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A UNIFORM STRONG LAW OF LARGE NUMBERS FOR PARTIAL SUM PROCESSES OF FUZZY RANDOM SETS

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Abstract. In this paper, we consider fuzzy random sets as (measurable) mappings from a probability space into the set of fuzzy sets and prove a uniform strong law of large numbers for sequences of independent and identically distributed fuzzy random sets. Our results generalize those of Bass and Pyke(1984)and Jang and Kwon(1998).

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1. Introduction

The study of the fuzzy random sets, defined as measurable mappings on a probability space, was initiated by Kwakernaak(1978) where useful basic properties were developed. Puri and Ralescu (1983) used the concept of fuzzy random variables in generating results for random sets to fuzzy random sets. Kruse(1978) proved a strong law of large numbers for independent identically distributed fuzzy random variables. Artstein and Vitale(1975) proved a strong law of large numbers(SLLN) for R^p-valued random sets and Cressie(1978) proved a SLLN for some paticular class of R^p-valued random sets. Using Rådstrom embedding(e.g. Rådstrom(1952)), Puri and Ralescu(1983) proved a SLLN for Banach space valued random sets and they also proved SLLN for fuzzy random sets, which generalized all of previous SLLN for random sets. Jang and Kwon(1998) proved a uniform strong law of large numbers for sequences of independent and identically distributed random sets which generalized that of Bass and Pyke(1984).

In this paper we consider fuzzy random sets as (measurable) mappings from a probability space into the set of fuzzy sets of a Euclidean space and we prove

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a uniform strong law of large numbers for sequences of independent and identically distributed fuzzy random sets. Our results generalize that of Bass and Pyke(1984) and Jang and Kwon(1998).

2. Definitions and Preliminaries

Let $\mathbf{C}(R^p) = \{A \subset R^p : A \quad \text{nonempty, compact}\}$ and $\mathbf{K}(R^p) = \{A \in C(R^p) : A \quad \text{convex}\}$. The space $\mathbf{C}(R^p)(\mathbf{K}(R^p))$ has a linear structure induced by the operations

$$A + B = \{a + b : a \in A, \quad b \in B\}, \qquad \lambda A = \{\lambda a : a \in A\}$$

for $A, B \in \mathbf{C}(R^p)(\mathbf{K}(R^p))$, $\lambda \in R$. The Hausdorff distance between two sets $A, B \in \mathbf{C}(R^p)$ is defined as

$$d(A, B) = \inf\{\lambda > 0 : A \subset B + \lambda U, \quad B \subset A + \lambda U\},$$

where $U = \{u : \|u\| \leq 1\}$. Denote the norm of $A \in \mathbf{C}(R^p)$ by $\|A\| = d(A, \{0\}) = \sup_{a \in A} \|a\|$. Since the space $\mathbf{K}(R^p)$ is a subset of $\mathbf{C}(R^p)$ these definitions induce a topological structure on $\mathbf{K}(R^p)$. It is well known that neither $\mathbf{C}(R^p)$ nor $\mathbf{K}(R^p)$ is a linear space, but they are separable complete metric spaces with the metric d. If (Ω, P) is a probability space, a random set is defined as a Borel measurable function $X : \Omega \to (\mathbf{C}(R^p), d)$. The expected value EX (Aumann(1965) and Hiai and Umegaki(1977)) is defined by $EX = \{E\phi | \phi : \Omega \to R^p, E\|\phi\| < \infty, \quad \phi(\omega) \in X(\omega) \,\text{a.e.}\}$. Note that if $E\|X\| < \infty$, the Bochner integral can be defined as $E(coX) = \int coX dP$ and $E(coX) \in \mathbf{K}(R^p)$.

Let $X_n, n \geq 1$ be a family of independent identically distributed random sets. Puri and Ralescu(1983) proved the following:

Theorem 2.1 (Puri and Ralescu(1983))**.** *If $X_n, n \geq 1$ is a sequence of independent and identically distributed random sets such that $E(\|X\|) < \infty$, then $\sum_{i=1}^n X_i/n \to E(X_1)$ a.s., the convergence being in the metric d.*

A fuzzy sets of R^p is a function of $u : R^p \to [0, 1]$. For each such fuzzy set u, we denote by $L_\alpha u = \{x \in R^p | u(x) \geq \alpha\}$, $0 \leq \alpha \leq 1$. its $\alpha-$ level sets. By *suppu* we denote the support of u, i.e. the closure of of the set $\{x \in R^p | u(x) > 0\}$. We consider the collection $\mathbf{F}(R^p)$ of those fuzzy sets $u : R^p \to [0, 1]$ with the following properties ; (1) u is upper semi-continuous (2) *suppu* is compact (3) $\{x \in R^p | u(x) = 1\} \neq \emptyset$.

