J. Appl. Math. & Informatics Vol. **30**(2012), No. 3 - 4, pp. 655 - 663
Website: http://www.kcam.biz

PERTURBATION ANALYSIS FOR THE POSITIVE DEFINITE SOLUTION OF THE NONLINEAR MATRIX EQUATION $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ †

XUE-FENG DUAN*, QING-WEN WANG AND CHUN-MEI LI

ABSTRACT. Based on the elegant properties of the spectral norm and Thompson metric, we firstly give two perturbation estimates for the positive definite solution of the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q (0 < |\delta_i| < 1)$ which arises in an optimal interpolation problem.

AMS Mathematics Subject Classification : 15A24, 65J15.
Key words and phrases : Nonlinear matrix equation, Positive definite solution, Perturbation estimate, Spectral norm, Thompson metric.

1. Introduction

We consider the nonlinear matrix equation

$$X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q, \quad 0 < |\delta_i| < 1, \tag{1.1}$$

where $A_1, A_2, \cdots, A_m$ are $n \times n$ nonsingular complex matrices, Q is an $n \times n$ positive definite matrix, and m is a positive integer. Here, A_i^* denotes the conjugate transpose of the matrix A_i. This type of nonlinear matrix equation arises in an optimal interpolation problem (see [17, Chapter 7] for more details).

In the last few years there has been a constantly increasing interest in developing the theory and numerical approaches for positive definite solutions to the nonlinear matrix equation of the form (1.1) [1,3-13,15,16,19,20]. Recently, Duan-Liao-Tang [4] showed that Eq.(1.1) always has a unique positive definite solution by using the fixed point theorem of mixed monotone operators, and

Received August 24, 2011. Revised November 28, 2011. Accepted December 5, 2011.
*Corresponding author.
†This work was supported by National Natural Science Foundation of China (11101100), and the Natural Science Foundation of Guangxi Province (0991238; 2011GXNSFA018138).

proposed a multi-step stationary iterative method to compute the unique positive definite solution. By making use of Thompson metric, Lim [13] provides a new proof for the existence and uniqueness of the positive definite solution for Eq.(1.1). A lot of results have been reported on the uniqueness and existence of positive definite solution and numerical methods for Eq.(1.1) in the special case $m = 1$ [8-10].

In the practical problem, we need to know that whether the optimal interpolation problem is ill-posed, that is to say, we often want to know that how the perturbation of coefficients influence on the solutions. For this purpose we study the perturbation analysis of Eq.(1.1). However, the perturbation analysis of Eq.(1.1) isn't still studied as far as we know. The main difficulty of studying the perturbation analysis of Eq.(1.1) is that how to deal with X^{δ_i}, when $\delta_i \in (0,1)$ or $\delta_i \in (-1,0)$. In this paper, we overcome this difficulty by using the elegant properties of the spectral norm and Thompson metric, and we give two perturbation estimates for the positive definite solution of Eq.(1.1) respectively. Now we consider the perturbed equation

$$\widetilde{X} - \sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i = \widetilde{Q}, \tag{1.2}$$

where $\widetilde{A}_i$ and $\widetilde{Q}$ are small perturbations of A_i and Q in Eq.(1.1). Here, we assume that $\widetilde{Q}$ be also positive definite. By Theorem 3.1 in Duan-Liao-Tang [3], we know that Eq.(1.1) has a unique positive definite solution $\widehat{X}$ and the perturbed equation (1.2) has a unique positive definite solution $\widetilde{X}$. In this paper, we will give an upper bound for $\delta(\widetilde{X}, \widehat{X})$, where $\delta(\cdot,\cdot)$ is a metric. In order to develop this paper, we need the symbols $\lambda_{max}(B)(\lambda_{min}(B))$, which denote the maximal (minimal) eigenvalue of an $n \times n$ Hermitian matrix B, and the symbol $M(\Omega, \alpha)$, which denotes the set of all strict contraction maps on Ω with the contraction constant $\alpha \in (0,1)$, that is to say, for arbitrary $f \in M(\Omega, \alpha)$, then

$$\delta(f(x), f(y)) \leq \alpha\delta(x, y), \quad \forall x, y \in \Omega.$$

2. The perturbation estimate by the spectral norm

In this section, we give an upper bound for $\|\widetilde{X} - \widehat{X}\|$, where the symbol $\|\cdot\|$ stands for the spectral norm. We begin with lemmas.

