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SOMEWHAT FUZZY PRECONTINUOUS MAPPINGS

YOUNG BIN IM, JOO SUNG LEE* AND YUNG DUK CHO

Abstract. We define and characterize a somewhat fuzzy precontinuous mapping and a somewhat fuzzy preopen mapping on a fuzzy topological space. Besides, some interesting properties of those mappings are given.

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1. Introduction

The class of somewhat continuous mappings was first introduced by Gentry and others [3]. Later, the concept of "somewhat" in classical topology has been extended to fuzzy topological spaces. In fact, somewhat fuzzy continuous mappings and somewhat fuzzy semicontinuous mappings on fuzzy topological spaces were introduced and studied by G. Thangaraj and G. Balasubramanian in [5] and [6] respectively.

Meanwhile, the concept of fuzzy precontinuous mapping on fuzzy topological space was introduced and studied by A. S. Bin Shahna in [2].

In this paper, the concepts of somewhat fuzzy precontinuous mappings and somewhat fuzzy preopen mappings on a fuzzy topological space are introduced and we characterize those mappings. Besides, some interesting properties of those mappings are also given.

2. Preliminaries

Throughout this paper, we denote μ^c with the complement of the fuzzy set μ on a nonempty set X, which is defined by $\mu^c(x) = (1-\mu)(x) = 1-\mu(x)$ for all $x \in X$. If μ is a fuzzy set on a nonempty set X and if ν is a fuzzy set on a nonempty set Y, then $\mu \times \nu$ is a fuzzy set on $X \times Y$, defined by $(\mu \times \nu)(x, y) = \min(\mu(x), \nu(y))$ for every $(x, y) \in X \times Y$. Let $f : X \to Y$ be a

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mapping and let μ be a fuzzy set on X. Then $f(\mu)$ is a fuzzy set on Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset,\ y \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Let ν be a fuzzy set on Y. Then $f^{-1}(\nu)$ is a fuzzy set on X, defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for each $x \in X$. The graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$ for each $x \in X$. Then $g^{-1}(\mu \times \nu) = \mu \wedge f^{-1}(\nu)$. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each $(x_1, x_2) \in X_1 \times X_2$ [1].

Now let X and Y be fuzzy topological spaces. We denote Intμ and Clμ with the interior and with the closure of the fuzzy set μ on a fuzzy topological space X respectively. Then (i) $1 - \text{Cl}\mu = \text{Int}(1 - \mu)$ and (ii) $\text{Cl}(1 - \mu) = 1 - \text{Int}\mu$.

We say that a fuzzy topological space X is *product related* to a fuzzy topological space Y if, for a fuzzy set μ on X and ν on Y, $\gamma \not\geq \mu$ and $\delta \not\geq \nu$ (in which case $(\gamma^c \times 1) \vee (1 \times \delta^c) \geq (\mu \times \nu)$) where γ is fuzzy open set on X and δ is a fuzzy open set on Y, then there exists a fuzzy open set γ_1 on X and a fuzzy open set δ_1 on Y such that $\gamma_1^c \geq \mu$ or $\delta_1^c \geq \nu$ and $(\gamma_1^c \times 1) \vee (1 \times \delta_1^c) = (\gamma^c \times 1) \times (1 \times \delta^c)$.

A mapping $f : X \to Y$ is called *fuzzy continuous* if $f^{-1}(\nu)$ is a fuzzy open set on X for any fuzzy open set ν on Y. And a mapping $f : X \to Y$ is called *fuzzy open* if $f(\mu)$ is a fuzzy open set on Y for any fuzzy open set μ on X.

A fuzzy set μ on a fuzzy topological space X is called *fuzzy semiopen* if $\mu \leq \text{ClInt}\mu$ and μ is called *fuzzy semiclosed* if μ^c is a fuzzy semiopen set on X.

A mapping $f : X \to Y$ is called *fuzzy semicontinuous* if $f^{-1}(\nu)$ is a fuzzy semiopen set on X for any fuzzy open set ν on Y. And a mapping $f : X \to Y$ is called *fuzzy semiopen* if $f(\mu)$ is a fuzzy semiopen set on Y for any fuzzy open set μ on X [1].

A fuzzy set μ on a fuzzy topological space X is called *fuzzy preopen* if $\mu \leq \text{IntCl}\mu$ and μ is called *fuzzy preclosed* if μ^c is a fuzzy preopen set on X.

