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APPROXIMATE PEXIDERIZED EXPONENTIAL TYPE FUNCTIONS

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Abstract. We show that every unbounded approximate Pexiderized exponential type function has the exponential type. That is, we obtain the superstability of the Pexiderized exponential type functional equation

$$f(x+y) = e(x,y)g(x)h(y).$$

From this result, we have the superstability of the exponential functional equation

$$f(x+y) = f(x)f(y).$$

1. Introduction

In 1940, S.M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [17]). Among those there was the question concerning the stability of homomorphisms : *Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?* In the next year, D.H. Hyers [5] answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Futhermore, the result of Hyers has been generalized by Th.M. Rassias [15]. Since then, the stability problems of various functional equations has been investigated by many authors (see [3-16]).

The superstability of the functional equation $f(x+y) = f(x)f(y)$ was studied by J. Baker, J. Lawrence and F. Zorzitto [2]. They proved that if f is a functional on a real vector space W satisfying $|f(x+y) - f(x)f(y)| \le \delta$ for some fixed $\delta > 0$ and all $x, y \in W$, then either f is bounded or else $f(x+y) = f(x)f(y)$ for all $x, y \in W$.

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This result was genealized with a simplified proof by J. Baker [1] as following : *Let* $\delta > 0$*, S be a semigroup and* $f : S \to C$ *satisfy* $|f(x+y) - f(x)f(y)| \leq \delta$ *for all* $x, y \in S$*. Put* $\beta := (1+\sqrt{1+4\delta})/2$*. Then either* $f(x) \leq \beta$ *for all* $x \in S$ *or else* $f(x+y) = f(x)f(y)$ *for all* $x, y \in S$*.*

The author [14] proved the superstability of the Pexiderized multiplicative functional equation

$$f(x+y) = g(x)h(x)$$

and G.H. Kim and the author [10] also obtained the superstability of the gamma-beta type functional equation

$$\beta(x,y)f(x+y) = f(x)f(y)$$

where $\beta(x,y)$ is of beta type function.

In this paper, we consider the Pexiderized exponential type functional equation

$$f(x+y) = e(x,y)g(x)h(y). \tag{1.1}$$

And then we prove the superstability of (1.1). Theorem 1 with $\phi(x) = \delta$ states that every unbounded approximate Pexiderized exponential type function is an exponential type function.

2. Definitions and Solutions

Definition 1. A function $e : [0,\infty) \times [0,\infty) \to [1,\infty)$ is *pseudo exponential* if $e(x,y)$ satisfies as follows;

(a) $e(x,y) = e(y,x)$, for all $x, y \in [0,\infty)$,
(b) $\frac{e(x,y)e(z,x+y)}{e(x,y+z)e(y,z)} = 1$, for all $x, y \in [0,\infty)$,
(c) $e(x,n) \to \infty$, as $n \to \infty$ for $n \in N^+$ and fixed $x \in [0,\infty)$,
(d) $e(0,x) = 1$, for all $x \in [0,\infty)$.

Definition 2. A function $f : [0,\infty) \to R$ is of *an approximate exponential type* if there is a $\delta > 0$ and a pseudo exponential function $e : [0,\infty) \times [0,\infty) \to [1,\infty)$ such that

$$|\ f(x+y) - e(x,y)f(x)f(y)\ | \leq \delta$$

for all $(x,y) \in [0,\infty) \times [0,\infty)$. In the case of $\delta = 0$, we call f an exponential type function.

Definition 3. A function $f : [0,\infty) \to R$ is of *an approximate Pexiderized exponential type* if there is a $\delta > 0$, a pseudo exponential function $e : [0,\infty) \times [0,\infty) \to [1,\infty)$ and some functions $g, h : [0,\infty) \to R$ such that

$$| f(x+y) - e(x,y)g(x)h(y) | \leq \delta$$

for all $(x,y) \in [0,\infty) \times [0,\infty)$. In the case of $\delta = 0$, we call f a Pexiderized exponential type function.

Examples and Solutions. If $f, g, h : R \to R$ are functions satisfying the equation (1.1) and $e(x,y) = a^{xy}$ $(a > 1)$ then e is a pseudo exponential function and $f(x) = a^{\frac{x^2}{2}+3}, g(x) = a^{\frac{x^2}{2}+2}, h(x) = a^{\frac{x^2}{2}+1}$ are solutions of it.

Now we consider the gamma-beta functional equation. If $f, g, h : (0,\infty) \to R$ are functions satisfying the equation (1.1) and $\beta(x,y)$ is the beta function then β^{-1} satisfies the conditions $(a) \sim (c)$ except (d) (see, Corollary 4 in [12]) and $f(x) = 6a^{x+1}\Gamma(x), g(x) = 3a^x\Gamma(x), h(x) = 2a^{x+1}\Gamma(x)$ are solutions of the equation (1.1).

3. Superstability of an Exponential Type Functional Equation

Theorem 1. *Let a function $\phi : [0,\infty) \to [0,\infty)$ be given and $e : [0,\infty) \times [0,\infty) \to [1,\infty)$ be a pseudo exponential function. Assume that $f, g, h : [0,\infty) \to R$ are nonzero functions with $|g(m)| \geq max(2, \frac{\phi(m)+\phi(0)}{|h(m)|})$ for some positive integer m and $g(0) = 1$ such that*

$$| f(x+y) - e(x,y)g(x)h(y) | \leq min\{\phi(x), \phi(y)\} \tag{3.1}$$

for all $(x,y) \in [0,\infty) \times [0,\infty)$. Then

$$g(x+y) = e(x,y)g(x)g(y)$$

for all $(x,y) \in [0,\infty) \times [0,\infty)$.

