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ON n-FOLD IMPLICATIVE VAGUE FILTERS IN BE-ALGEBRAS

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Abstract. In this paper, we introduce the notion of an implicative vague filter in BE-algebras, and investigate some properties of them. Also we give conditions for a vague set to be an implicative vague filter, and we characterize implicative vague filters in BE-algebras. We define the notion of n-fold implicative vague filters in BE-algebras and we give characterizations of n-fold implicative vague filters and n-fold implicative BE-algebras.

1. Introduction

Several authors from time to time have made a number of generalizations of Zadeh's fuzzy set theory [12]. Of course, the notion of vague set theory introduced by Gau and Buehrer [4] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [3] studied vague groups. Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [5, 6]. In [8], H.S. Kim and Y.H. Kim introduced the notion of a BE-algebra as a generalization of a BCK-algebra. Jun and Park [7, 11] studied vague ideals and vague deductive systems in subtraction algebras. Sun Shin Ahn and Jung Mi Ko [2] introduced the notion of a vague filter in BE-algebra, and investigate some properties of it.

In this paper, we use introduce the notion of an implicative vague filter in BE-algebras, and investigate some properties of it. Also we give conditions for a vague set to be a positive implicative vague filter, and we characterize implicative vague filters in BE-algebras. We define the notion of n-fold implicative vague filters in BE-algebras and we give characterizations of n-fold implicative vague filters and n-fold implicative BE-algebras.

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2. Preliminaries

We recall some definitions and results discussed in [8].

An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$;

(BE2) $x * 1 = 1$ for all $x \in X$;

(BE3) $1 * x = x$ for all $x \in X$;

(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x*x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Definition 2.1 ([8]). Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1 \in F$;

(F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

Example 2.2 ([8]). Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $F_1 := \{1, a, b\}$ is a filter of X, but $F_2 := \{1, a\}$ is not a filter of X, since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

Proposition 2.3. *Let $(X; *, 1)$ be a BE-algebra and let F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

Proposition 2.4. *Let $(X; *, 1)$ be a self distributive BE-algebra. Then following hold: for any $x, y, z \in X$,*

(i) *if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.*

(ii) *$y * z \leq (z * x) * (y * z)$.*

(iii) $y * z \leq (x * y) * (x * z)$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.4(iii).

Definition 2.5 ([3])**.** A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(1) A truth membership function

$$t_A : U \to [0, 1],$$

and

(2) A false membership function

$$f_A : U \to [0, 1],$$

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the "evidence for u", and $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle | u \in U\},$$

where the interval $[t_A(u), 1-f_A(u)]$ is called the *vague value* of u in A and is denoted by $V_A(u)$.

3. Implicative Vague Filters

For our discussion, we shall use the following notations, which are given in [3], on interval arithmetic.

Let $I[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) = [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1], I_2 = [a_2, b_2] \in I[0, 1]$. Similarly, we define "imin". The concept of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$. It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ ([3]).

In what follows let X be a BE-algebra unless otherwise specified.

Definition 3.1 ([2]). A vague set A of X is called a *vague filter* of X if the following conditions are true:

(c1) $(\forall x \in X)\ (V_A(1) \succeq V_A(x))$,
(c2) $(\forall x, y \in X)\ (V_A(y) \succeq \text{imin}\{V_A(x*y), V_A(x)\})$,

that is,

$$t_A(1) \geq t_A(x), 1 - f_A(1) \geq 1 - f_A(x)$$

and

$$t_A(y) \geq \min\{t_A(x*y), t_A(x)\},$$
$$1 - f_A(y) \geq \min\{1 - f_A(x*y), 1 - f_A(x)\}$$

for all $x, y \in X$.

Proposition 3.2 ([2]). *Every vague filter A of a BE-algebra X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow V_A(x) \preceq V_A(y))$,
(ii) $(\forall x, y, z \in X)(V_A(x*z) \succeq imin\{V_A(x*(y*z)), V_A(y)\}$.

