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DYNAMICAL SYSTEMS AND GROUPOID ALGEBRAS ON HIGHER RANK GRAPHS

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Abstract. For a locally compact higher rank graph Λ, we construct a two-sided path space Λ^Δ with shift homeomorphism σ and its corresponding path groupoid Γ. Then we find equivalent conditions of aperiodicity, cofinality and irreducibility of Λ in (Λ^Δ, σ), Γ, and the groupoid algebra $C^*(\Gamma)$.

1. Preliminary

Since Kumjian and Pask's ground breaking paper for higher rank graph ([3]), main interest of higher rank graph (or k-graph) algebras has been the 'one-sided' path space Λ^Ω of k-graph Λ and its corresponding path groupoid $\mathcal{G}_\Lambda$. On the other hand, as in the case of subshift of finite types and Cuntz-Krieger algebras ([1]), two-sided path space Λ^Δ was constructed by Kumjian and Pask in [4] to study Smale space structure on k-graphs. This paper is a partial result of the author's attempt to understand Kumjian and Pask's results on $\mathbb{Z}^k$-actions on k-graphs ([4]).

Although one-sided path space is easier to use combinatorially, in the view point of dynamical systems, two sided path space Λ^Δ is more natural for shift map σ is a homeomorphism on Λ^Δ comparing to the fact that σ is a local homeomorphism on Λ^Ω. We construct a dynamical system (Λ^Δ, σ) and its corresponding groupoid Γ from the two-sided path space of a k-graph Λ. Then we show that some basic properties of Λ are naturally transferred to properties of (Λ^Δ, σ), Γ, and $C^*(\Gamma)$ the groupoid C^*-algebra of Γ.

For this purpose, we make an assumption on our higher rank graph Λ that it is locally compact with no sources to assure that Λ^Δ is a locally compact Hausdorff space with infinitely many elements. Under this assumption, we show that

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aperiodicity of Λ, topological freeness of (Λ^Δ, σ), and essential principality of Γ are equivalent to each other (Proposition 3.1).

This result may need a little explain: In k-graphs, obtaining aperiodicity with combinatorial method is not an easy task. But, in dynamical systems, topological freeness is a relatively mild restriction, e.g., every minimal system is topologically transitive, and every topologically transitive system is topologically free ([7]). And, in groupoids, essentially principal property implies that there is an order preserving bijective relation between the open invariant subsets of the unit space of a groupoid and its groupoid C^*-algebra ([5]). Because our groupoid Γ comes from the dynamical system (Λ^Δ, σ), invariant subsets of Γ^0 are strongly related to orbits and invariant subsets of (Λ^Δ, σ). So dynamical properties and groupoid properties will interdispaly those of k-graphs. After we give relevant definitions of k-graphs, dynamical systems and groupoids in Section 2, we use this property to find equivalent conditions of cofinality and irreducibility of Λ on (Λ^Δ, σ), Γ, and $C^*(\Gamma)$ in Section 3.

2. Higher Rank Graphs

We briefly review definitions and basic properties of k-graphs, dynamical systems and groupoids. All materials in this section are taken from [3, 4, 5, 7].

Definition 2.1 ([3, 4])**.** A k-graph is a pair (Λ, d) where

$$\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Hom}(\Lambda), r, s)$$

is a countable small category and $d\colon \Lambda \to \mathbb{N}^k$ is a morphism, called the degree map, satisfying the *factorization property*: For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there exist unique elements $\mu, \nu \in \Lambda$ such that

$$d(\mu) = m,\ d(\nu) = n \text{ and } \lambda = \mu\nu.$$

Every $\lambda \in \Lambda$ is called a *path*. For every nonzero $n \in \mathbb{N}^k$, define $\Lambda^n = d^{-1}(n)$ and identify Λ^0 with $\mathrm{Obj}(\Lambda)$. Let $r, s\colon \Lambda \to \Lambda^0$ denote the range and source maps. We abbreviate (Λ, d) to Λ when there is no confusion.

