} \bmod k}} N/d + \sum_{\substack{d|N \\ d\equiv \frac{k+l}{2} \bmod k}} N/d,$$

$$\tau_{2;k,l}(N) = \sum_{\substack{d|N \\ d\equiv \frac{k-l}{2} \bmod k}} (N/d)^2 - \sum_{\substack{d|N \\ d\equiv \frac{k+l}{2} \bmod k}} (N/d)^2.$$

In [1, 2], Cho et al. considered various identities for q-series whose coefficients were given by divisor functions.

In §2 we will give formulas of divisor functions for the Weierstrass $\wp(N)$ function with $N = \frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{\tau+1}{2}$.

In §3 we shall give a theorem for divisor functions that is similar to Farkas results. Farkas [3] provided a relation between the sums of divisors satisfying congruence conditions and the sums of numbers of divisors satisfying congruence conditions. In the proof he took logarithmic derivative to theta functions and used the heat equation. In this section, however, we obtain a similar result by differentiating further. For any $n \geq 1$, we have

$$k \cdot \tau_{2;k,l}(n) = 2n \cdot E_{\frac{k-l}{2}}(n;k) + l \cdot \tau_{1;k,l}(n) + 2k \cdot \sum_{j=1}^{n-1} E_{\frac{k-l}{2}}(j;k)\tau_{1;k,l}(n-j).$$

Finally, we shall give a table for $E_1(N;3), \sigma(N), \tau_{1;3,1}(N)$ and $\tau_{2;3,1}(N)$ $(1 \leq N \leq 50)$ and state simulation results for them.

2. Divisor functions for Weierstrass $\wp$ function

N. J. Fine's list of identities of the basic hypergeometric series type appeared in [5]. In this section, we shall state two identities in [5, p. 78, p. 79]:

$$\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8 (q^4;q^4)_\infty^8} = 1 + 8\sum_{N=1}^{\infty} q^N (2 + (-1)^N) \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega, \tag{1}$$

$$\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} = \sum_{N \text{ odd}} \sigma(N) q^N. \tag{2}$$

Throughout this section, we shall fix the following notations: $K = \mathbb{Q}(\sqrt{D})$ with $D < 0$ is an imaginary quadratic field, $\mathfrak{H}$ the complex upper half plane and

$\tau \in K \cap \mathfrak{H}$. Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathfrak{H}$) be a lattice and $z \in \mathbb{C}$. The Weierstrass $\wp$ function relative to Λ_τ is defined by the series

$$\wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

and the Eisenstein series of weight $2k$ for Λ_τ with $k > 1$ is the series

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \omega^{-2k}.$$

We shall use the notations $\wp(z)$ and G_{2k} instead of $\wp(z; \Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when the lattice Λ_τ is fixed. Then the Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}.$$

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60 G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140 G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau).$$

For our purposes, we need the following propositions for the Weierstrass functions.

Proposition 2.1 ([9, 10])**.** *Let $\tau \in K \cap \mathfrak{H}$. Then*

(a) $\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2 (q^2; q^2)_\infty^4 (-q; q^2)_\infty^8.$

(b) $\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2 (q^2; q^2)_\infty^4 (q; q^2)_\infty^8.$

(c) $\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{\tau}{2}\right) = 16\pi^2 q (q^2; q^2)_\infty^4 (-q^2; q^2)_\infty^8.$

Proposition 2.2 ([6, 7, 8])**.** *Let $\tau \in K \cap \mathfrak{H}$. Then*

(a) $\wp\left(\frac{\tau}{2}\right) = -\frac{\pi^2}{3}((q^2; q^2)_\infty^4 (-q; q^2)_\infty^8 + 16q (q^2; q^2)_\infty^4 (-q^2; q^2)_\infty^8).$

(b) $\wp\left(\frac{\tau+1}{2}\right) = -\frac{\pi^2}{3}((q^2; q^2)_\infty^4 (-q; q^2)_\infty^8 - 32q (q^2; q^2)_\infty^4 (-q^2; q^2)_\infty^8).$