Lemma 2.1 [2, Theorem X.3.8]**.** *Let f be an operator monotone function on $(0,\infty)$ and let A, B be two positive operators bounded below by a, i.e. $A > aI$ and $B > aI$ for a positive number a. If there exists $f'(a)$, then for every unitary invariant norm $\|\cdot\|$, we have*

$$\|f(A) - f(B)\| \leq f'(a)\|A - B\|.$$

Lemma 2.2. *If* $0 < |\theta| < 1$, *and* X *and* Y *are positive definite matrices of the same order with* $X, Y \geq bI > 0$, *then*

$$\|X^{\theta} - Y^{\theta}\| \leq |\theta| b^{\theta-1} \|X - Y\|.$$

Proof. We first consider the case $0 < \theta < 1$. From Lemma 2.1 it follows that

$$\|X^{\theta} - Y^{\theta}\| \leq \theta b^{\theta-1} \|X - Y\| = |\theta| b^{\theta-1} \|X - Y\|. \tag{2.1}$$

Consider the other case $-1 < \theta < 0$. Since $X, Y \geq bI > 0$, then we have

$$X^{\theta} \leq b^{\theta} I, \quad Y^{\theta} \leq b^{\theta} I. \tag{2.2}$$

By (2.2) and (2.1), we have

$$\begin{array}{rcl} \|X^{\theta} - Y^{\theta}\| & = & \|X^{\theta}(Y^{-\theta} - X^{-\theta})Y^{\theta}\| \\ & \leq & \|X^{\theta}\|\|Y^{\theta}\|\|X^{-\theta} - Y^{-\theta}\| \\ & \leq & b^{2\theta}\|X^{-\theta} - Y^{-\theta}\| \\ & \leq & b^{2\theta}| - \theta|b^{-\theta-1}\|X - Y\| \\ & = & |\theta|b^{\theta-1}\|X - Y\|. \end{array} \tag{2.3}$$

Combining (2.1) and (2.3), we have

$$\|X^{\theta} - Y^{\theta}\| \leq |\theta| b^{\theta-1} \|X - Y\|, \quad 0 < |\theta| < 1.$$ □

Theorem 2.1. *Let*

$$b = min\{\lambda_{min}(Q), \lambda_{min}(\widetilde{Q})\}.$$

If

$$t = 1 - \sum_{i=1}^{m} (\|A_i\|^2 |\delta_i| b^{\delta_i - 1}) > 0,$$

then we have

$$\|\widetilde{X} - \widehat{X}\| \leq \frac{1}{t}[\|\Delta Q\| + \sum_{i=1}^{m} (\|\Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i\| + \|A_i^* \widetilde{X}^{\delta_i} \Delta A_i\|)],$$

where

$$\Delta A_i = \widetilde{A}_i - A_i, \quad i = 1, 2, \cdots, m \quad and \quad \Delta Q = \widetilde{Q} - Q.$$

Proof. Since $\widehat{X}$ and $\widetilde{X}$ are the unique positive definite solution of Eq. (1.1) and its perturbed equation (1.2) respectively, then we have

$$\widehat{X} - \sum_{i=1}^{m} A_i^* \widehat{X}^{\delta_i} A_i = Q, \tag{2.4}$$

and

$$\widetilde{X} - \sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i = \widetilde{Q}. \tag{2.5}$$

From (2.4) and (2.5) it is easy to obtain that

$$\widehat{X} \geq Q \geq \lambda_{min}(Q) I \geq bI, \tag{2.6}$$

$$\widetilde{X} \geq \widetilde{Q} \geq \lambda_{min}(\widetilde{Q})I \geq bI. \tag{2.7}$$

Subtracting (2.4) from (2.5) we get

$$\widetilde{X} - \widehat{X} - (\sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i - \sum_{i=1}^{m} A_i^* \widehat{X}^{\delta_i} A_i) = \widetilde{Q} - Q,$$

which implies that

$$\widetilde{X} - \widehat{X} = (\widetilde{Q} - Q) + \sum_{i=1}^{m} [A_i^* (\widetilde{X}^{\delta_i} - \widehat{X}^{\delta_i}) A_i + \Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i + A_i^* \widetilde{X}^{\delta_i} \Delta A_i]. \tag{2.8}$$