The spaces extends $\mathbf{K}(R^p)$ in the sense that for each $A \in \mathbf{K}(R^p)$, its characteristic function $\chi_A \in \mathbf{K}(R^p)$. Define, for $u, v \in \mathbf{F}(R^p)$, $\lambda \in R$, $(u + v)(x) = \sup_{y+z=x} \min[u(y), v(z)]$. $(\lambda u)(x) = u(\lambda^{-1}x)$ if $\lambda \neq 0$ and $(\lambda u)(x) = \chi_{\{}0\}(x)$ if $\lambda = 0$. The by simple topological arguments and properties of upper semi continuous functions, it is easy to se that $u+v, \lambda \in \mathbf{F}(R^p)$ and $L_\alpha(u+v) = L_\alpha u + L_\alpha v$ and $L_\alpha(\lambda u) = \lambda L_\alpha u$ for $0 \leq \alpha \leq 1$. In this paper we will use two metrics as follow:

$$d_1(u, v) = \int_0^1 d(L_\alpha u, L_\alpha v) d\alpha$$

and

$$d_\infty(u,v) = \sup_{\alpha>0} d(L_\alpha, L_\alpha v)$$

which was originally defined in Puri and Ralescu(1983). Clearly $d_1(\chi_A, \chi_B) = d_\infty(\chi_A, \chi_B) = d(A,B)$, for $A, B \in \mathbf{K}(R^p)$. Then $\mathbf{F}(R^p, d_1)$ is a separable metric spaces.

A fuzzy random set is a Borel measurable function $X : \Omega : \mathbf{F}((R^p), d_\infty)$. If X is a fuzzy random variable such that $E||suppX|| < \infty$, then the expected value EX is the fuzzy set satisfying $L_\alpha(EX) = E(L_\alpha X)$, $0 < \alpha \leq 1$. It is well known that $E(X) : R^p \to [0,1]$ is a upper semi continuous, $L_1(EX) \neq \emptyset$ and $(EX) \in \mathbf{F}(R^p)$. we will also need the concept of a convex hull of a fuzzy set. If $u \in \mathbf{F}(R^p)$, then $cou \in \mathbf{F}_c(R^p)$, the convex hull of u is defined by $cou = \inf\{v \in \mathbf{F}_c(R^p) | v \geq u\}$. If $X : \Omega \to \mathbf{F}(R^p)$ is a fuzzy random set, then $coX : \Omega \to \mathbf{F}_c(R^p)$ is defined by $coX(\omega) = coX(\omega)$.

Theorem 2.2 (Klement and Ralescu(1986)). *If $X_n, n \geq 1$ is a sequence of independent and identically distributed fuzzy random sets such that $E(\|suppX_1\|) < \infty$, then $\sum_{i=1}^n X_i/n \to E(coX_1)$ a.s., the convergence being in the metric d_1.*

If $B \subset [0,\infty)^d$ is Borel measurable, define

$$S(B) = \sum_{j\in B} X_j$$

to be the partial sum of random sets whose index is in B. Let $|B|$ denote the Lebesgue measure of B. Then our question in the paper is : if $\{B_n\}_{n=1}^\infty$ is a sequence of sets (not necessarily nested) with $|B_n| \to \infty$, will $\frac{S(B_n)}{|B_n|}$ converge to $EcoX$ a.s. and will this convergence be uniform over a large family of such sets? We provide some answers to the questions under the condition(listed below) on an index family $\mathbf{A}$ where $\mathbf{A}$ denote a subfamily of $[0,1]^d$.

Given a set B, let $nB = \{nx : x \in B\}$ and $B(\delta) = \{x : \rho(x, \partial B) < \delta\}$ be the δ-annulus of ∂B, where $\rho(\cdot,\cdot)$ is Euclidean distance and ∂B is the boundary of B. Define $r(\delta) = \sup_{A\in\mathbf{A}} |A(\delta)|$. It is said that $\mathbf{A}$ satisfies the smooth boundary condition(SBC) when $r(\delta) \to 0$ as $n \to \infty$. Under the SBC on $\mathbf{A}$, Bass and Pyke(1984) proved uniform strong law of large numbers for random variables as follows;

Theorem 2.3 (Bass and Pyke(1984)). *Let $X_n, n \geq 1$ be a sequence of independent and identically distributed random variables such that $E(|X|) < \infty$. Suppose $\mathbf{A}$ is a collection of Lebesgue measurable subsets of $[0,1]^d$ such that $r(\delta) = \sup_{A\in\mathbf{A}} |A(\delta)| \to 0$ as $\delta \to 0$, then*

$$\sup_{A\in\mathbf{A}} d\left(\frac{S(nA)}{n^d}, |A|EX\right) \to 0 \quad a.s. \quad as \quad n \to \infty.$$

Theorem 2.4 (Jang and Kwon(1998)). *Let $X, X_n, n \geq 1$ be a sequence of independent and identically distributed random sets such that $E(\|X\|) < \infty$.*

