J. Appl. Math. & Informatics Vol. **30***(2012), No. 3 - 4, pp. 693 - 697*
Website: http://www.kcam.biz

SOME QUOTIENT STRUCTURES OF A SEMIRING†

YOUNG HEE KIM

ABSTRACT. In this note, we discuss $N(R, <_R)$-structure in a semiring, and show that if $(R, +, \cdot)$ is a cancellative $<_+$-stable semiring, then $(R/N(R, <_+), +, \cdot)$ is a semiring.

AMS Mathematics Subject Classification : 16Y60.
Key words and phrases : semiring, quotient structure, $<_+$-stable, cancellative.

1. Introduction

The notion of a semiring was first introduced by H. S. Vandiver in 1934, but implicitly semirings had appeared earlier in studies on the theory of ideals of rings [2]. Semirings occur in different mathematical fields, such as ideals of a ring, as positive cones of partially ordered rings and fields, in the context of topological considerations, and in the foundations of arithmetic, including questions raided by school education. Semirings have become of great interest as a tool in many other branches of computer science [3].

By a *semiring* [1] we shall mean a set R endowed with two associative binary operations called an *addition* and a *multiplication* (denoted by $+$ and $\cdot$, respectively) satisfying the following conditions:

(i) addition is a commutative operation,
(ii) there exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$, and
(iii) multiplication distributes over addition both from the left and from the right.

From now on we write R and S for semirings. A subset A of R is a *left* (resp., *right*) *ideal* if $x, y \in A$ and $r \in R$ imply that $x + y \in A$ and $rx \in A$ (resp., $xr \in A$). If A is both a left and a right ideal of R, we say that A is a two sided ideal, or simply, an *ideal* of R.

Received September 27, 2011. Revised December 6, 2011. Accepted December 9, 2011.
†This work was supported by the research grant of the Chungbuk National University in 2011.

We note that if $f : R \to S$ is an onto homomorphism and if A is a left (resp., right) ideal of R, then $f(A)$ is a left (resp., right) ideal of S.

In this note, we discuss $N(R, <_R)$-structure in a semiring, and show that if $(R, +, \cdot)$ is a cancellative $<_+$-stable semiring, then $(R/N(R, <_+), +, \cdot)$ is a semiring.

2. Preliminaries

There are two ways to define a partially ordered set on a set: (i) weak inclusion: reflexive, anti-symmetric and transitive; (ii) strong inclusion: irreflexive and transitive, and they are equivalent (see [7]). J. Neggers et al. [6] discussed the notion of semiring order in semirings, and obtained some results related with the notion of fuzzy left ideals of the semirings.

Suppose that R is a semiring. Define a relation $<_R$ on R as follows:

$$x <_R y \quad \text{provided} \quad x + y = y \quad \text{and} \quad xy = x, x \neq y.$$

Thus, since $0 + y = y$ and $0y = 0$, it follows that if $y \neq 0$, then $0 <_R y$ always, i.e., 0 is a unique miminal element.

L1. *$x <_R y$ and $y <_R x$ are impossible.*

L2. *$x <_R y$ and $y <_R z$ imply $x <_R z$.*

The set $(R, <_R)$ is a poset with unique miminal element 0. We shall refer to it as the *semiring order* of R. A non-empty subset I of a semiring order $(R, <_R)$ is called an *order ideal* if $x \in I, y <_R x$ imply $y \in I$.

Example 2.1 ([6])**.** Let $\mathbf{R}^+$ be the collection of non-negative real numbers with the usual operations " $+$ " and " $\cdot$ ". Then $(\mathbf{R}^+, +, \cdot)$ is a semiring. Also, if $x <_{\mathbf{R}^+} y$ then $x + y = y$ means $x = 0$ and $y \neq 0$. In particular, if $x \neq y$, and $x \neq 0, y \neq 0$, then x and y are incomparable. Hence, $(\mathbf{R}^+ - \{0\}, <_{\mathbf{R}^+})$ is an antichain. We shall consider $\mathbf{R}^+$ to be an *antichain semiring*.

Example 2.2 ([6])**.** Let $\mathbf{R}^+$ be the collection of non-negative real numbers. Define operations " $\oplus$ " and " $\odot$ " by $x \oplus y := \max\{x, y\}$, $x \odot y := \min\{x, y\}$. Then we have $(\mathbf{R}^+, \oplus, \odot)$ is a semiring. Indeed, suppose that $x <_{\mathbf{R}^+} y$. Then $x \oplus y = y$ and $x \odot y = x$. If $r \in \mathbf{R}^+$, then $x <_{\mathbf{R}^+} y$ implies $r \odot x <_{\mathbf{R}^+} r \odot y$ as well. Hence $(r \odot x) \oplus (r \odot y) = r \odot y = r \odot (x \oplus y)$. Thus $(\mathbf{R}^+, \oplus, \odot)$ is a semiring.