Combining Lemma 2.2 and (2.6)-(2.8) we have

$$\begin{aligned}
\|\widetilde{X} - \widehat{X}\| &= \|(\widetilde{Q} - Q) + \sum_{i=1}^{m} [A_i^* (\widetilde{X}^{\delta_i} - \widehat{X}^{\delta_i}) A_i + \Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i + A_i^* \widetilde{X}^{\delta_i} \Delta A_i]\| \\
&\leq \|\Delta Q\| + \| \sum_{i=1}^{m} [A_i^* (\widetilde{X}^{\delta_i} - \widehat{X}^{\delta_i}) A_i]\| + \| \sum_{i=1}^{m} (\Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i)\| + \\
&\quad \| \sum_{i=1}^{m} (A_i^* \widetilde{X}^{\delta_i} \Delta A_i)\| \\
&\leq \|\Delta Q\| + \sum_{i=1}^{m} [\|A_i\|^2 \|\widetilde{X}^{\delta_i} - \widehat{X}^{\delta_i}\|] + \sum_{i=1}^{m} [\|\Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i\|] + \\
&\quad \sum_{i=1}^{m} [\|A_i^* \widetilde{X}^{\delta_i} \Delta A_i\|] \\
&\leq \|\Delta Q\| + \sum_{i=1}^{m} [\|A_i\|^2 |\delta_i| b^{\delta_i - 1}] \|\widetilde{X} - \widehat{X}\| + \sum_{i=1}^{m} [\|\Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i\| + \\
&\quad \|A_i^* \widetilde{X}^{\delta_i} \Delta A_i\|],
\end{aligned}$$

which implies that

$$(1 - \sum_{i=1}^{m} \|A_i\|^2 |\delta_i| b^{\delta_i - 1}) \|\widetilde{X} - \widehat{X}\| \leq \|\Delta Q\| + \sum_{i=1}^{m} [\|\Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i\| + \|A_i^* \widetilde{X}^{\delta_i} \Delta A_i\|].$$

Since

$$t = 1 - \sum_{i=1}^{m} (\|A_i\|^2 |\delta_i| b^{\delta_i - 1}) > 0,$$

then we have

$$\|\widetilde{X} - \widehat{X}\| \leq \frac{1}{t} [\|\Delta Q\| + \sum_{i=1}^{m} (\|\Delta A_i^* \widetilde{X}^{\delta_i} \widetilde{A}_i\| + \|A_i^* \widetilde{X}^{\delta_i} \Delta A_i\|)]. \qquad \square$$

3. The perturbation estimate by Thompson Metric

In this section, we first review the Thompson metric on the set of all $n \times n$ positive definite matrix $P(n)$. Obviously, it is an open convex cone. And then we discuss the perturbation bound of the unique positive definite solution of Eq.(1.1) by using the perturbation theorem of contraction map.

The Thompson metric on $P(n)$ is defined by

$$d(A, B) = max\{logW(A/B), \quad logW(B/A)\},$$

where $W(A/B) = inf\{\lambda > 0 : A \leq \lambda B\} = \lambda_{max}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$. From Nussbaum [14] we obtain that $P(n)$ is a complete metric space with respect to the Thompson metric and $d(A, B) = \|log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|$, where the symbol $\|\cdot\|$ stands for the spectral norm. Now we shortly introduce the elegant properties of the Thompson metric on $P(n)$ (see [14,18] for more details). It is invariant under the matrix inversion and congruence transformations

$$d(A, B) = d(A^{-1}, B^{-1}) = d(N^*AN, N^*BN) \tag{3.1}$$

for any $n \times n$ nonsingular matrix N. The other useful result is the nonpositive curvature property of the Thompson metric

$$d(X^r, Y^r) \leq rd(X, Y), \quad r \in [0, 1]. \tag{3.2}$$

According to (3.1) and (3.2), we have

$$d(N^*X^rN, N^*Y^rN) \leq |r|d(X, Y), \quad r \in [-1, 1]. \tag{3.3}$$

We begin with some lemmas.