Standing Assumption. Throughout this paper, every k-graph is locally finite and has no sources in the sense of Kumjian and Pask ([3, 4]), every groupoid is a topological groupoid, and an ideal of a C^*-algebra means a closed two-sided ideal.

$\mathbb{Z}^k$-actions on k-graphs ([4])**.** Suppose that (Δ, d) is a k-graph defined by

$$\Delta = \{(m, n) \mid m, n \in \mathbb{Z}^k \text{ and } m \le n\}$$

with the structure maps

$$(l,m)\cdot(m,n)=(l,n),\ \ r(m,n)=m,\ \ s(m,n)=n \text{ and } d(m,n)=m-n.$$

Let Λ be a k-graph, and the corresponding two-sided infinite path space be set by

$$\Lambda^{\Delta}=\{x\colon \Delta\to\Lambda \mid \ x \text{ is a } k\text{-graph homomorphism }\}$$

Then Λ^{Δ} is a zero-dimensional space consisting of 'two-sided' paths on Λ. A topology is endowed on Λ^{Δ} where its basis is given by

$$Z(\lambda,n)=\{x\in\Lambda^{\Delta}\mid x(n,n+d(\lambda))=\lambda\}$$

with $n\in\mathbb{Z}^k$ and $\lambda\in\Lambda$. It is not difficult to check that Λ^{Δ} is compact (locally compact, respectively) if Λ^0 is finite (infinite, respectively) so that λ^{Δ} is a metrizable space. A metric ρ on Λ^{Δ} is defined as follows: For $e=(1,\dots,1)\in\mathbb{Z}^k$ and $j\in\mathbb{N}$, let $\theta_j\in\Delta$ be the element $(-je,je)$. Given $x,y\in\Lambda^{\Delta}$, set

$$h(x,y)=\begin{cases}1 & x(0)\neq y(0)\\ 1+\sup\{j\mid x(\theta_j)=y(\theta_j)\} & \text{otherwise.}\end{cases}$$

Then, for a fixed number $r\in(0,1)$, a metric ρ is defined by the formula

$$\rho(x,y)=r^{h(x,y)} \text{ for } x,y\in\Lambda^{\Delta}.$$

Let σ be the action of $\mathbb{Z}^k$ on Λ^{Δ} by the homeomorphism $\sigma^p\colon\Lambda^{\Delta}\to\Lambda^{\Delta}$, $p\in\mathbb{Z}^k$, defined by

$$(\sigma^p x)(m,n)=x(m+p,n+p).$$

Definition 2.2 ([3, 4])**.** A k-graph Λ is called *irreducible* if, for every $u,v\in\Lambda^0$, there is $\lambda\in\Lambda$ with $d(\lambda)\neq 0$ such that $u=r(\lambda)$ and $v=s(\lambda)$. And Λ is called *two-sided cofinal* if, for every $x\in\Lambda^{\Delta}$ and $v\in\Lambda^0$, there are $\alpha,\beta\in\Lambda$ such that $s(\alpha)=x(m,m)$, $r(\alpha)=v=s(\beta)$ and $r(\beta)=x(n,n)$ for some $m,n\in\mathbb{Z}^k$.

Note that Kumjian and Pask defined cofinality of Λ for one-sided path space as follows ([3]): For every $x\in\Lambda^{\Omega}$ and $v\in\Lambda^0$, there is an $\alpha\in\Lambda$ such that $v=s(\alpha)$ and $r(\alpha)=x(n,n)$ for some $n\in\mathbb{N}^k$. We modified their definition for two-sided case.

Remark 2.3. In our definition of two-sided cofinality, we can set $m\leq n$: If $m\neq n$, let $p\in\mathbb{Z}^k$ be such that $m\leq p$ and $n\leq p$. Then $\gamma=\beta\cdot x(n,p)$ is a path from v to $x(p,p)$ such that $m\leq p$.