And let $\mathbf{A}$ *be a collection of Lebesgue measurable subsets of* $[0,1]^d$ *such that* $r(\delta) = \sup_{A\in\mathbf{A}} |A(\delta)| \to 0$ *as* $\delta \to 0$. *Then*

$$\sup_{A\in\mathbf{A}} d\left(\frac{S(nA)}{n^d}, |A|EX\right) \to 0 \quad a.s. \quad as \quad n \to \infty.$$

3. Main Results

Theorem 3.1. *Let* $X, X_n, n \geq 1$ *be a sequence of independent and identically distributed fuzzy random sets such that* $E(\|suppX\|) < \infty$. *And let* $\mathbf{A}$ *be a collection of Lebesgue measurable subsets of* $[0,1]^d$ *such that* $r(\delta) = \sup_{A\in\mathbf{A}} |A(\delta)| \to 0$ *as* $\delta \to 0$. *Then*

$$\sup_{A\in\mathbf{A}} d_1\left(\frac{S(nA)}{n^d}, |A|EcoX\right) \to 0 \quad a.s. \quad as \quad n \to \infty.$$

Before getting into the proof of theorem 3.1 we need a lemma as follows;

Lemma 3.1. *For* $u,\, v \in \mathbf{F}(R^p)$ *and* $a,\, b > 0$

(1) $d_1(au, bu) \leq |a-b|\|u\|$
(2) $d_1(u, u+v) \leq \|v\|$
(3) $d_1(au, av) = |a| d_1(u,v)$

Proof. $\|u\| = d_1(u, \chi_{\{0\}}) = \int_0^1 d(L_\alpha u, L_\alpha(0))d\alpha = \int_0^1 \|L_\alpha u\| d_\alpha$
For (1)

$$\begin{aligned} d_1(au, bv) &= \int_0^1 d(L_\alpha(au), L_\alpha(bu))d\alpha = \int_0^1 d(aL_\alpha(u), bL_\alpha(u))d\alpha \\ &\leq \int_0^1 |a-b|\|L_\alpha(u)\|d\alpha \leq |a-b|\int_0^1 \|L_\alpha u\|d\alpha \\ &\leq \int_0^1 |a-b|\|L_\alpha(u)\|d\alpha \leq |a-b|\int_0^1 \|L_\alpha u\|d\alpha \end{aligned}$$

For(2)

$$\begin{aligned} d_1(u, u+v) &= \int_0^1 d(L_\alpha(u), L_\alpha(u+v))d\alpha = \int_0^1 d(L_\alpha(u), L_\alpha(v+v))d\alpha \\ &= \int_0^1 d(L_\alpha(u), L_\alpha u + L_\alpha v)d\alpha \leq \int_0^1 \|L_\alpha v\|d\alpha \leq \|v\| \end{aligned}$$

For (3)

$$d_1(au, av) = \int_0^1 d(L_\alpha(au), L_\alpha(av))d\alpha = \int_0^1 |a| d(L_\alpha(u), L_\alpha(v))d\alpha = |a|d(u,v)$$

□

Proof of the theorem 3.1 First of all, if $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ is fixed, denote $(0, \mathbf{x}] = \{(y_1, y_2, \cdots, y_d) : 0 < y_i \le x_i, i = 1, 2, \cdots d\}$ and $\sharp$ cardinality, then by theorem 2.1,

$$\frac{S(n(0,\mathbf{x}])}{\sharp(J \cap n(0,\mathbf{x}])} \to EcoX \quad \text{a.s..}$$

Hence, we have

$$n^{-d} S(n(0,\mathbf{x}]) = \frac{\sharp(J \cap n(0,\mathbf{x}])}{n^d} \cdot \frac{S(n(0,\mathbf{x}])}{\sharp(J \cap n(0,\mathbf{x}])} \to |(0,\mathbf{x}]| EcoX.$$

Secondly, if A can be obtained by a finite number of unions and differences of rectangles of the form $(0, \mathbf{x}]$, then by linearity we have

$$n^{-d} S(nA) \to |A| EcoX \quad \text{a.s..}$$

If m is an integer, let $C_j = m^{-1}(j-1, j]$, and for any $A \in \mathbf{A}$, let $R_m^-(A) = \cup_{C_j \subset A} C_j$ and $R_m^+(A) = \cup_{C_j \cap A \neq \emptyset} C_j$. Then $R_m^-(A)$ and $R_m^+(A)$ are inner and outer fits of A by cubes of size $1/m$. Since the furthest any point of $R_m^+(A) \setminus R_m^-(A)$ can be from the boundary of A is the diameter of a cube of size $1/m$, we have by assumption

$$\sup_{A \in \mathbf{A}} |R_m^+(A) \setminus R_m^-(A)| \le r(\frac{d^{1/2}}{m}).$$