Lemma 3.1 [13, Lemma 2.1]**.** *For any $A, B, C, D \in P(n)$,*

$$d(A + B, C + D) \leq max\{d(A, C), d(B, D)\}.$$

Especially,

$$d(A + B, A + C) \leq d(B, C).$$

Lemma 3.2 [16, Theorem 2.1]**.** *Let $\phi \in M(\Omega, \alpha)$. Then the map ϕ has a unique fixed point $x^*(\phi)$ on Ω.*

Lemma 3.3 [16, Theorem 2.2] (Perturbation Theorem of Contraction Map)**.** *Let the map $\phi \in M(\Omega, \alpha)$. Then for every $\varepsilon > 0$ and for all maps $\psi \in M(\Omega, \alpha)$ satisfying*

$$\sup_{X\in\Omega} \delta(\psi(X), \phi(X)) < min\{\frac{1-\alpha}{3}\varepsilon, 1\},$$

we have the inequality

$$\delta(x^*(\psi), x^*(\phi)) < \varepsilon,$$

where the symbols $x^*(\psi)$ and $x^*(\phi)$ denote the unique fixed point of ψ and ϕ on Ω, respectively.

Theorem 3.1. *Let*

$$\delta = max\{|\delta_i|, \quad i = 1, 2, \cdots, m\};$$

$$q = d(\widetilde{Q}, Q) = \|log(\widetilde{Q}^{-\frac{1}{2}}Q\widetilde{Q}^{-\frac{1}{2}})\|;$$

$a_i = d(\widetilde{A}_i^*X^{\delta_i}\widetilde{A}_i, A_i^*X^{\delta_i}A_i) = \|log((\widetilde{A}_i^*X^{\delta_i}\widetilde{A}_i)^{-\frac{1}{2}}(A_i^*X^{\delta_i}A_i)(\widetilde{A}_i^*X^{\delta_i}\widetilde{A}_i)^{-\frac{1}{2}})\|, i = 1, 2, \cdots, m$. *For every $\varepsilon > 0$, if*

$$\sup_{X\in P(n)} max\{q, a_1, a_2, \cdots, a_m\} \leq min\{\frac{1-\delta}{3}\varepsilon, 1\}, \tag{3.4}$$

then we have

$$d(\widetilde{X}, \widehat{X}) < \varepsilon,$$

where $\widehat{X}$ and $\widetilde{X}$ are the unique positive definite solution of Eq.(1.1) and its perturbed equation (1.2), respectively.

Proof. Let

$$G(X) = Q + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i, \quad X \in P(n),$$

and

$$\widetilde{G}(X) = \widetilde{Q} + \sum_{i=1}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \quad X \in P(n).$$

Observe that the solution of Eq.(1.1) and its perturbed equation (1.2) are the fixed points of G and $\widetilde{G}$, respectively. Now we will prove that $G, \widetilde{G} \in M(P(n), \delta)$.

It is easy to vertify that

$$G, \widetilde{G} : P(n) \to P(n).$$

For arbitrary $X, Y \in P(n)$, according to Lemma 3.1 and (3.3), we have

$$\begin{array}{rcl} d(G(X), G(Y)) & = & d(Q + \sum\limits_{i=1}^{m} A_i^* X^{\delta_i} A_i, Q + \sum\limits_{i=1}^{m} A_i^* Y^{\delta_i} A_i) \\ & \leq & d(\sum\limits_{i=1}^{m} A_i^* X^{\delta_i} A_i, \sum\limits_{i=1}^{m} A_i^* Y^{\delta_i} A_i) \\ & \leq & max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), d(\sum\limits_{i=2}^{m} A_i^* X^{\delta_i} A_i, \sum\limits_{i=2}^{m} A_i^* Y^{\delta_i} A_i)\} \\ & \leq & max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), max\{d(A_2^* X^{\delta_2} A_2, A_2^* Y^{\delta_2} A_2), \\ & & d(\sum\limits_{i=3}^{m} A_i^* X^{\delta_i} A_i, \sum\limits_{i=3}^{m} A_i^* Y^{\delta_i} A_i)\}\} \\ & = & max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), d(A_2^* X^{\delta_2} A_2, A_2^* Y^{\delta_2} A_2), \\ & & d(\sum\limits_{i=3}^{m} A_i^* X^{\delta_i} A_i, \sum\limits_{i=3}^{m} A_i^* Y^{\delta_i} A_i)\} \\ & \leq & \cdots\cdots \\ & \leq & max\{d(A_1^* X^{\delta_1} A_1, A_1^* Y^{\delta_1} A_1), d(A_2^* X^{\delta_2} A_2, A_2^* Y^{\delta_2} A_2), \\ & & \cdots, d(A_m^* X^{\delta_m} A_m, A_m^* Y^{\delta_m} A_m)\} \\ & \leq & max\{|\delta_1| d(X,Y), |\delta_2| d(X,Y), \cdots, |\delta_m| d(X,Y)\} \\ & = & \delta d(X,Y), \end{array}$$