A mapping $f : X \to Y$ is called *fuzzy precontinuous* if $f^{-1}(\nu)$ is a fuzzy preopen set on X for any fuzzy open set ν on Y and a mapping $f : X \to Y$ is called *fuzzy preopen* if $f(\mu)$ is a fuzzy preopen set on Y for any fuzzy open set μ on X.

A fuzzy set μ on a fuzzy topological space X is called *fuzzy α-open* if $\mu \leq \text{IntClInt}\mu$ and μ is called *fuzzy α-closed* if μ^c is a fuzzy α-open set on X.

A mapping $f : X \to Y$ is called *fuzzy α-continuous* if $f^{-1}(\nu)$ is a fuzzy α-open set on X for any fuzzy open set ν on Y and a mapping $f : X \to Y$ is called *fuzzy α-open* if $f(\mu)$ is a fuzzy α-open set on Y for any fuzzy open set μ on X [2].

A fuzzy set μ on a fuzzy topological space X is called *fuzzy dense* if there exists no fuzzy closed set ν such that $\mu < \nu < 1$.

A mapping $f : X \to Y$ is called *somewhat fuzzy continuous* if there exists a fuzzy open set $\mu \neq 0_X$ on X such that $\mu \leq f^{-1}(\nu) \neq 0_X$ for any fuzzy open set

ν on Y . It is clear that every fuzzy continuous mapping is a somewhat fuzzy continuous mapping. But the converse is not true in general.

A mapping $f : X \to Y$ is called *somewhat fuzzy open* if there exists a fuzzy open set $\nu \neq 0_Y$ on Y such that $\nu \leq f(\mu) \neq 0_Y$ for any fuzzy open set μ on X. Note that every fuzzy open mapping is a somewhat fuzzy open mapping but the converse is not true in general [5].

A fuzzy set μ on a fuzzy topological space X is called *fuzzy semidense* if there exists no fuzzy semiclosed set ν such that $\mu < \nu < 1$.

A mapping $f : X \to Y$ is called *somewhat fuzzy semicontinuous* if there exists a fuzzy semiopen set $\mu \neq 0_X$ on X such that $\mu \leq f^{-1}(\nu) \neq 0_X$ for any fuzzy open set ν on Y. It is clear that every fuzzy continuous mapping is a somewhat fuzzy semicontinuous mapping. But the converse is not true in general.

A mapping $f : X \to Y$ is called *somewhat fuzzy semiopen* if there exists a fuzzy semiopen set $\nu \neq 0_Y$ on Y such that $\nu \leq f(\mu) \neq 0_Y$ for any fuzzy open set μ on X. Every fuzzy open mapping is a somewhat fuzzy semiopen mapping but the converse is not true in general [6].

A fuzzy set μ on a fuzzy topological space X is called *fuzzy α-dense* if there exists no fuzzy α-closed set ν such that $\mu < \nu < 1$.

A mapping $f : X \to Y$ is called *somewhat fuzzy α-continuous* if there exists a fuzzy α-open set $\mu \neq 0_X$ on X such that $\mu \leq f^{-1}(\nu) \neq 0_X$ for any fuzzy open set ν on Y. It is clear that every fuzzy continuous mapping is a somewhat fuzzy α-continuous mapping. But the converse is not true in general.

A mapping $f : X \to Y$ is called *somewhat fuzzy α-open* if there exists a fuzzy α-open set $\nu \neq 0_Y$ on Y such that $\nu \leq f(\mu) \neq 0_Y$ for any fuzzy open set μ on X. Every fuzzy open mapping is a somewhat fuzzy α-open mapping but the converse is not true in general [4].

3. Somewhat fuzzy precontinuous mapping

In this section, we introduce a somewhat fuzzy precontinuous mapping and a somewhat fuzzy preopen mapping which are weaker than a somewhat fuzzy continuous mapping and a somewhat fuzzy open mapping respectively. And we characterize those mappings.

Definition 3.1. A mapping $f : X \to Y$ is called *somewhat fuzzy precontinuous* if there exists a fuzzy preopen set $\mu \neq 0_X$ on X such that $\mu \leq f^{-1}(\nu) \neq 0_X$ for any fuzzy open set ν on Y.

From the definitions, it is clear that every somewhat fuzzy continuous mapping is a fuzzy precontinuous mapping and every fuzzy precontinuous mapping is a somewhat fuzzy precontinuous mapping. But the converses are not true in general as the following example shows.