(c) $\wp\left(\frac{1}{2}\right) = \frac{2\pi^2}{3}((q^2; q^2)_\infty^4 (-q; q^2)_\infty^8 - 8q (q^2; q^2)_\infty^4 (-q^2; q^2)_\infty^8).$

Using (1), (2) and Proposition 2.2 we obtain the identity for $\wp$:

$$K_1(q) := -\frac{3}{\pi^2}\wp\left(\frac{\tau}{2}\right) \;=\; \frac{(q^2; q^2)_\infty^{20}}{(q)_\infty^8 (q^4; q^4)_\infty^8} + 16 \frac{q (q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4}$$

$$= 1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N) \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega + 16 \sum_{N \text{ odd}} \sigma(N)q^N$$

$$= 1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega + 24 \sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega + 16 \sum_{N \text{ odd}} \sigma(N)q^N$$

$$= 1 + 24\sum_{N=1}^{\infty} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega$$

$$= 1 + 24\sum_{N=1}^{\infty} \sigma_1(N;2)q^N,$$

$$K_2(q) := -\frac{3}{\pi^2}\wp\left(\frac{\tau+1}{2}\right) = \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 32\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}$$

$$= 1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N) \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega - 32 \sum_{N \text{ odd}} \sigma(N)q^N$$

$$= 1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega + 24 \sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega - 32 \sum_{N \text{ odd}} \sigma(N)q^N$$

$$= 1 + 24\sum_{N=1}^{\infty} (-1)^N q^N \sum_{\substack{\omega|N \\ \omega\text{odd}}} \omega$$

$$= 1 + 24\sum_{N=1}^{\infty} (-1)^N \sigma_1(N;2)q^N,$$

and

$$K_3(q) := \frac{3}{2\pi^2}\wp\left(\frac{1}{2}\right) = \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 8\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}$$

$$= 1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N) \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega - 8 \sum_{N \text{ odd}} \sigma(N)q^N$$

$$= 1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega + 24 \sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega - 8 \sum_{N \text{ odd}} \sigma(N)q^N$$

$$= 1 + 24\sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega$$

$$= 1 + 24\sum_{N \text{ even}} \sigma_1(N;2)q^N.$$

We summarize the above as follows.

Theorem 2.3. *Let* $S_1 := \sum_{N\ odd} \sigma_1(N;2)q^N$ *and* $S_2 := \sum_{N\ even} \sigma_1(N;2)q^N$. *Then*

(a) $K_1(q) = 1 + 24S_1 + 24S_2$.
(b) $K_2(q) = 1 - 24S_1 + 24S_2$.
(c) $K_3(q) = 1 + 24S_2$.

3. A relation of divisor functions

Theta functions with characteristics are defined as

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i \left(n+\frac{\epsilon}{2}\right)^2 \tau + 2\pi i \left(n+\frac{\epsilon}{2}\right)\left(z+\frac{\epsilon'}{2}\right)\right),$$

where $\epsilon, \epsilon' \in \mathbb{R}$, $z \in \mathbb{C}$ and $\tau \in \mathfrak{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Then

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} \text{ with } q = e^{2\pi i\tau}.$$

Note that for $q_z := e^{2\pi i z}$, we have

$$\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (z, 3\tau) = e^{\pi i/6} q^{1/24} q_z^{1/6} \prod_{n=0}^{\infty} (1-q^{3n+3})(1-q^{3n+2}q_z)(1-q^{3n+1}q_z^{-1}),$$

it is true by the Jacobi triple product identity. Thus

$$\begin{aligned} \frac{\theta' \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (0,3\tau)}{\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (0,3\tau)} &= \frac{\partial}{\partial z} \log \theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (z, 3\tau)\bigg|_{z=0} \\ &= 2\pi i \left(\frac{1}{6} + \sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right) \\ &= 2\pi i \left(\frac{1}{6} + \sum_{N=1}^{\infty} E_1(N;3) q^N\right). \end{aligned}$$