Definition 2.4 ([3])**.** For $x\in\Lambda^{\Delta}$ and $p\in\mathbb{Z}^k$, p is called a *period* of x if, for every $(m,n)\in\Delta$, $\sigma^p x(m,n)=x(m,n)$. That x is called *periodic* if it has a nonzero period, *eventually periodic* if $\sigma^n x$ is periodic for some $n \in \mathbb{N}^k$. Otherwise x is called to be *aperiodic*. A k-graph Λ is said to satisfy *aperiodic condition* if, for every $v \in \Lambda^0$, there is an aperiodic path $x \in \Lambda^\Delta$ such that $x(0,0) = v$.

Two-sided path groupoids ([4])**.** Suppose that Λ is a k-graph with its corresponding two-sided path space Λ^Δ. We define the 'two-sided' path groupoid of Λ by

$$\Gamma = \{(x,n,y) \colon x,y \in \Lambda^\Delta, n \in \mathbb{Z}^k, \sigma^l x = \sigma^m y, n = l - m \text{ for some } l,m \in \mathbb{Z}^k\}$$

with the set of composable pairs

$$\Gamma^{(2)} = \{((x,n,y),(w,m,z)) \in \Gamma \times \Gamma : y = w\}$$

and the structure maps

$$s(x,n,y) = (x,0,x),\ r(x,n,y) = (y,0,y),$$
$$(x,n,y)(y,m,z) = (x,n+m,z), \text{ and } (x,n,y)^{-1} = (y,-n,x).$$

The unit space of Γ, denoted Γ^0, is identified with Λ^Δ via the diagonal map, and the isotropy group bundle is given by

$$I = \{(x,n,x) \in \Gamma\}.$$

Theorem 2.5 ([2, 3])**.** *Suppose that Λ is a k-graph and that Γ is its two-sided path groupoid as defined above. Then there is a topology on Γ that makes Γ a second countable, r-discrete, locally compact, Hausdorff groupoid with the Haar system given by the counting measures.*

Groupoids and dynamical systems.

Definition 2.6 ([5])**.** Let G be a topological groupoid with open range map and G^0 its unit space. A subset E of Γ^0 is said to be *invariant* if $r \circ s^{-1}(E) = E$. Then we say that G is

(1) *minimal* if the only open invariant subsets of G^0 are the empty set $\emptyset$ and G^0 itself,
(2) *irreducible* if every nonempty open invariant subset of G^0 is dense, and
(3) *essentially principal* if G is locally compact and, for every closed invariant subset F of G^0, $\{u \in F : r^{-1}(u) \cap s^{-1}(u) = \{u\}\}$ is dense in F.

Notation 2.7. For a groupoid G, we denote $C^*(G)$ the groupoid C^*-algebra of G.

The following theorem gives a relation between gropoids and their groupoid algerba:

Theorem 2.8 ([5, II.4.5 and 4.6])**.** *Suppose that G is a groupoid with its groupoid algebra $C^*(G)$. Let $\mathcal{O}(G)$ be the lattice of invariant open subsets of the unit space G^0 of G and $\mathcal{I}(C^*(G))$ the lattice of ideals of $C^*(G)$. Then there is a one-to-one order preserving relation from $\mathcal{O}(G)$ to $\mathcal{I}(C^*(G))$. Moreover, if G is essentially principal, then the correspondence is bijective.*

Definition 2.9 ([7])**.** Suppose that X is a locally compact Hausdorff space and that, for every $p \in \mathbb{Z}^k$, $h^p\colon X \to X$ is a homeomorphism. Then the dynamical system (X, h) is called

(1) *minimal* if every orbit is dense in X,
(2) *topologically transitive* if for every pair of open sets $\{U, V\}$ there is an $n \in \mathbb{Z}^k$ such that $h^n(U) \cap V \neq \emptyset$, and
(3) *topologically free* if $\mathrm{Per}^\infty(X)$ is dense in X.

3. Main Results

We will find equivalent conditions of aperiodicity of k-graphs in their corresponding dynamical systems and two-sided path groupoids. Then we use this property to investigate cofinality and irreducibility of k-graphs from dynamical systems, groupoids, and groupoid C^*-algebras.

Before go further, we may need to mention that similar relations between dynamical systems and groupoids are already proved in [8] under a little different conditions.