Example 3.2. Let μ_1, μ_2 and μ_3 be fuzzy sets on $X = \{a, b, c\}$ with

$$\mu_1(a) = 0.1, \mu_1(b) = 0.1, \mu_1(c) = 0.1,$$

$$\mu_2(a) = 0.2, \mu_2(b) = 0.2, \mu_2(c) = 0.2,$$

$$\mu_3(a) = 0.5, \mu_3(b) = 0.5, \mu_3(c) = 0.5.$$

And let

$$\tau_1 = \{0_X, \mu_1, 1_X\},\ \tau_2 = \{0_X, \mu_2, 1_X\} \text{ and } \tau_3 = \{0_X, \mu_3, 1_X\}$$

be fuzzy topologies on X.

Since $i_X^{-1}(\mu_2) = \mu_2$ is a fuzzy preopen set on (X, τ_3), $i_X : (X, \tau_3) \to (X, \tau_2)$ is fuzzy precontinuous. But there is no non-zero fuzzy open set smaller than $i_X^{-1}(\mu_2) = \mu_2 \neq 0$. Hence $i_X : (X, \tau_3) \to (X, \tau_2)$ is not somewhat fuzzy continuous. Now, since $\mu_1 \leq i_X^{-1}(\mu_2) = \mu_2 \neq 0$ and μ_1 is a preopen set on (X, τ_1), $i_X : (X, \tau_1) \to (X, \tau_2)$ is somewhat fuzzy precontinuous. But $i_X^{-1}(\mu_2) = \mu_2$ is not a fuzzy preopen set on (X, τ_1). i_X is not fuzzy precontinuous. □

Definition 3.3. A fuzzy set μ on a fuzzy topological space X is called *fuzzy predense* if there exists no fuzzy preclosed set ν such that $\mu < \nu < 1$.

Theorem 3.1. *Let $f : X \to Y$ be a mapping. Then the following are equivalent:*

(1) f is somewhat fuzzy precontinuous.

(2) If ν is a fuzzy closed set on Y such that $f^{-1}(\nu) \neq 1_X$, then there exists a fuzzy preclosed set $\mu \neq 1_X$ of X such that $f^{-1}(\nu) \leq \mu$.

(3) If μ is a fuzzy predense set on X, then $f(\mu)$ is a fuzzy predense set on Y.

Proof. (1) implies (2): Let ν be a fuzzy closed set on Y such that $f^{-1}(\nu) \neq 1_X$. Then ν^c is a fuzzy open set in Y and $f^{-1}(\nu^c) = (f^{-1}(\nu))^c \neq 0_X$. Since f is somewhat fuzzy precontinuous, there exists a fuzzy preopen set $\mu^c \neq 0_X$ on X such that $\mu^c \leq f^{-1}(\nu^c)$. Hence there exists a fuzzy preclosed set $\mu \neq 1_X$ on X such that $f^{-1}(\nu) = 1 - f^{-1}(\nu^c) \leq 1 - \mu^c = \mu$.

(2) implies (3): Let μ be a fuzzy predense set on X and suppose that $f(\mu)$ is not fuzzy predense on Y. Then there exists a fuzzy preclosed set ν on Y such that $f(\mu) < \nu < 1$. Since $\nu < 1$ and $f^{-1}(\nu) \neq 1_X$, there exists a fuzzy preclosed set $\delta \neq 1_X$ such that $\mu \leq f^{-1}(f(\mu)) < f^{-1}(\nu) \leq \delta$. This contradicts to the assumption that μ is a fuzzy predense set on X. Hence $f(\mu)$ is a fuzzy predense set on Y.

(3) implies (1): Let ν be a fuzzy open set on Y with $f^{-1}(\nu) \neq 0_X$. Suppose that there exists no fuzzy preopen set $\mu \neq 0_X$ on X such that $\mu \leq f^{-1}(\nu)$. Then $(f^{-1}(\nu))^c$ is a fuzzy set on X such that there is no fuzzy preclosed set δ on X with $(f^{-1}(\nu))^c < \delta < 1$. In fact, if there exists a fuzzy preopen set δ^c such that $\delta^c \leq f^{-1}(\nu)$, then it is a contradiction. So $(f^{-1}(\nu))^c$ is a fuzzy predense set on X. Hence $f((f^{-1}(\nu))^c)$ is a fuzzy predense set on Y. But $f((f^{-1}(\nu))^c) = f(f^{-1}(\nu^c)) \neq \nu^c < 1$. This is a contradiction to the fact that $f((f^{-1}(\nu))^c)$ is fuzzy predense on Y. Hence there exists a preopen set $\mu \neq 0_X$ in X such that $\mu \leq f^{-1}(\nu)$. Consequently, f is somewhat fuzzy precontinuous. □