Farkas showed in [3, 4] that for $N, k, l \geq 1$ with $k \equiv l \pmod 2$, $k \geq 3$ and $l \geq k-2$,

$$\begin{aligned} k \cdot \tau_{1;k,l}(N) = 2\sigma_{0,\frac{k-l}{2},\frac{k+l}{2}}(N;k) + l \cdot E_{\frac{k-l}{2}}(N;k) \\ + k \cdot \sum_{j=1}^{N-1} E_{\frac{k-l}{2}}(j;k) E_{\frac{k-l}{2}}(N-j;k). \end{aligned}$$

Finally, we obtain a similar result by differentiating further. More precisely,

Theorem 3.1. *For any $n \geq 1$, we have*

$$k \cdot \tau_{2;k,l}(n) = 2n \cdot E_{\frac{k-l}{2}}(n;k) + l \cdot \tau_{1;k,l}(n) + 2k \cdot \sum_{j=1}^{n-1} E_{\frac{k-l}{2}}(j;k)\tau_{1;k,l}(n-j).$$

Proof. Let $q_z = e^{2\pi i z}$, $q = e^{2\pi i \tau}$ with $z \in \mathbb{C}$ and $\tau \in \mathfrak{H}$, and $s = \frac{l}{k} \in \mathbb{Q}$ with positive integers l, k such that $k \equiv l \bmod 2$, $k \geq 3$, and $l \leq k-2$. Then we define theta functions as

$$\theta\begin{bmatrix} s \\ 1 \end{bmatrix}(z,\tau) = e^{\pi i s/2} q^{s^2/8} q_z^{s/2} (q)_\infty (q^{\frac{1+s}{2}} q_z)_\infty (q^{\frac{1-s}{2}} q_z^{-1})_\infty.$$

By taking logarithmic derivative with respect to z we have

$$\frac{1}{2\pi i}\frac{\partial}{\partial z}\log\theta\begin{bmatrix} s \\ 1 \end{bmatrix}(z,\tau) = \frac{s}{2} + \sum_{n=0}^{\infty}\left(\frac{-q^{n+\frac{1+s}{2}} q_z}{1-q^{n+\frac{1+s}{2}} q_z} + \frac{q^{n+\frac{1-s}{2}} q_z^{-1}}{1-q^{n+\frac{1-s}{2}} q_z^{-1}}\right).$$

If we differentiate more with respect to z, we get

$$\begin{aligned} &\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix} s \\ 1 \end{bmatrix}(z,\tau) \\ = &-\sum_{n=0}^{\infty}\left(\frac{q^{n+\frac{1+s}{2}} q_z}{(1-q^{n+\frac{1+s}{2}} q_z)^2} + \frac{q^{n+\frac{1+s}{2}} q_z^{-1}}{(1-q^{n+\frac{1+s}{2}} q_z^{-1})^2}\right), \\ &\frac{1}{(2\pi i)^3}\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix} s \\ 1 \end{bmatrix}(z,\tau) \\ = &-\sum_{n=0}^{\infty}\left(\frac{q^{n+\frac{1+s}{2}} q_z + (q^{n+\frac{1+s}{2}} q_z)^2}{(1-q^{n+\frac{1+s}{2}} q_z)^3} - \frac{q^{n+\frac{1-s}{2}} q_z^{-1} + (q^{n+\frac{1-s}{2}} q_z^{-1})^2}{(1-q^{n+\frac{1-s}{2}} q_z^{-1})^3}\right). \end{aligned}$$