Proposition 3.1. *For a k-graph Λ, the following are equivalent:*

(1) *Λ satisfies the aperiodic condition.*
(2) *(Λ^Δ, σ) is topologically free.*
(3) *Γ is essentially principal.*

Proof. (1) $\iff$ (2). If Λ satisfies the aperiodic condition, then, for every $\lambda \in \Lambda$ with $d(\lambda) = n$, there are aperiodic paths $x, y \in \Lambda^\Delta$ such that $x(0,0) = r(\lambda)$ and $y(0,0) = s(\lambda)$. Then $z = \sigma^n y \cdot \lambda \cdot x$ defined by

$$z(p,q) = \begin{cases} y(p+n,0) & q = -d(\lambda) \\ \lambda & p = n = -d(\lambda) \text{ and } q = 0 \\ x(0,q) & p = 0 \end{cases}$$

is an aperiodic path, and, for $Z(\lambda, m)$ with $m \in \mathbb{Z}^k$, we have $\sigma^{-n-m} z(m, m+n) = s(-n, 0) = \lambda$. So the aperiodic points are dense in Λ^Δ, and (Λ^Δ, σ) is a topologically free system.

If aperiodic points are dense in Λ^Δ, then, for $v \in \Lambda^0$ and $\lambda \in \Lambda$ such that $s(\lambda) = v$, there is an aperiodic path $x \in Z(\lambda, 0)$. Then we have $x(0, d(\lambda)) = \lambda$ and $x(0,0) = s(\lambda) = v$. Therefore Λ satisfies the aperiodic condition.

(2) $\Longleftrightarrow$ (3). Let $A = \{x \in \Lambda^\Delta : \sigma^k(x) = \sigma^l(x) \text{ implies } k = l\}$, the set of aperiodic points in Λ^Δ, and $B = \{b = (x, 0, x) \in \Gamma^0 : \{b\} = r^{-1}(b) \cap s^{-1}(b) \subset \Gamma\}$, the set of elements in Γ^0 with trivial isotopy. Then it is trivial that $x \in A$ if and only if $(x, 0, x) \in B$. Hence A is dense in Λ^Δ if and only if B is dense in Γ^0.

For a closed invariant subset F of Γ^0, we note $\{u \in F : r^{-1}(u) \cap s^{-1}(u) = \{u\}\} = F \cap B$. Thus B is dense in Γ^0 implies that, for every closed invariant subset F of Γ^0, $F \cap B$ is dense in F. Conversely, density of $F \cap B$ in F implies B is dense in Γ^0 when we set $F = \Gamma^0$. □

Proposition 3.2. *For a k-graph Λ, the followings are equivalent:*

(1) Λ *is a two-sided cofinal graph.*
(2) (Λ^Δ, σ) *is a minimal system.*
(3) Γ *is a minimal groupoid.*
(4) $C^*(\Gamma)$ *is a simple algebra.*

Proof. (1) $\Longrightarrow$ (2). Suppose that Λ is a two-sided cofinal graph and $\lambda \in \Lambda$. Then for $s(\lambda)$ and $r(\lambda)$, there are $\alpha, \beta \in \Lambda$ and $m, n \in \mathbb{Z}^k$ such that $s(\alpha) = x(m, m)$, $r(\alpha) = s(\lambda)$, $r(\lambda) = s(\beta)$, and $r(\beta) = x(n, n)$. As in the case of Remark 2.3, we may set $m \le n$. Then $y \in Z(\lambda, 0)$ defined by

$$y(p,q) = \begin{cases} x(m+p+d(\alpha), m) & q = -d(\alpha) \\ \alpha & p = -d(\alpha), q = 0 \\ \lambda & p = 0, q = d(\lambda) \\ \beta & p = d(\lambda), q = d(\lambda) + d(\beta) \\ x(n, n+q-d(\lambda)-d(\beta)) & p = d(\lambda) + d(\beta) \end{cases}$$

has the same orbit as that of x, and $\sigma^{-l}(y) \in Z(\lambda, l)$ for every $l \in \mathbb{Z}^k$. Hence (Λ^Δ, σ) is a minimal system.