Theorem 3.2. *Let X_1 be product related to X_2 and let Y_1 be product related to Y_2. If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is a somewhat fuzzy precontinuous mappings, then the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also somewhat fuzzy precontinuous.*

Proof. Let $\lambda = \bigvee_{i,j}(\mu_i \times \nu_j)$ be a fuzzy open set on $Y_1 \times Y_2$ where μ_i and ν_j are a fuzzy open set on Y_1 and Y_2 respectively. Then $(f_1 \times f_2)^{-1}(\lambda) = \bigvee_{i,j}(f_1^{-1}(\mu_i) \times f_2^{-1}(\nu_j))$. Since f_1 is somewhat fuzzy precontinuous, there exists a fuzzy preopen set $\delta_i \neq 0_{X_1}$ such that $\delta_i \leq f_1^{-1}(\mu_i) \neq 0_{X_1}$. And since f_2 is somewhat fuzzy precontinuous, there exists a fuzzy preopen set $\eta_j \neq 0_{X_2}$ such that $\eta_j \leq f_2^{-1}(\nu_j) \neq 0_{X_2}$. Now $\delta_i \times \eta_j \leq f_1^{-1}(\mu_i) \times f_2^{-1}(\nu_j) = (f_1 \times f_2)^{-1}(\mu_i \times \nu_j)$ and $\delta_i \times \eta_j \neq 0_{X_1 \times X_2}$. Hence $\delta_i \times \eta_j$ is a fuzzy preopen set on $X_1 \times X_2$. Moreover, $\bigvee_{i,j}(\delta_i \times \eta_j) \neq 0_{X_1 \times X_2}$ is a fuzzy preopen set on $X_1 \times X_2$ such that $\bigvee_{i,j}(\delta_i \times \eta_j) \leq \bigvee_{i,j}(f_1^{-1}(\mu_i) \times f_2^{-1}(\nu_j)) = (f_1 \times f_2)^{-1}(\bigvee_{i,j}(\mu_i \times \nu_j)) = (f_1 \times f_2)^{-1}(\lambda) \neq 0_{X_1 \times X_2}$. Therefore, $f_1 \times f_2$ is somewhat fuzzy precontinuous. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping. If the graph $g : X \to X \times Y$ of f is somewhat fuzzy irresolute continuous, then f is also somewhat fuzzy irresolute continuous.*

Proof. Let ν be a fuzzy open set on Y. Then $f^{-1}(\nu) = 1 \wedge f^{-1}(\nu) = g^{-1}(1 \times \nu)$. Since g is somewhat fuzzy precontinuous and $1 \times \nu$ is a fuzzy open set on $X \times Y$, there exists a fuzzy preopen set $\mu \neq 0_X$ on X such that $\mu \leq g^{-1}(1 \times \nu) = f^{-1}(\nu) \neq 0_X$. Therefore, f is somewhat fuzzy precontinuous. □

Definition 3.4. A mapping $f : X \to Y$ is called *somewhat fuzzy preopen* if there exists a fuzzy preopen set $\nu \neq 0_Y$ on Y such that $\nu \leq f(\mu) \neq 0_Y$ for any fuzzy open set μ on X.

From the definitions, it is clear that every somewhat fuzzy open mapping is a fuzzy preopen mapping and every fuzzy preopen mapping is a somewhat fuzzy preopen mapping. But the converses are not true in general as the following example shows.

Example 3.5. Let μ_1, μ_2 and μ_3 be fuzzy sets on $X = \{a, b, c\}$ with

$$\mu_1(a) = 0.1, \mu_1(b) = 0.1, \mu_1(c) = 0.1,$$
$$\mu_2(a) = 0.2, \mu_2(b) = 0.2, \mu_2(c) = 0.2 \text{ and}$$
$$\mu_3(a) = 0.5, \mu_3(b) = 0.5, \mu_3(c) = 0.5.$$

And let

$$\tau_1 = \{0_X, \mu_1, 1_X\}, \tau_2 = \{0_X, \mu_2, 1_X\} \text{ and}$$
$$\tau_3 = \{0_X, \mu_3, 1_X\}$$

be fuzzy topologies on X.