Proof. If we replace x by m and also y by m in (3.1), respectively, we get

$$| f(2m) - e(m,m)g(m)h(m) | \leq \phi(m).$$

Also if we replace x by 0 in (3.1) then we have

$$|f(y) - h(y)| \leq min\{\phi(0), \phi(y)\} \leq \phi(0)$$

for all $y \in [0,\infty)$. An induction argument implies that for all $n \geq 2$

$$\begin{aligned} &| f(nm) - \prod_{i=1}^{n-1} e(im,m)g(m)^{n-1}h(m) | \\ &\leq (\phi(m)+\phi(0)) \left(1 + \sum_{i=1}^{n-2} \left(|g(m)|^i \prod_{k=1}^{i} e(m,(n-k)n)\right)\right). \end{aligned} \tag{3.2}$$

Indeed, if the inequality (3.2) holds, we have

$$\begin{aligned}
&| \, f((n+1)m) - \prod_{i=1}^{n} e(m,im) g(m)^n h(m) \, | \\
&\le | \, |f((n+1)m) - e(m,nm)g(m)h(nm) \, | \\
&\quad + |g(m)| e(m,nm) \, | \, (h(nm) - f(nm) \, | \\
&\quad + |g(m)| e(m,nm) \left| f(nm) - \prod_{i=1}^{n-1} e(im,m) g(m)^{n-1} h(m) \right| \\
&\le \phi(m) + \phi(0) |g(m)| e(m,nm) \\
&\quad + |g(m)| e(m,nm) (\phi(m) + \phi(0)) \\
&\quad \cdot \left(1 + \sum_{i=1}^{n-2} \left(|g(m)|^i \prod_{k=1}^{i} e(m,(n-k)n) \right) \right) \\
&\le (\phi(m) + \phi(0)) \left(1 + \sum_{i=1}^{n-1} \left(|g(m)|^i \prod_{k=1}^{i} e(m,(n-k+1)n) \right) \right)
\end{aligned}$$

for all $n \ge 2$. By (3.2) we get

$$\begin{aligned}
&\left| \frac{f(nm)}{\left(\prod_{i=1}^{n-1} e(m,im) \right) g(m)^{n-1} h(m)} - 1 \right| \\
&\le \left(\frac{1}{|g(m)|^{n-1}} + \frac{1}{|g(m)|^{n-2}} + \cdots + \frac{1}{|g(m)|^{1}} \right) \cdot \frac{\phi(m) + \phi(0)}{|h(m)|} \\
&< \frac{\phi(m) + \phi(0)}{|h(m)||g(m)} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right) = 2 \frac{\phi(m) + \phi(0)}{|g(m)||h(m)|} \le \frac{1}{2}
\end{aligned}$$

for all positive integer n. Thus we can easily show that

$$|f(nm)| \to \infty \quad as \quad n \to \infty \quad and \quad |h(nm)| \to \infty \quad as \quad n \to \infty. \tag{3.3}$$

Since $\frac{e(x,y)e(z,x+y)}{e(x,y+z)e(y,z)} = 1$ and $\frac{e(x,y+nm)}{e(nm,x+y)} = \frac{e(x,y)}{e(y,nm)}$,

$$\begin{aligned}
&|h(nm)| \, | \, g(x+y) - e(x,y)g(x)g(y) \, | \\
&\le | \, e(nm,x+y)h(nm)g(x+y) - f(nm+x+y) \, | \, \frac{1}{e(nm,x+y)} \\
&\quad + \frac{1}{e(nm,x+y)} \, | \, f(x+y+nm) - e(x,y+nm)g(x)h(y+nm) \, |
\end{aligned} \tag{3.4}$$

$$+ \mid h(y+nm) - f(y+nm) \mid\mid g(x) \mid \cdot \frac{e(x, y+nm)}{e(nm, x+y)}$$
$$+ \mid f(y+nm) - e(y, nm)g(y)h(nm) \mid |g(x)| \cdot \frac{e(x, y+nm)}{e(nm, x+y)}$$
$$\leq \frac{\phi(m)+\phi(0)}{e(nm, x+y)} + (\phi(0)+\phi(y))|g(x)| \cdot \frac{e(x,y)}{e(y,nm)} < \infty$$

for all sufficiently large n and $(x, y) \in [0, \infty) \times [0, \infty)$. It follows from (3.3) and (3.4) by dividing $|h(nm)|$ that

$$g(x, y) = e(x, y)g(x)g(y)$$

for all $(x, y) \in [0, \infty) \times [0, \infty)$. □

Corollary 1. *Let* $\delta > 0$ *and* $a > 1$ *be given. Suppose that* $f : [0, \infty) \to R$ *be a nonzero function with* $|f(m)| \geq max(2, \sqrt{2\delta})$ *for some positive integer* m *and* $g(0) = 1$ *such that*

$$\mid f(x+y) - a^{xy}g(x)h(y) \mid \leq \delta$$

for all $x,\ y \in R$. *Then*

$$g(x+y) = a^{xy}g(x)g(y)$$

for all $x,\ y \in R$.

Proof. Let $e(x, y) = a^{xy}$ for all $x,\ y \in [0, \infty)$. Then $e(x, y)$ is a pseudo exponential function. Also let $\phi(x) = \delta$ then $\phi(m) + \phi(0) = 2\delta$ for any $m \in N$. By Theorem 1, we complete the proof. □

Corollary 2. *Let* $\delta > 0$ *be given. Suppose that* $f : [0, \infty) \to R$ *be a function with* $|f(m)| \geq max(2, \sqrt{2\delta})$ *for some positive integer* m *and* $f(0) = 1$ *such that*

$$\mid f(x+y) - f(x)f(y) \mid \leq \delta$$

for all $x,\ y \in R$. *Then*

$$f(x+y) = f(x)f(y)$$

for all $x,\ y \in R$.

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