(2) $\Longrightarrow$ (1). Suppose that (Λ^Δ, σ) is a minimal system. For every $x \in \Lambda^\Delta$ and $v \in \Lambda^0$, let α and β be paths such that $r(\alpha) = v = s(\beta)$. Since the orbit of x is dense in Λ^Δ, there is an $n \in \mathbb{Z}^k$ such that $\sigma^n x \in Z(\alpha\beta, 0)$. Then we have $x(n, n + d(\alpha)) = \alpha$ and $x(n + d(\alpha), n + d(\alpha) + d(\beta)) = \beta$ such that $s(\alpha) = x(n, n)$ and $r(\beta) = x(n + d(\alpha) + d(\beta), n + d(\alpha) + d(\beta))$ Thus Λ is a two-sided cofinal graph.

(2) $\Longrightarrow$ (3). Remark that for a subset E of Γ^0, $(y, 0, y) \in r \circ s^{-1}(E)$ if and only if $y = \sigma^n(x)$ for some $(x, 0, x) \in E$ and $n \in \mathbb{Z}^k$, i.e., $r \circ s^{-1}(E)$ is identified

as $\cup_{(x,0,x)\in E}\mathrm{Orb}(x)$. So minimality of (Λ^Δ, σ) implies that the only open invariant subsets of Γ^0 are empty set and Γ^0.

(3) $\Longrightarrow$ (2). Assume that (Λ^Δ, σ) is not a minimal system. Then there is an $x \in \Lambda^\Delta$ such that $\overline{\mathrm{Orb}(x)} \subsetneq \Lambda^\Delta$. Let $Y = \Lambda^\Delta - \overline{\mathrm{Orb}(x)}$ and $E = \{(y, 0, y) \colon y \in Y\}$. We show that E is an invariant open subset of Γ^0.

First we remark that E is an open subset of Γ^0 ([5]) and $E \subset r \circ s^{-1}(E)$. We also note that, for $(a, 0, a) \in r \circ s^{-1}(E)$, there are $y \in Y$ and $n \in \mathbb{Z}^k$ such that $\sigma^n(y) = a$.

Assume that E is not an invariant open subset of Γ^0, and obtain a contradiction. Then we have, as $E \subset r \circ s^{-1}(E)$ and $r \circ s^{-1}(E)$ is open in Γ^0, $r \circ s^{-1}(E) \cap (\Gamma^0 - E) \neq \emptyset$ and $r \circ s^{-1}(E) \cap \mathrm{Int}(\Gamma^0 - E) \neq \emptyset$. Thus there exists an

$$(a, 0, a) \in r \circ s^{-1}(E) \cap \mathrm{Int}(\Gamma^0 - E)$$

such that $a \in \mathrm{Orb}(x)$ and $a \in \mathrm{Orb}(y)$ for some $y \in E$. So there are $n, m \in \mathbb{Z}^k$ such that $\sigma^n(y) = a$ and $\sigma^m(x) = a$. But this is a contradiction to the fact that $y \in Y = \Lambda^\Delta - \overline{\mathrm{Orb}(x)}$. Therefore we have a nontrivial invariant open subset of Γ^0, and Γ is not a minimal groupoid.

(3) $\Longrightarrow$ (4). We recall that every minimal system is topologically free ([7]). Thus Γ is essentially principal by Proposition 3.1, and there is a bijective relation between the lattice of invariant open subsets of the unit space of Γ and the lattice of closed two-sided ideal of $C^*(\Gamma)$. Then $C^*(\Gamma)$ is a simple algebra as Γ is a minimal groupoid.

(4) $\Longrightarrow$ (3) If $C^*(\Gamma)$ is simple, then $C^*(\Gamma)$ does not have a nontrivial ideal, and Γ cannot have any nontrivial open invariant subset by Theorem 2.8. Thence Γ is a minimal groupoid. □

Remark 3.3. In the above proposition, we need aperiodicty of Λ only for (3) $\Longrightarrow$ (4). Even in one-sided case ([3, Proposition 4.8]), because of Renault's theorem (Theorem 2.8), simplicity of $C^*(\Lambda)$ implies cofinality of Λ does not require the aperiodic condition of Λ.