Since $i_X(\mu_2) = \mu_2$ is a fuzzy preopen set on (X, τ_3), $i_X : (X, \tau_2) \to (X, \tau_3)$ is fuzzy preopen. But there is no non-zero fuzzy open set smaller than $i_X(\mu_2) = \mu_2 \neq 0$. Hence $i_X : (X, \tau_2) \to (X, \tau_3)$ is not somewhat fuzzy open. Now, since $\mu_1 \leq i_X(\mu_2) = \mu_2 \neq 0$ and μ_1 is a preopen set in (X, τ_1), $i_X : (X, \tau_2) \to (X, \tau_1)$ is somewhat fuzzy preopen. But $i_X(\mu_2) = \mu_2$ is not a fuzzy preopen set on (X, τ_1). □

Theorem 3.4. *Let $f : X \to Y$ be a bijection. Then the following are equivalent:*

(1) f is somewhat fuzzy preopen.

(2) If μ is a fuzzy closed set on X such that $f(\mu) \neq 1_Y$, then there exists a fuzzy preclosed set $\nu \neq 1_Y$ on Y such that $f(\mu) < \nu$.

Proof. (1) implies (2): Let μ be a fuzzy closed set on X such that $f(\mu) \neq 1_Y$. Since f is bijective and μ^c is a fuzzy open set on X, $f(\mu^c) = (f(\mu))^c \neq 0_Y$. From the definition, there exists a preopen set $\delta \neq 0_Y$ on Y such that $\delta < f(\mu^c) = (f(\mu))^c$. Consequently, $f(\mu) < \delta^c = \nu \neq 1_Y$ and ν is a fuzzy preclosed set on Y.

(2) implies (1): Let μ be a fuzzy open set on X such that $f(\mu) \neq 0_Y$. Then μ^c is a fuzzy closed set on X and $f(\mu^c) \neq 1_Y$. Hence there exists a fuzzy preclosed set $\nu \neq 1_Y$ on Y such that $f(\mu^c) < \nu$. Since f is bijective, $f(\mu^c) = (f(\mu))^c < \nu$. Thus $\nu^c < f(\mu)$ and $\nu^c \neq 0_X$ is a fuzzy preopen set on Y. Therefore, f is somewhat fuzzy preopen. □

Theorem 3.5. *Let $f : X \to Y$ be a surjection. Then the following are equivalent:*

(1) f is somewhat fuzzy preopen.

(2) If ν is a fuzzy predense set on Y, then $f^{-1}(\nu)$ is a fuzzy predense set on X.

Proof. (1) implies (2): Let ν be a fuzzy predense set on Y. Suppose $f^{-1}(\nu)$ is not fuzzy predense on X. Then there exists a fuzzy preclosed set μ on X such that $f^{-1}(\nu) < \mu < 1$. Since f is somewhat fuzzy preopen and μ^c is a fuzzy preopen set on X, there exists a fuzzy preopen set $\delta \neq 0_Y$ on Y such that $\delta \leq f(\text{Int}\mu^c) \leq f(\mu^c)$. Since f is surjective, $\delta \leq f(\mu^c) < f(f^{-1}(\nu^c)) = \nu^c$. Thus there exists a preclosed set δ^c on Y such that $\nu < \delta^c < 1$. This is a contradiction. Hence $f^{-1}(\nu)$ is fuzzy predense on X.

(2) implies (1): Let μ be a fuzzy open set on X and $f(\mu) \neq 0_Y$. Suppose there exists no fuzzy preopen set $\nu \neq 0_Y$ on Y such that $\nu \leq f(\mu)$. Then $(f(\mu))^c$ is a fuzzy set on Y such that there exists no fuzzy preclosed set δ on Y with $(f(\mu))^c < \delta < 1$. This means that $(f(\mu))^c$ is fuzzy predense on Y. Thus $f^{-1}((f(\mu))^c)$ is fuzzy predense on X. But $f^{-1}((f(\mu))^c) = (f^{-1}(f(\mu)))^c \leq \mu^c < 1$. This is a contradiction to the fact that $f^{-1}((f(\nu))^c$ is fuzzy predense on X. Hence there exists a preopen set $\nu \neq 0_Y$ on Y such that $\nu \leq f(\mu)$. Therefore, f is somewhat fuzzy preopen. □

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