Theorem 3.3 ([2]). *Let A be a vague set of a BE-algebra X. Then A is a vague filter of X if and only if it satisfies*

$$(\forall x, y, z \in X)(z \leq x*y \Rightarrow V_A(y) \succeq imin\{V_A(x), V_A(z)\}.$$

Definition 3.4. A vague set A of a BE-algebra X is called an *implicative vague filter* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)\ (V_A(x*z) \succeq \text{imin}\{V_A(x*(y*z)), V_A(x*y)\})$,

that is,

$$t_A(1) \geq t_A(x), 1 - f_A(1) \geq 1 - f_A(x)$$

and

$$t_A(x*z) \geq \min\{t_A(x*(y*z)), t_A(x*y)\},$$
$$1 - f_A(x*z) \geq \min\{1 - f_A(x*(y*z))), 1 - f_A(x*y)\}$$

for all $x, y, z \in X$.

Let us illustrate this definition using the following examples.

Example 3.5. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ as in Example 2.2. Let A be a vague set in X defined as follows:

$$A := \{\langle 1, [0.7, 0.2]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.7, 0.2]\rangle, \langle c, [0.5, 0.3]\rangle, \langle d, [0.5, 0.3]\rangle, \langle 0, [0.5, 0.3]\rangle\}.$$

It is routine to verify that A is an implicative vague filter of X.

Theorem 3.6. *Every implicative vague filter is a vague filter.*

Proof. Let A be an implicative vague filter of a BE-algebra X. If we take $x := 1$ in (c3) and use (BE3), then we obtain (c2). Hence A is a vague filter of X. □

The converse of Theorem 3.6 is not true in general as the following example.

Example 3.7. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ as in Example 2.2. Let B be a vague set in X defined as follows:

$$B := \{\langle 1, [0.7, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.5, 0.3]\rangle, \langle d, [0.5, 0.3]\rangle, \langle 0, [0.5, 0.3]\rangle\}.$$

It is routine to verify that B is a vague filter of X. But it is not an implicative vague filter, since

$$V_B(d * 0) \not\succeq \text{imin}\{V_B(d * (a * 0)), V_B(d * a)\}).$$

Now, we give an equivalent condition for every vague filter to be an implicative vague filter.

Theorem 3.8. *For any vague filter A of a self distributive BE-algebra X, the following are equivalent:*

(i) *A is an implicative vague filter of X.*
(ii) $(\forall x, y \in X)(V_A(x * y) \succeq V_A(x * (x * y))$.
(iii) $(\forall x, y, z \in X)(V_A((x * y) * (x * z)) \succeq V_A(x * (y * z))$.

Proof. (i)⇒(ii) Assume that A is an implicative vague filter of X. Putting $z := y, y := x$ in (c3), we have

$$\begin{aligned} V_A(x * y) &\succeq \text{imin}\{V_A(x * (x * y)), V_A(x * x)\} \\ &= \text{imin}\{V_A(x * (x * y)), V_A(1)\} \\ &= V_A(x * (x * y)). \end{aligned}$$

Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x * (y * z) \leq x * ((x * y) * (x * z)) = x * (x * ((x * y) * z))$ for all $x, y, z \in X$, we have $V_A(x * ((x * y) * (x * z))) =$

$V_A(x*(x*((x*y)*z))) \succeq V_A(x*(y*z))$ by Proposition 3.2(i). Using (ii), we have

$$\begin{aligned} V_A((x*y)*(x*z)) =& V_A(x*((x*y)*z)) \\ \succeq& V_A(x*(x*((x*y)*z))) \\ \succeq& V_A(x*(y*z)) \end{aligned}$$

for all $x, y, z \in X$. Thus (iii) holds.