As an application of our dynamical approach to k-graphs, it may be noteworthy to mention simplicity of $C^*(\Lambda)$, the C^*-algebra of a k-graph Λ obtained from one-sided path space Λ^Ω ([3]).

Corollary 3.4. *If a k-graph Λ is two-sided cofinal, then $C^*(\Lambda)$ is a simple algebra.*

Proof. By [3, Proposition 4.8], if Λ is a cofinal graph and satisfies the aperiodic condition, then $C^*(\Lambda)$ is simple. It is trivial that two-sided cofinality implies (one-sided) cofinality, and we just need to obtain aperiodic condition from two-sided cofinality: By Proposition 3.2, (Λ^Δ, σ) is a minimal system, and every minimal system is topologically free ([7]). Hence (Λ^Δ, σ) is a topologically free system, and Λ satisfies the aperiodic condition by Proposition 3.1. Therefore $C^*(\Lambda)$ is a simple algebra. □

Corollary 3.5. *If a k-graph Λ is an irreducible graph, then (Λ^Δ, σ) is a minimal system.*

Since every irreducible graph is two-sided cofinal by definition, the above Corollary is trivial. And there is a little more to say about irreducible graphs.

Definition 3.6 ([6])**.** A point x in a dynamical system (X, h) is called a *nonwandering point* if for every open neighborhood U of x, there is an $n \in \mathbb{Z}^k \backslash \{0\}$ such that $h^n(U) \cap U \neq \emptyset$.

Proposition 3.7. *Suppose that Λ is a k-graph. Then Λ is an irreducible graph if and only if the corresponding dynamical system $\left(\Lambda^\Delta, \sigma\right)$ is a topologically transitive system and every point in Λ^Δ is a nonwandering point.*

Proof. ($\Longrightarrow$) It suffices to show this for cylinder sets $U = Z(\lambda, l)$ and $V = Z(\nu, n)$. Since Λ is irreducible, for $s(\nu), r(\lambda) \in \Lambda^0$, there is a $\mu \in \Lambda$ such that $r(\mu) = s(\nu)$ and $s(\mu) = r(\lambda)$. Then, for every $x \in Z(\lambda\mu\nu, l)$,

$$\begin{aligned} x(l, l + d(\lambda\mu\nu)) &= x(l, l + d\lambda + d\mu + d\nu) \\ &= x(l, l + d\lambda) \cdot x(l + d\lambda, l + d\lambda + d\mu) \\ &\qquad\qquad \cdot x(l + d\lambda + d\mu, l + d\lambda + d\mu + d\nu) \\ &= \lambda\mu\nu \end{aligned}$$

and the factorization property imply that

$$x \in Z(\lambda, l) \cap Z(\nu, l + d\lambda + d\mu).$$

Since $\sigma^q\left(Z(\nu, n)\right) = Z(\nu, n - q)$ for every $q \in \mathbb{Z}^k$, we have

$$Z(\nu, l + d\lambda + d\mu) = Z\left(\nu, n - (n - l - d\lambda - d\mu)\right) = \sigma^{n-l-d\lambda-d\mu} Z(\nu, n).$$

Therefore we have

$$Z(\lambda\mu\nu, l) \subset Z(\lambda, l) \cap \sigma^{n-l-d\lambda-d\mu} Z(\nu, n),$$

and (Λ, σ) is a topologically transitive system.

To show that every point is nonwandering, let x be a point in Λ^Δ and $U = Z(\lambda, l)$ an open neighborhood of x. Then the irreducible condition implies that there is a path μ such that $s(\mu) = r(\lambda)$, $r(\mu) = s(\lambda)$ and $d(\mu) \neq 0$. It is not difficult to check

$$Z(\lambda\mu\lambda, l) \subset Z(\lambda, l) \cap \sigma^{-d\lambda - d\mu} Z(\lambda, l)$$

and that every point in Λ^{Δ} is a nonwandering point.