Since $0 < \delta < 1$, we know that the map G is strict contraction on P(n) with the contraction constant δ. In a similar manner mentioned above, we obtain that the map $\widetilde{G}$ be also a strict contraction on P(n) with the contraction constant δ. Hence,

$$G, \widetilde{G} \in M(P(n), \delta).$$

From Lemma 3.2 it follows that the map G and $\widetilde{G}$ have a unique fixed point $\widehat{X}$ and $\widetilde{X}$ on $P(n)$ respectively, which are the unique positive definite solution of Eq.(1.1) and its perturbed equation (1.2).

For arbitrary $X \in P(n)$, according to Lemma 3.1 and (3.3), we have

$$\begin{aligned}
d(\widetilde{G}(X), G(X)) &= d(\widetilde{Q} + \sum_{i=1}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, Q + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i) \\
&\leq max\{d(\widetilde{Q}, Q), d(\sum_{i=1}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i)\} \\
&\leq max\{d(\widetilde{Q}, Q), d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), \\
&\quad d(\sum_{i=2}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=2}^{m} A_i^* X^{\delta_i} A_i)\} \\
&\leq max\{d(\widetilde{Q}, Q), d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), max\{d(\widetilde{A}_2^* X^{\delta_2} \widetilde{A}_2, \\
&\quad A_2^* X^{\delta_2} A_2), d(\sum_{i=3}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=3}^{m} A_i^* X^{\delta_i} A_i)\}\} \\
&= max\{d(\widetilde{Q}, Q), d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), d(\widetilde{A}_2^* X^{\delta_2} \widetilde{A}_2, \\
&\quad A_2^* X^{\delta_2} A_2), d(\sum_{i=3}^{m} \widetilde{A}_i^* X^{\delta_i} \widetilde{A}_i, \sum_{i=3}^{m} A_i^* X^{\delta_i} A_i)\} \\
&\leq \cdots\cdots \\
&\leq max\{d(\widetilde{Q}, Q), d(\widetilde{A}_1^* X^{\delta_1} \widetilde{A}_1, A_1^* X^{\delta_1} A_1), d(\widetilde{A}_2^* X^{\delta_2} \widetilde{A}_2, \\
&\quad A_2^* X^{\delta_2} A_2), \cdots, d(\widetilde{A}_m^* X^{\delta_m} \widetilde{A}_m, A_m^* X^{\delta_m} A_m)\} \\
&= max\{q, a_1, a_2, \cdots, a_m\}.
\end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we have

$$\sup_{X \in P(n)} d(\widetilde{G}(X), G(X)) \leq \sup_{X \in P(n)} max\{q, a_1, a_2, \cdots, a_m\} \leq min\{\frac{1-\delta}{3}\varepsilon, 1\},$$

and from Lemma 3.3 it follows that

$$d(\widetilde{X}, \widehat{X}) < \varepsilon. \qquad \square$$

4. Conclusion

In this paper, we consider the positive definite solution of the nonlinear matrix equation

$$X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q, \quad 0 < |\delta_i| < 1,$$

which arises in an optimal interpolation problem. Two new perturbation estimates for the unique positive definite solution are derived by making use of the elegant properties of spectral norm and Thompson metric.

Acknowledgements

The authors wish to thank the Editors and anonymous referees for providing very useful suggestions for improving this paper. This work was done during the first author as a postdoctoral research fellow at Department of Mathematics, Shanghai University. He is very much grateful for their providing good working conditions.