Proposition 2.3 ([4])**.** *Let $(R, +, \cdot)$ be a semiring and $x, y \in R$. If we define a relation $<_+$ on R by $x <_+ y$ if and only if $x + y = y, x \neq y$, then it is a partial order.*

Since $0 + x = x$ for any $x \in R$, $0 <_+ x$ for any $x \in R - \{0\}$. Hence $(R, <_+)$ is a poset with unique minimal element 0.

3. $N(R, <_R)$-structure

Let R be a semiring and let $<_R$ be a semiring order of R discussed in section 2. We denote the set of non-maximal elements of R with respect to $<_R$ as follows:

$$N(R, <_R) := \{x \in R \,|\, \exists y \in R \; s.t. \; x <_R y\}$$

Since $0 <_R x$ for any non-zero $x \in R$, we have $0 \in N(R, <_R)$, i.e., $N(R, <_R) \neq \emptyset$ when $R \neq \{0\}$.

Define a relation "$\sim$" on R by $x \sim y$ if and only if $x + N(R, <_R) = y + N(R, <_R)$. Then it is easy to see that $\sim$ is an equivalence relation on R. Denote by $[x]$ the equivalence class of $x \in R$, and let $R/N(R, <_R) := \{[x] \,|\, x \in R\}$. Since $[0] = \{y \in R \,|\, y \sim 0\}$, we have

$$\begin{aligned} 0 \sim y \quad &\Leftrightarrow \quad 0 + N(R, <_R) = y + N(R, <_R) \\ &\Rightarrow \quad 0 \in y + N(R, <_R) \\ &\Rightarrow \quad \exists\, y^* \in N(R, <_R) \, s.t. \, 0 = y + y^* \end{aligned}$$

Proposition 3.1. *If $(R, +, \cdot)$ is a commutative semiring, then $(N(R, <_R), \cdot)$ is a groupoid.*

Proof. Given $a, b \in N(R, <_R)$, there exist $\tilde{a}, \tilde{b} \in R$ such that $a <_R \tilde{a}, b <_R \tilde{b}$. This means that $a + \tilde{a} = \tilde{a}, a\tilde{a} = a, a \neq \tilde{a}$ and $b + \tilde{b} = \tilde{b}, b\tilde{b} = b, b \neq \tilde{b}$. Since $(R, \cdot)$ is commutative, we obtain $(ab)(\tilde{a}\tilde{b}) = (a\tilde{a})(b\tilde{b}) = ab$. Moreover, $ab + \tilde{a}b = (a + \tilde{a})b = \tilde{a}b$. It follows that $ab + \tilde{a}\tilde{b} = ab + \tilde{a}(b + \tilde{b}) = ab + \tilde{a}b + \tilde{a}\tilde{b} = \tilde{a}b + \tilde{a}\tilde{b} = \tilde{a}(b + \tilde{b}) = \tilde{a}\tilde{b}$, proving that $ab \in N(R, <_R)$. □

Proposition 3.2. *If $(R, +, \cdot)$ is an idempotent commutative semiring, then $R \cdot N(R, <_R) \subseteq N(R, <_R)$.*

Proof. Let $x \in N(R, <_R)$ and let $r \in R$. Then $x <_R \tilde{x}$ for some $\tilde{x} \in R$. It follows that $x \cdot \tilde{x} = x, x + \tilde{x} = \tilde{x}$. Since $(R, +, \cdot)$ is idempotent and commutative, we obtain $rx + r\tilde{x} = r(x + \tilde{x}) = r\tilde{x}$ and $(rx)(r\tilde{x}) = r^2(x\tilde{x}) = rx$. Hence $rx < r\tilde{x}$, which shows that $rx \in N(R, <_R)$. □

Proposition 3.3. *If $(R, +, \cdot)$ is a ring, then $N(R, <_R) = \{0\}$.*

Proof. If $x \in N(R, <_R)$, then $x <_R y$ for some $y \in R$. It follows that $x + y = y$. Since R is a ring, we have $x = 0$, proving the proposition. □

4. $N(R, <_+)$-structure

Given the partial order $<_+$ on a semiring R discussed in section 2, we define a set

$$N(R, <_+) := \{x \in R \,|\, x <_+ y \text{ for some } y \in R\}$$

Since $0 <_+ x$ for any non-zero $x \in R$, we have $0 \in N(R, <_+)$, i.e., $N(R, <_+) \neq \emptyset$ when $R \neq \{0\}$.

A semiring $(R, +, \cdot)$ is said to be $<_+$*-stable* if $N(R, <_+)$ is closed under "+". A semiring $(R, +, \cdot)$ is said to be *cancellative* if $xa = xb, x \neq 0$ then $a = b$ where $a, b, x \in R$.