($\Longleftarrow$) For every $u, v \in \Lambda^0$, we need to show that there is a $\lambda \in \Lambda$ such that $s(\lambda) = u$, $r(\lambda) = v$ and $d(\lambda) \neq 0$. Since $(\Lambda^{\Delta}, \sigma)$ is transitive, there is $n \in \mathbb{Z}^k$ such that

$$\sigma^n \left(Z(u, 0)\right) \cap Z(v, 0) \neq \emptyset.$$

Then there exists a k-graph morphism $x \colon \Delta \to \Lambda$ such that

$$x \in Z(u, -n) \cap Z(v, 0) \implies x(-n, -n) = u \text{ and } x(0, 0) = v$$

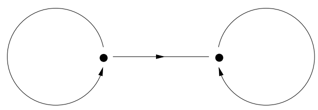
Let ℓ be a finite path in Δ between $(-n, -n)$ and $(0, 0)$ with $d(\ell) \neq 0$. Then $x(\ell)$ is a path in Λ with $d\left(x(\ell)\right) \neq 0$ such that either $s\left(x(\ell)\right) = u$ and $r\left(x(\ell)\right) = v$ or $s\left(x(\ell)\right) = v$ and $r\left(x(\ell)\right) = u$.

Suppose $s\left(x(\ell)\right) = v$ and $r\left(x(\ell)\right) = u$. Since every point in Λ^{Δ} is nonwandering, for an open neighborhood $Z\left(x(\ell), 0\right)$ of $x(\ell)$, there is an $m \in \mathbb{Z}^k$ such that

$$\sigma^m Z\left(x(\ell), 0\right) \cap Z\left(x(\ell), 0\right) = Z\left(x(\ell), -m\right) \cap Z\left(x(\ell), 0\right) \neq \emptyset.$$

Then, for every $y \in Z\left(x(\ell), -m\right) \cap Z\left(x(\ell), 0\right)$, $y\left(-m + d(\ell), 0\right)$ is a path whose source is u and range is v. Therefore the graph Λ is an irreducible graph. $\square$

Remark 3.8. We need the nonwandering condition for a graph to be irreducible. Consider the following 1-graph Λ. Then the corresponding dynamical system is



topologically transitive, but Λ is not irreducible.

To connect topologically transitive systems, irreducible groupoids and prime C^*-algebras, we need a few technical lemmas. Recall that a C^*-algebra is called a *prime* C^*-algebra if intersection of any two nonzero ideals is nonzero.

Lemma 3.9 ([5, I.4.1])**.** *Suppose that G is a groupoid and that $(s, r) : G \to G^0 \times G^0$ is given by the source map s and range map r of G. Then G is an irreducible groupoid if and only if Im(G) under (s, r) is dense in $G^0 \times G^0$.*

Lemma 3.10 ([8])**.** *Let G be a topological groupoid with open range map and G^0 its unit space. If U is an invariant subset of Γ^0, then $V = \Gamma^0 - U$ and $W = IntU$ are also invariant subsets of Γ^0.*

For a groupoid G and its groupoid algebra $C^*(G)$, the ideals of $C^*(G)$ related to open invariant subsets of G^0 mentioned in Theorem 2.8 are given as follow: For any open invariant subset U of G^0, let

$$I_c(U) = \{f \in C_c(G) : f(x, n, y) = 0 \text{ if } (x, n, y) \notin s^{-1}(U)\}$$

and $I(U)$ the closure of $I_c(U)$ in $C^*(G)$. Then $I(U)$ is an ideal of $C^*(G)$ [5, II.4.5].

Next property is certainly a well-known fact to experts, but we were unable to find any reference.