Let $\mathbf{R}_m^- = \{R_m^-(A) : A \in \mathbf{A}\}$ and $\mathbf{R}_m^{\triangle} = \{R_m^+(A) \setminus R_m^-(A) : A \in \mathbf{A}\}$. Since each $A \in \mathbf{A}$ is contained in $[0,1]^d$ it should be evident that $\sharp\mathbf{R}_m^-$ and $\sharp\mathbf{R}_m^{\triangle}$ are finite. Then, for m fixed, we have

$$\begin{aligned}
\limsup_{n\to\infty} \sup_{A \in \mathbf{A}} d_1(n^{-d} S(nA),\ |A| EcoX) \le & \limsup_{n\to\infty} \sup_{A \in \mathbf{A}} n^{-d} d_H(S(nA), S(nR_m^-(A))) \\
& + \limsup_{n\to\infty} \sup_{A \in \mathbf{A}} d_1(n^{-d} S(nR_m^-(A)), |R_m^-(A)| EcoX) \\
& + \limsup_{n\to\infty} \sup_{A \in \mathbf{A}} d_1(|A| EX, |R_m^-(A)| EcoX) \\
= & I_1 + I_2 + I_3
\end{aligned}$$

For (I_1), notice that by lemma (2)

$$d_1(S(nA), S(nR_m^-(A))) = d_1(\sum_{j \in nA} X_j, \sum_{j \in nR_m^-(A)} X_j) \le \sum_{j \in (nR_m^+(A) \setminus nR_m^-(A))} \|suppX_j\|.$$

Therefore,

$$\begin{aligned}
I_1 & \le \limsup_{n\to\infty, A \in \mathbf{A}} n^{-d} \sum_{j \in (nR_m^+(A) \setminus nR_m^-(A))} \|suppX_j\| \\
& \le \limsup_{n\to\infty} \max_{B \in \mathbf{R}_m^{\triangle}} n^{-d} \sum_{j \in B} \|suppX_j\| \qquad (3.1) \\
& \le E\|suppX\| \max_{B \in \mathbf{R}_m^{\triangle}} |B| \le E\|suppX\| r(\frac{d^{1/2}}{m}) \quad \text{a.s.}
\end{aligned}$$

where in the third inequality we used the classical Kolmogorov strong law of large numbers for real random variables.

For (I_2), since $\sharp\mathbf{R}_m^- < \infty$ and every set $B \in \mathbf{R}_m^-$ can be obtained by a finite number of unions and differences of rectangles of the form $(0, \mathbf{x}]$, by theorem 2.1 we have

$$\begin{aligned} I_2 &= \limsup_{n\to\infty, A\in\mathbf{A}} d_1(n^{-d}S(nR_m^-(A)), |R_m^-(A)|EcoX) \\ &\le \limsup_{n\to\infty, B\in\mathbf{R_m^-}} d_1(n^{-d}S(nB), |B|EcoX) \longrightarrow 0 \quad \text{a.s..} \end{aligned} \tag{3.2}$$

Finally, let us do I_3. By lemma (1), $d_1(|A|EcoX, |R_m^-(A)|EcoX) \le |A \setminus R_m^-(A)|\|EcoX\|$. Therefore,

$$I_3 \le \|EcoX\| r(\frac{d^{1/2}}{m}). \tag{3.3}$$

Summing (3.1), (3.2) and (3.3) up, we have

$$\limsup_{n\to\infty} \sup_{A\in\mathbf{A}} d_1(n^{-d}S(nA), |A|EcoX) = (\|EcoX\| + E\|coX\|) r(\frac{d^{1/2}}{m}) \quad \text{a.s..} \tag{3.4}$$

Letting $m \to \infty$, the right side of (3.4) approaches to 0, which completes the proof of theorem 3.1.

Remark. Suppose that we are given a sequence of sets B_n such that $|B_n| \to \infty$, as in section 2. Let $A_n = n^{-1}B_n$ and let $\mathbf{A} = \{A_n\}$. If $\mathbf{A}$ satisfies SBC and $|A_n|$ is bounded away from 0, then

$$\begin{aligned} \limsup_{n\to\infty} d_1\left(\frac{S(B_n)}{|B_n|}, EcoX\right) &= \limsup_{n\to\infty} d_1\left(\frac{S(nA_n)}{n^d|A_n|}, EcoX\right) \\ &= \limsup_{n\to\infty} |A_n|^{-1} d_1\left(\frac{S(nA_n)}{n^d}, |A_n|EcoX\right) \\ &\le \limsup_{n\to\infty} |A_n|^{-1} \limsup_{A\in\mathbf{A}} d_1\left(\frac{S(nA)}{n^d}, |A|EcoX\right) \\ &= 0 \quad \text{a.s..} \end{aligned}$$

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