References

1. W. N. Anderson, Jr., T. D. Morley and G. E. Trapp, *Positive solutions to* $X = A - BX^{-1}B^*$, Linear Algebra Appl. **134**(1990), 53-62.
2. R. Bhatia, *Matrix analysis, Graduate Texts in Mathematics,* Springer-Verlag, Berlin, 1997.
3. M. S. Chen and S. F. Xu, *Perturbation analysis of the Hermitian positive definite solution of the matrix equation* $X - A^*X^{-2}A = I$, Linear Algebra Appl. **394**(2005), 39-51.
4. X. F. Duan, A. P. Liao and B. Tang, *On the nonlinear matrix equation* $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$, Linear Algebra Appl. **429**(2008), 110-121.
5. X. F. Duan and A. P. Liao, *On the existence of Hermitian positive definite solutions of the matrix equation* $X^s + A^*X^{-t}A = Q$, Linear Algebra Appl. **429**(2008), 673-687.
6. X. F. Duan and A. P. Liao, *On the nonlinear matrix equation* $X + A^*X^{-q}A = Q (q \geq 1)$, Math. Comput. Mod. **49**(2009), 936-945.
7. C. H. Guo and P. Lancaster, Iterative solution of two matrix equations, Math. Comput. **68**(1999), 1589-1603.
8. D. J. Gao and Y. H. Zhang, *On the Hermitian positive definite solutions of the matrix equation* $X - A^*X^qA = Q (q > 0)$, Mathematica Numerical Sinica **29**(2007), 73-80.
9. V. I. Hasanov, *Solutions and perturbation theory of nonlinear matrix equations*, Ph. D. Thesis, Sofia 2003.
10. V. I. Hasanov, *Positive definite solutions of the matrix equations* $X \pm A^*X^{-q}A = Q$, Linear Algebra Appl. **404**(2005), 166-182.
11. I. G. Ivanov, V. I. Hasanov, F. Uhilg, *Improved methods and starting values to solve the matrix equations* $X \pm A^*X^{-1}A = I$ *iteratively*, Math. comput. **74**(2004), 263-278.
12. X. G. Liu and H. Gao, *On the positive definite solutions of the matrix equation* $X^s \pm A^TX^{-t}A = I_n$, Linear Algebra Appl. **368**(2003), 83-97.
13. Y. Lim, *Solving the nonlinear matrix equation* $X = Q + \sum_{i=1}^{m} M_i^* X^{\delta_i} M_i$, *via a contraction principle*, Linear Algebra Appl. **430**(2009), 1380-1383.
14. R. D. Nussbaum, *Hilbert's projective metric and iterated nonlinear maps*, Memoirs of Amer. Math. Soc. 391, 1988.
15. Z. Y. Peng and S. M. El-Sayed, *On positive definite solution of a nonlinear matrix equation,* Numer. Linear Algebra Appl. **14**(2007), 99-113.
16. A. C. M. Ran, M. C. B. Reurings and A. L. Rodman, *A perturbation analysis for nonlinear selfadjoint operators*, SIAM J. Matrix Anal. Appl. **28**(2006), 89-104.
17. L. A. Sakhnovich, *Interpolation Theory and Its Applications, Mathematics and Its Applications*, Kluwer Academic, Dordrecht, 1997.
18. A. C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Pro. Amer. Math. Soc. **14**(1963), 438-443.
19. X. X. Zhan and J. J. Xie, *On the matrix equation* $X + A^TX^{-1}A = I$, Linear Algebra Appl. **247**(1996), 337-345.
20. Y. H. Zhang, *On Hermitian positive definite solutions of matrix equation* $X + A^*X^{-2}A = I$, Linear Algebra Appl. **372**(2003), 295-304.

Xue-feng Duan is a associate Professor of Guilin University of Electronic Technology. He received the M.S., and Ph.D. degrees from the college of Mathematics and Econometrics, Hunan University, in 2005 and 2008, respectively. His research interests focus on Numerical algebra, Numerical analysis for the matrix equation and Matrix low rank approximation.

College of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, P.R. China.
e-mail: duanxuefeng@guet.edu.cn

Qing-wen Wang was awarded Ph.D. in Mathematics from University of Science and Technology of China. He is currently a full professor and a supervisor of PhD students of Shanghai University. Dr. Wang has published more than sixty articles in the fields of matrix algebra; topics cover mainly matrix equations and quaternion matrices. He authored four books. Research has been sponsored by NWO of The Netherlands, the National Natural Science Foundation of China, etc.

Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China.
e-mail: wqw@shu.edu.cn

Chun-mei Li was awarded B.S. in Mathematics from Changsha University in 2008. She is currently a postgraduate student in Guilin University of Electronic Technology. Her research interests focus on the theory and numerical methods of the nonlinear matrix equations.

College of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, P.R. China.
e-mail: flx@guet.edu.cn