We now evaluate at $z = 0$ to obtain

$$\begin{aligned} &\frac{1}{2\pi i}\frac{\partial}{\partial z}\log\theta\begin{bmatrix} s \\ 1 \end{bmatrix}(z,k\tau)\Big|_{z=0} \\ = &\frac{l}{2k} + \sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} q^{(kn+\frac{k-l}{2})m} - \sum_{m=1}^{\infty} q^{(kn+\frac{k+l}{2})m}\right) \\ = &\frac{l}{2k} + \sum_{n=1}^{\infty} E_{\frac{k-l}{2}}(n;k) q^n, \end{aligned} \tag{3}$$

$$\begin{aligned} &\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix} s \\ 1 \end{bmatrix}(z,k\tau)\Big|_{z=0} \\ = &-\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} m q^{(kn+\frac{k-l}{2})m} + \sum_{m=1}^{\infty} m q^{(kn+\frac{k+l}{2})m}\right) \\ = &-\sum_{n=0}^{\infty} \tau_{1;k,l}(n) q^n \end{aligned} \tag{4}$$

and

$$
\begin{aligned}
&\frac{1}{(2\pi i)^3}\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,k\tau)\Big|_{z=0}\\
&= -\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} m^2 q^{(kn+\frac{k+l}{2})m} - \sum_{m=1}^{\infty} m^2 q^{(kn+\frac{k-l}{2})m}\right) \qquad (5)\\
&= \sum_{n=1}^{\infty}\tau_{2;k,l}(n)q^n.
\end{aligned}
$$

Here we recall that theta functions satisfy the heat equation, that is,

$$\frac{\partial^2}{\partial z^2}\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau) = 4\pi i\frac{\partial}{\partial\tau}\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau).$$

Hence we achieve

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau) &= \frac{\theta''\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)} - \left(\frac{\theta'\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)}\right)^2\\
&= 4\pi i\frac{\partial}{\partial\tau}\frac{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)} - \left(\frac{\theta'\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,\tau)}\right)^2\\
&= 4\pi i\frac{\partial}{\partial\tau}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau) - \left(\frac{\partial}{\partial z}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau)\right)^2.
\end{aligned}
$$

Here the symbol $'$ stands for the partial derivative with respect to z. If we take logarithmic derivative to theta function with respect to τ, then we derive

$$
\begin{aligned}
&\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau)\\
&= \frac{s^2}{8} - \sum_{n=0}^{\infty}\left(\frac{(n+1)q^{n+1}}{1-q^{n+1}} + \frac{(n+\frac{1+s}{2})q^{n+\frac{1+s}{2}}q_z}{1-q^{n+\frac{1+s}{2}}q_z} + \frac{(n+\frac{1-s}{2})q^{n+\frac{1-s}{2}}q_z^{-1}}{1-q^{n+\frac{1-s}{2}}q_z^{-1}}\right).
\end{aligned}
$$

So we establish

$$
\begin{aligned}
&\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau)\\
&= -2(2\pi i)^3\sum_{n=0}^{\infty}\left(\left(n+\frac{1+s}{2}\right)\frac{q^{n+\frac{1+s}{2}}q_z}{(1-q^{n+\frac{1+s}{2}}q_z)^2} - \left(n+\frac{1-s}{2}\right)\frac{q^{n+\frac{1-s}{2}}q_z^{-1}}{(1-q^{n+\frac{1-s}{2}}q_z^{-1})^2}\right)\\
&\quad -2\frac{\partial}{\partial z}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau)\cdot\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,\tau).
\end{aligned}
$$

Thus by evaluating at $z=0$ we have

$$\frac{1}{(2\pi i)^3}\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,k\tau)\Big|_{z=0}$$

$$= 2\left(\frac{1}{k}\sum_{n=0}^{\infty}\left(kn+\frac{k-l}{2}\right)\frac{q^{kn+\frac{k-l}{2}}}{(1-q^{kn+\frac{k-l}{2}})^2}-\left(kn+\frac{k+l}{2}\right)\frac{q^{kn+\frac{k+l}{2}}}{(1-q^{kn+\frac{k+l}{2}})^2}\right)$$
$$+\left(\frac{l}{2k}+\left(\sum_{n=1}^{\infty}E_{\frac{k-l}{2}}(n;k)q^n\right)\left(\sum_{n=0}^{\infty}\tau_{1;k,l}(n)q^n\right)\right),$$