Lemma 3.11. *Suppose that G is a groupoid and that U and V are open invariant subsets of G^0. Then $I(U) \cap I(V) = I(U \cap V)$.*

Proof. We just need to check $I_c(U) \cap I_c(V) = I_c(U \cap V)$:

$$\begin{aligned} f \in I_c(U) \cap I_c(V) &\iff f(x, n, y) = 0 \text{ for } (x, 0, x) \in U^c \cup V^c = (U \cap V)^c \\ &\iff f \in I_c(U \cap V). \end{aligned}$$

□

Proposition 3.12. *Suppose that Λ is a k-graph. Then the followings are equivalent:*

(1) *(Λ^Δ, σ) is a topologically transitive system.*
(2) *Γ is an irreducible groupoid.*
(3) *$C^*(\Gamma)$ is a prime C^*-algebra.*

Proof. (1) $\Longrightarrow$ (2). Suppose that (Λ^Δ, σ) is a topologically transitive system and that U is a nontrivial open invariant subset of Γ^0. As in the proof of Proposition 3.2, U is an invariant subset of Γ^0 implies

$$U = r \circ s^{-1}(U) = \{(\sigma^n(x), 0, \sigma^n(x)) \colon (x, 0, x) \in U \text{ and } n \in \mathbb{Z}^k\}.$$

So, when $\pi \colon \Gamma^0 \to \Lambda^\Delta$ given by $(y, 0, y) \to y$ is the identification map of Γ^0, U is an invariant subset of Γ^0 means that $\sigma^n(\pi(U)) \subset U$ for every $n \in \mathbb{Z}^k$.

If U is not dense in Γ^0, then $\Gamma^0 - \overline{U} \neq \emptyset$ and (Λ^Δ, σ) is topologically transitive imply that there is $n \in \mathbb{Z}^k$ such that $\sigma^n(\pi(U)) \cap \pi(\Gamma^0 - \overline{U}) \neq \emptyset$. Hence we have $U \cap (\Gamma^0 - \overline{U}) \neq \emptyset$, which is a contradiction. Thus V is an empty set, and a nontrivial open invariant subset U of Γ^0 is dense in Γ^0.

(2) $\Longrightarrow$ (1). Suppose that (Λ^Δ, σ) is not topologically transitive. then there exist open subsets $U, V \subset \Lambda^\Delta$ such that $\sigma^n U \cap V = \emptyset$ for every $n \in \mathbb{Z}^k$. So, for any $x \in U$, $y \in V$ and $n \in \mathbb{Z}^k$, we have $(x, n, y) \notin \Gamma$ and $(x, 0, x) \times (y, 0, y) \notin \mathrm{Im}(\Gamma)$. Thus $\{(x, 0, x) \times (y, 0, y) \colon x \in U, y \in V\}$ is a nonempty open subset $\Gamma^0 \times \Gamma^0$ that is disjoint to $\mathrm{Im}(\Gamma)$, and Γ is not irreducible by Lemma 3.9.

(3) $\Longrightarrow$ (2). First we remind that, by Lemma 3.11, $I(E\cap F)=I(E)\cap I(F)$ when E and F are open invariant subsets of Γ^0. Let U be a nonempty open invariant subset of Γ^0, and $V=\mathrm{Int}(\Gamma^0-U)$. Then V is also an open invariant subset of Γ^0 by Lemma 3.10.

If U is not dense in Γ^0, then V is also a nonempty open invariant subset, and we have two nonzero ideals $I(U)$ and $I(V)$. So we have from prime property of $C^*(\Gamma)$ that $I(U)\cap I(V)=I(U\cap V)$ is a nonzero ideal, which is a contradiction to the fact that $U\cap V$ is an empty set. Therefore every nonempty open invariant subset of Γ^0 is dense in Γ^0, and Γ is an irreducible groupoid.

(2) $\Longrightarrow$ (3). Since every topologically transitive system is topologically free, Γ is an essentially principal groupoid, and there is a bijective relation between the set of open invariant subsets of Γ^0 and that of ideals in $C^*(\Gamma)$ by Theorem 2.8.

Suppose that I and J are nonzero ideals in $C^*(\Gamma)$. Then there are nonempty open invariant subsets $U(I)$ and $U(J)$ of Γ^0. As Γ is irreducible, $U(I)\cap U(J)\neq\emptyset$ whose corresponding ideal in $C^*(\Gamma)$ is $I\cap J$ by Lemma 3.11. Thus $C^*(\Gamma)$ is a prime algebra. □

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