Example 4.1 ([5], p.12)**.** Let $\mathbf{R}^+$ be the set of all non-negative real numbers and $x, y \in \mathbf{R}^+$. If we define $x + y := \max\{x, y\}$ and $x \cdot y := xy$ (the usual product), then $(\mathbf{R}^+, +, \cdot)$ is a cancellative semiring. It is easy to see that $(\mathbf{R}^+, +, \cdot)$ is $<_+$-stable.

Let $(R, +, \cdot)$ be a $<_+$-stable semiring. Define a relation "$\sim$" on R by

$$x \sim y \iff x + i_1 = y + i_2 \text{ for some } i_1, i_2 \in N(R, <_+)$$

Then it is easy to see that $\sim$ is an equivalence relation on R. In fact, If $x \sim y, y \sim z$, then $x + i_1 = y + i_2, y + i_3 = z + i_4$ for some $i_k \in N(R, <_+)$ $(k = 1, 2, 3, 4)$. It follows that $x + i_1 + i_3 = y + i_2 + i_3 = z + i_4 + i_2$. Since R is $<_+$-stable, we have $x \sim z$.

Lemma 4.2. *Let $(R, +, \cdot)$ be a semiring. For any $x, y, a, b \in R$, if $x \sim y, a \sim b$, then $a + x \sim b + y$.*

Proof. Suppose $x \sim y$ and $a \sim b$. Then $a + i_1 = b + i_2$, $x + i_3 = y + i_4$ for some $i_k \in N(R, <_+)$ where $k = 1, 2, 3, 4$. Hence $(a + x) + (i_1 + i_3) = (b + y) + (i_2 + i_4)$. Since R is $<_+$-stable, $i_1 + i_3, i_2 + i_4 \in N(R, <_+)$, which proves the lemma. □

Lemma 4.3. *Let $(R, +, \cdot)$ be a cancellative semiring. If $i \in N(R, <_+)$ and $x \in R$, then $xi, ix \in N(R, <_+)$.*

Proof. If $i \in N(R, <_+)$, then $i <_+ n$ for some $n \in R$, i.e., $i + n = n, i \neq n$. Hence $xi + xn = x(i + n) = xn$. Since R is cancellative, $xi \neq xn$ for any non-zero x in R. If $x = 0$, then $xi = 0i = 0 \in N(R, <_+)$, $ix = i0 = 0 \in N(R, <_+)$. Hence $xi, ix \in N(R, <_+)$ for any $x \in R$. □

Lemma 4.4. *Let $(R, +, \cdot)$ be a cancellative $<_+$-stable semiring. If $a \sim b$ and $x \sim y$, then $ax \sim by$.*

Proof. Suppose that $a \sim b$ and $x \sim y$. Then $a + i_1 = b + i_2, x + i_3 = y + i_4$ for some $i_1, i_2, i_3, i_4 \in N(R, <_+)$. Obviously $(a + i_1)(x + i_3) = ax + (i_1x + ai_3 + i_1i_3)$ and $(b + i_2)(y + i_4) = by + (i_2y + bi_4 + i_2i_4)$. But since $(i_1x + ai_3 + i_1i_3)$ and $(i_2y + bi_4 + i_2i_4)$ belong to $N(R, <_+)$, by Lemmas 4.2 and 4.3, we obtain $ax \sim by$. □

Let $(R, +, \cdot)$ be a cancellative $<_+$-stable semiring and let $x, y \in R$. Define a set $[x] := \{y \in R \,|\, x \sim y\}$ and a set $R/N(R, <_+) := \{[x] \,|\, x \in R\}$. If we define $[x] + [y] := [x + y]$ and $[x] \cdot [y] := [xy]$, then the operations are well-defined by Lemmas 4.2 and 4.4, i.e., $(R/N(R, <_+), +, \cdot)$ is a semiring. We summarize:

Theorem 4.5. *If $(R, +, \cdot)$ is a cancellative $<_+$-stable semiring, then $(R/N(R, <_+), +, \cdot)$ is a semiring.*

References

1. P. J. Allen, *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc. **21** (1969), 412–416.
2. K. Głazek, A guide to the literature on semirings and their applications in mathematics and information sciences with complete bibliograph, Kluwer Academic, Dordrecht, 2002.
3. J. S. Golan, Semirings and their applications, Kluwer Academic, Dordrecht, 1999.
4. J. S. Han, H. S. Kim and J. Neggers, *Semiring orders in a semirings* (submitted).
5. U. Hebisch and H. J. Weinert, Semirings, algebraic theory and applications in computer science, World Scientific, Singapore, 1993.
6. J. Neggers, Y. B. Jun and H. S. Kim, *On L-fuzzy ideals in semirings* II, Czech. Math. J. **49** (1999), 127–133.
7. J. Neggers and H. S. Kim, Basic posets, World Scientific Pub., Singapore, 1998.

Young Hee Kim is working as a professor in Department of Mathematics and is interested in BE-algebras.

Department of Mathematics, College of Natural Sciences, Chungbuk National University, Cheongju 361-763, Korea.
e-mail: yhkim@chungbuk.ac.kr