(iii)⇒(i) Assume that (iii) holds. Using (c2) and (iii), we have

$$\begin{aligned} V_A(x*z) \succeq& \text{imin}\{V_A((x*y)*(x*z)), V_A(x*y)\} \\ \succeq& \text{imin}\{V_A(x*(y*z)), V_A(x*y)\} \end{aligned}$$

for all $x, y, z \in X$. Thus A is an implicative vague filter of X. □

Theorem 3.9. *Let X be a self distributive BE-algebra. Then A is an implicative vague filter of X if and only if A is a vague filter of X.*

Proof. By Theorem 3.6, every implicative vague filter is a vague filter.

Conversely, let A be a vague filter of X. Since X is self distributive, $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$. Using (c2), we have

$$\begin{aligned} V_A(x*z) \succeq& \text{imin}\{V_A((x*y)*(x*z)), V_A(x*y)\} \\ =& \text{imin}\{V_A(x*(y*z)), V_A(x*y)\}. \end{aligned}$$

Thus A is an implicative vague filter of X. □

4. n-fold Implicative Vague Filters

For any element x and y of a BE-algebra X and positive integer n, let $x^n * y$ denote $x*(\cdots*(x*(x*y))\cdots)$ in which x occurs n times, and $x^0 * y = y$.

Definition 4.1. A vague set A of a BE-algebra X is called an *n-fold implicative vague filter* of X if it satisfies (c1) and

(c4) $(\forall x, y, z \in X)$ $(V_A(x^n * z) \succeq \text{imin}\{V_A(x^n*(y*z)), V_A(x^n*y)\})$.

that is,

$$t_A(1) \geq t_A(x), 1 - f_A(1) \geq 1 - f_A(x)$$

and

$$t_A(x^n * z) \geq \min\{t_A(x^n*(y*z)), t_A(x^n*y)\},$$
$$1 - f_A(x^n*z) \geq \min\{1 - f_A(x^n*(y*z))), 1 - f_A(x^n*y)\}$$

for all $x, y, z \in X$.

Note that 1-fold implicative vague filter is an implicative vague filter.

Example 4.2. Let $X := \{1, a, b, c, d, 0\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Then X is a transitive BE-algebra. Let B be a vague set in X defined as follows:

$$A := \{\langle 1, [0.7, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.7, 0.2]\rangle, \langle c, [0.7, 0.2]\rangle, \langle d, [0.5, 0.3]\rangle, \langle 0, [0.5, 0.3]\rangle\}.$$

It is routine to verify that A is an n-fold implicative vague filter of X.

Theorem 4.3. *Every n-fold implicative vague filter of a BE-algebra X is a vague filter of X.*

Proof. Let A be an n-fold implicative vague filter of a BE-algebra X. Taking $x = 1$ in (c4) and using (BE3), we conclude

$$V_A(z) \succeq \text{imin}\{V_A(y * z), V_A(y)\},$$

that is, (c2) holds. Hence A is a vague filter of X. □

The converse of Theorem 4.3 is not true in general as seen in the following example.

Example 4.4. Let X be a BE-algebra as in Example 4.2. Let B be a vague set in X defined as follows:

$$B := \{\langle 1, [0.7, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.5, 0.3]\rangle, \langle d, [0.5, 0.3]\rangle, \langle 0, [0.5, 0.3]\rangle\}.$$

Then B is a vague filter of X, but B is not a 1-fold implicative vague filter of X because

$$V_B(d * c) = V_B(b) \not\succeq V_A(1) = \text{imin}\{V_B(d * (b * c)), V_B(d * b)\}.$$

We give conditions for a vague filter to be an n-fold implicative vague filter of X.

Theorem 4.5. *Let A be a vague filter of a self distributive BE-algebra. Then the following are equivalent:*

(i) *A is an n-fold implicative vague filter of X.*
(ii) $(\forall x, y \in X)(V_A(x^n * y) \succeq V_A(x^{n+1} * y))$.
(iii) $(\forall x, y, z \in X)(V_A((x^n * y) * (x^n * z)) \succeq V_A(x^n * (y * z))$.