which implies

$$\sum_{n=1}^{\infty}\tau_{2;k,l}(n)q^n$$
$$=\frac{2}{k}\left(\sum_{n=0}^{\infty}n\left(\sum_{\substack{0<d|n\\ d\equiv\frac{k-l}{2} \bmod k}}1\right)q^n-\sum_{n=0}^{\infty}n\left(\sum_{\substack{0<d|n\\ d\equiv\frac{k+l}{2} \bmod k}}1\right)q^n\right)$$
$$+\left(\frac{l}{2k}+\sum_{n=1}^{\infty}E_{\frac{k-l}{2}}(n;k)q^n\right)\left(\sum_{n=0}^{\infty}\tau_{1;k,l}(n)q^n\right)$$
$$=\frac{2}{k}\sum_{n=1}^{\infty}nE_{\frac{k-l}{2}}(n;k)q^n+\frac{l}{k}\sum_{n=1}^{\infty}\tau_{1;k,l}(n)q^n$$
$$+2\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n-1}E_{\frac{k-l}{2}}(j;k)\tau_{1;k,l}(n-j)\right)q^n.$$

This completes the proof of theorem. □

Remark 3.2. Let $k=3, l=1, n=3m+2$. Then $\tau_{1;3,1}(3m+2)=\sigma(3m+2)$ and $\tau_{2;3,1}(3m+2)=\sum_{0<d|(3m+2)}d^2\chi(d)$, where σ is the usual divisor sum function and $\chi(d)$ is defined by $\chi(d)=1,-1,0$ according as $d\equiv 1,-1,0 \bmod 3$. Since $E_1(3j;3)=E_1(j;3)$ and $E_1(3j+2;3)=0$,

$$3\tau_{2;3,1}(3m+2)$$
$$=\tau_{1;3,1}(3m+2)+6\sum_{j=0}^{m}(E_1(3j;3)\tau_{1;3,1}(3(m-j)+2)$$
$$+E_1(3j+1;3)\tau_{1;3,1}(3(m-j)+1)+E_1(3j+2;3)\tau_{1;3,1}(3(m-j)))$$
$$=\sigma(3m+2)+6\sum_{j=0}^{m}E_1(j;3)\sigma(3(m-j)+2)$$
$$+6\sum_{j=0}^{m}E_1(3j+1;3)\sigma(3(m-j)+1).$$

Thus the equation above is equivalent to

$$
\begin{aligned}
&3\sum_{m=0}^{\infty}\tau_{2;3,1}(3m+2)q^m\\
&=\sum_{m=0}^{\infty}\sigma(3m+2)q^m+6\left(\sum_{n=0}^{\infty}E_1(n;3)q^n\right)\left(\sum_{n=0}^{\infty}\sigma(3n+2)q^n\right)\\
&\quad+6\left(\sum_{n=0}^{\infty}E_1(3n+1;3)q^n\right)\left(\sum_{n=0}^{\infty}\sigma(3n+1)q^n\right).
\end{aligned}
$$

Remark 3.3. As is well known $E_1(N;3)$, $E_1(N;4)$, and $E_{1,3}(N;8)$ are important arithmetical functions in the study of sums of squares (see [5]). In particular, when $p_i \equiv 1 \bmod 3$, $q_i \equiv 2 \bmod 3$, $u_i \equiv 1 \bmod 4$, $w_i \equiv 3 \bmod 4$, $g_i \equiv 1 \bmod 8$, $x_i \equiv 3 \bmod 8$, $y_i \equiv 5 \bmod 8$ and $z_i \equiv 7 \bmod 8$ are primes, we can readily get that