Proof. (i)⇒(ii) Assume that A is an n-fold implicative vague filter of X. Putting $z := y, y := x$ in (c4), we have

$$\begin{aligned} V_A(x^n * y) \succeq & \text{imin}\{V_A(x^n * (x * y)), V_A(x^n * x)\} \\ = & \text{imin}\{V_A(x^{n+1} * y), V_A(1)\} \\ = & V_A(x^{n+1} * y). \end{aligned}$$

Hence (ii) holds.

(ii)⇒(iii) Suppose that (ii) holds. Since $x^n * (y * z) \le x^n * ((x^n * y) * (x^n * z))$, we have

$$V_A(x^n * ((x^n * y) * (x^n * z))) \succeq V_A(x^n * (y * z))$$

by Proposition 3.2(i). Since $x^{n+1} * (x^{n-1} * ((x^n * y) * z)) = x^n * (x^n * ((x^n * y) * z)) = x^n * ((x^n * y) * (x^n * z))$ and using (ii), we have

$$\begin{aligned} V_A(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) = & V_A(x^n * (x^{n-1} * ((x^n * y) * z))) \\ \succeq & V_A(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\ = & V_A(x^n * ((x^n * y) * (x^n * z))) \\ \succeq & V_A(x^n * (y * z)). \qquad (4.1) \end{aligned}$$

It follows from (ii) and (4.1) that

$$\begin{aligned} V_A(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) = & V_A(x^n * (x^{n-2} * ((x^n * y) * z))) \\ \succeq & V_A(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) \\ \succeq & V_A(x^n * (y * z)). \end{aligned}$$

Repeating this process, we conclude that $V_A((x^n * y) * (x^n * z)) = V_A(x^n * ((x^n * y) * z)) \succeq V_A(x^n * (y * z))$. Hence (iii) holds.

(iii)⇒(i) Assume that (iii) holds. Using (c2) and (iii), we have

$$\begin{aligned} V_A(x^n * z) \succeq & \text{imin}\{V_A((x^n * y) * (x^n * z)), V_A(x^n * y)\} \\ \succeq & \text{imin}\{V_A(x^n * (y * z)), V_A(x^n * y)\}. \end{aligned}$$

Thus A is an n-fold implicative vague filter of X. □

Definition 4.6. Let n be a positive integer. A BE-algebra X is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in X$.

Corollary 4.7. *In an n-fold implicative BE-algebra, the notion of vague filters and n-fold implicative vague filters coincide.*

Proof. Straightforward. □

The following is a characterization of n-fold implicative vague filters.

Theorem 4.8. *A non-empty subset A of a BE-algebra X is an n-fold implicative vague filter of X if and only if it satisfies (c1) and*

(c5) $(\forall x, y, z \in X)(V_A(y^n * z) \succeq imin\{V_A(x * (y^{n+1} * z)), V_A(x)\})$.

Proof. Suppose that A is an n-fold implicative vague filter of X. By Theorem 4.3, A is a vague filter of X. Using Theorem 4.5 and (c2), we have

$$\begin{aligned} V_A(y^n * z) \succeq & V_A(y^{n+1} * z) \\ \succeq & \text{imin}\{V_A(x * (y^{n+1} * z)), V_A(x)\} \end{aligned}$$

for any $x, y, z \in X$. Hence (c5) holds.

Conversely, assume that A satisfies (c1) and (c5). Using (BE3), we have

$$\begin{aligned} V_A(y) =& V_A(1^n * y) \\ \succeq & \text{imin}\{V_A(x * (1^{n+1} * y)), V_A(x)\} \\ =& \text{imin}\{V_A(x * y), V_A(x)\}. \end{aligned}$$

Hence (c2) holds and so A is a vague filter of X. Using (c5), (c1) and (BE3) we have

$$\begin{aligned} V_A(x^n * y) \succeq & \min\{V_A(1 * (x^{n+1} * y)), V_A(1)\} \\ =& V_A(x^{n+1} * y). \end{aligned}$$

By Theorem 4.5, A is an n-fold implicative vague filter of X. □

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