$$
\begin{aligned}
&E_1(3^n p_1^{e_1}\cdots p_r^{e_r} q_1^{f_1}\cdots q_s^{f_s};3)\\
&=\begin{cases}(e_1+1)\cdots(e_r+1) & \text{if } f_i(1\le i\le s)\equiv 0 \bmod 2\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&E_1(2^n u_1^{c_1}\cdots u_a^{c_a} w_1^{d_1}\cdots w_b^{d_b};4)\\
&=\begin{cases}(c_1+1)\cdots(c_a+1) & \text{if } d_i(1\le i\le b)\equiv 0 \bmod 2\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&E_{1,3}(2^n g_1^{\alpha_1}\cdots g_s^{\alpha_s} x_1^{\beta_1}\cdots x_a^{\beta_a} y_1^{\gamma_1}\cdots y_b^{\gamma_b} z_1^{\delta_1}\cdots z_c^{\delta_c};8)\\
&=\begin{cases}(\alpha_1+1)\cdots(\alpha_s+1)(\beta_1+1)\cdots(\beta_a+1)\\ \text{if } \gamma_i(1\le i\le b)\equiv\delta_j(1\le j\le c)\equiv 0 \bmod 2,\\ 0\\ \text{otherwise.}\end{cases}
\end{aligned}
$$

Remark 3.4. In computer networking and computer science, bandwidth, network bandwidth, data bandwidth or digital bandwidth is a bit rate measure or available or consumed data communication resources expressed in bits/second or multiples of it. When we send some data transmission in computer networking, they want to find small computation algorithms for them. We think our study is very important about a simple computation in networking. Using MATLAP, we find $E_1(N;3)$, $\sigma(N)$, $\tau_{1;3,1}(N)$ and $\tau_{2;3,1}(N)$ $(1\le N\le 50)$. See Figures 1-5.

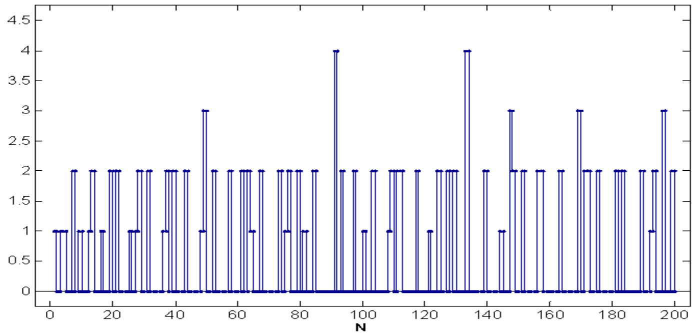

Figure 1. $E_1(N;3)$ $(1 \leq N \leq 200)$

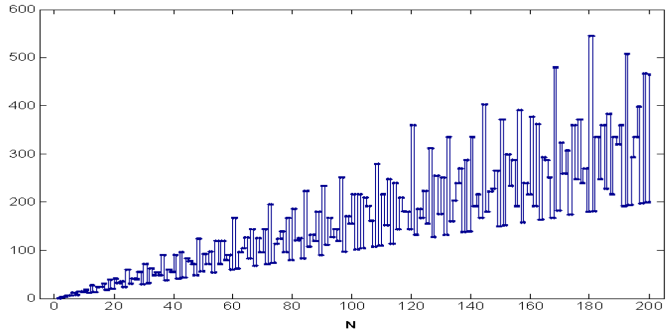

Figure 2. $\sigma(N)$ $(1 \leq N \leq 200)$

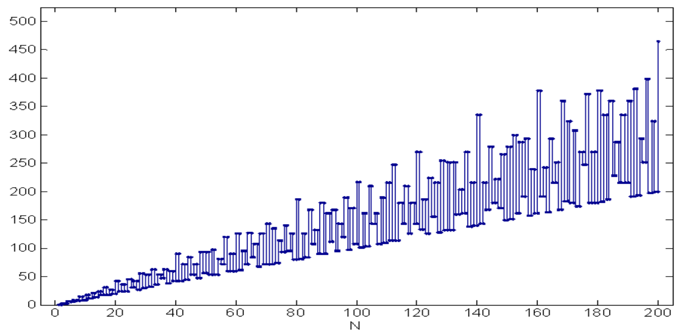

Figure 3. $\tau_{1;3,1}(N)$ $(1 \leq N \leq 200)$

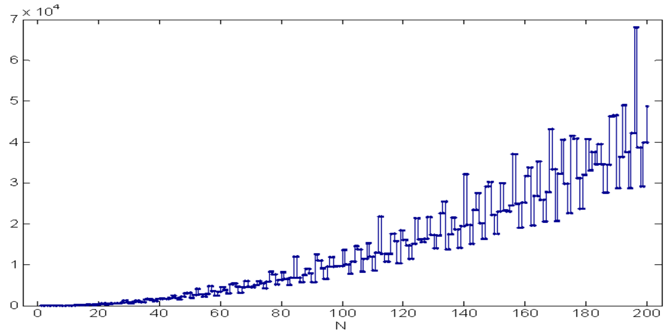

FIGURE 4. $\tau_{2;3,1}(N)$ $(1 \leq N \leq 200)$

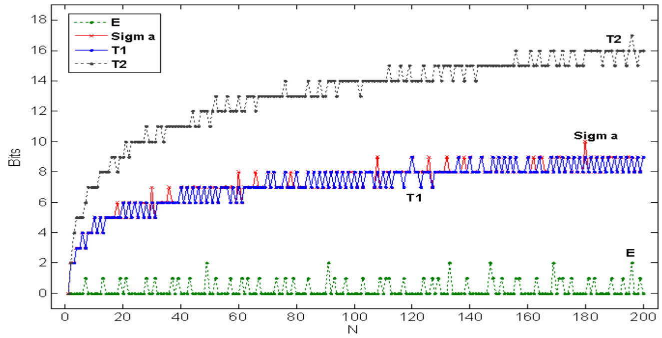

FIGURE 5. The state simulation result for bits $(1 \leq N \leq 200)$

N	$E_1(N;3)$	$\sigma(N)$	$\tau_{1;3,1}(N)$	$\tau_{2;3,1}(N)$	N	$E_1(N;3)$	$\sigma(N)$	$\tau_{1;3,1}(N)$	$\tau_{2;3,1}(N)$
1	1	1	1	1	26	0	42	42	588
2	0	3	3	3	27	1	40	27	729
3	1	4	3	9	28	2	56	56	1344
4	1	7	7	21	29	0	30	30	840
5	0	6	6	24	30	0	72	54	648
6	0	12	9	27	31	2	32	32	1024
7	2	8	8	64	32	0	63	63	1323
8	0	15	15	75	33	0	48	36	1080
9	1	13	9	81	34	0	54	54	864
10	0	18	18	72	35	0	48	48	1536
11	0	12	12	120	36	1	91	63	1701
12	1	28	21	189	37	2	38	38	1444
13	2	14	14	196	38	0	60	60	1200
14	0	24	24	192	39	2	56	42	1764
15	0	24	18	216	40	0	90	90	1800
16	1	31	31	341	41	0	42	42	1680
17	0	18	18	288	42	0	96	72	1728
18	0	39	27	243	43	2	44	44	1936
19	2	20	20	400	44	0	84	84	2520
20	0	42	42	504	45	0	78	54	1944
21	2	32	24	576	46	0	72	72	1584
22	0	36	36	360	47	0	48	48	2208
23	0	24	24	528	48	1	124	93	3069
24	0	60	45	675	49	3	57	57	3249
25	1	31	31	651	50	0	93	93	1953

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Daeyeoul Kim
National Institute for Mathematical Sciences
Daejeon 305-811, Korea
E-mail address: daeyeoul@nims.re.kr

Min-Soo Kim
Division of Cultural Education
Kyungnam University
Changwon 631-701, Korea
E-mail address: mskim@kyungnam.ac.kr