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ON HàJEK-RèNYI-TYPE INEQUALITY FOR CONDITIONALLY NEGATIVELY ASSOCIATED RANDOM VARIABLES AND ITS APPLICATIONS†

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ABSTRACT. Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and $\{X_n | n \geq 1\}$ be a sequence of random variables defined on it. A finite sequence of random variables $\{X_n | n \geq 1\}$ is said to be conditionally negatively associated given $\mathcal{F}$ if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$, $Cov^{\mathcal{F}}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$ *a.s.* whenever f_1 and f_2 are coordinatewise nondecreasing functions. We extend the Hàjek-Rènyi-type inequality from negative association to conditional negative association of random variables. In addition, some corollaries are given.

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1. Introduction

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and $\{X_n | n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $\{\Omega, \mathcal{F}, P\}$.

A finite sequence of random variables $\{X_i | 1 \leq i \leq n\}$ is said to be negatively associated(NA) if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinatewise nondecreasing functions f_1 on R^A, f_2 on R^B,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$$

whenever covariance exists. An infinite sequence of random variables $\{X_n | n \geq 1\}$ is NA if every finite subfamily is NA. This concept was first introduced by Alam and Saxena[1]. Joag-Dev and Proschan[6] showed that many well known multivariate distributions possess the NA property, and concepts of NA random variables are of considerable uses in system reliability theory, percolation theory

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and multivariate analysis of statistics. We refer to Joag-Dev and Proschan[6] for fundamental properties, Matula[11] for the three series theorem, Su et al.[19] and Shao[18] for moment inequalities, Shao and Su[17] for the law of the iterated logarithm, Liang and Su[7] for complete convergence, Newman[12] for the central limit theorem, Lin[8] for the invariance principle, among others.

Let X and Y be random variables with $EX^2 < \infty$ and $EY^2 < \infty$. Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. Prakasa Rao[14] defined the notion of the conditional covariance of X and Y given $\mathcal{F}$ as

$$Cov^{\mathcal{F}}(X,Y) = E^{\mathcal{F}}((X - E^{\mathcal{F}}X)(Y - E^{\mathcal{F}}Y)),$$

where $E^{\mathcal{F}}Z$ denotes the conditional expectation of a random variable Z given $\mathcal{F}$.

On the basis of the above definition of conditional covariance, Yuan et al.[21] introduced a new kind of dependence called conditional negative association, which is an extension the corresponding non-conditional case.

Definition 1.1. A finite family of random variables $\{X_i | 1 \leq i \leq n\}$ is said to be conditional negatively associated given $\mathcal{F}(\mathcal{F} - NA)$ if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$, and any real coordinatewise nondecreasing functions f_1 on R^A, f_2 on R^B,

$$Cov^{\mathcal{F}}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0 \ a.s.$$

whenever $\mathcal{F}$-covariance exists. An infinite sequence of random variables $\{X_n | n \geq 1\}$ is $\mathcal{F} - NA$ if every finite subfamily is $\mathcal{F} - NA$. Yuan et al. presented the relation between negative association and conditional negative association, that is, the negative association does not imply the conditional negative association, and vice versa.

Hàjek-Rènyi[4] proved the following important inequality. If $\{X_n, n \geq 1\}$ is sequence of independent random variables with mean zero and finite second moments, and $\{b_n, n \geq 1\}$ is a sequence of positive nondecreasing real numbers, then for any $\varepsilon > 0$,

$$P\left(\max_{m \leq k \leq n} \left| \frac{\sum_{j=1}^{k} X_j}{b_k} \right| > \varepsilon\right) \leq \varepsilon^{-2} \left(\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \frac{1}{b_m^2} \sum_{j=1}^{m} EX_j^2 \right).$$

The Hàjek-Rènyi- type inequality was studied by many authors (see Gan[3], Liu et al.[9], Cai[2], Prakasa Rao [13], Hu et al.[5], Qiu et al.[15], Rao[16], and Sung[20], etc.) The main purpose of this paper is to extend the Hàjek-Rènyi inequality from negative association to $\mathcal{F} - NA$ random variables, and this motivate our original interest in conditional negative association of random variables. In particular, we proved the conditional Hàjek-Rènyi -type inequality, which is conditional versions of the earlier results for NA random variables. As an applications, we obtain the integrability of supremum and strong law of large numbers for conditionally negatively associated random variables. Finally, throughout this paper, $\{X_n|n \geq 1\}$ will be called conditionally centered if $E^{\mathcal{F}}X_n = 0$ for every $n \geq 1$ and c will be represented positive constants which their value may change from one place to another.

2. The Hàjek-Rènyi-type inequality for $\mathcal{F} - NA$ random variables

To prove the Hàjek-Rènyi-type inequality for $\mathcal{F} - NA$ random variables, we need the following conditional Rosenthal type inequalities for $\mathcal{F} - NA$ random variables which is extended the corresponding results for NA random variables (see, Shao[18] and Su et al.[19]).

Lemma 2.1 ([21])**.** Let $\{X_n|n \geq 1\}$ be a sequence of conditionally centered $\mathcal{F} - NA$ random variables with $E^{\mathcal{F}}|X_n|^p < \infty$ $a.s.$, $n \geq 1$ and $p \geq 1$. Then there exists a positive constant c such that for all $n \geq 1$,

$$E^{\mathcal{F}}\left(\max_{1\leq k\leq n}|\sum_{i=1}^{k}X_i|^p\right) \leq c\sum_{i=1}^{n}E^{\mathcal{F}}|X_i|^p \ a.s.\ for\ 1 \leq p \leq 2$$

and

$$E^{\mathcal{F}}\left(\max_{1\leq k\leq n}|\sum_{i=1}^{k}X_i|^p\right) \leq c\left(\sum_{i=1}^{n}E^{\mathcal{F}}|X_i|^p + (\sum_{i=1}^{n}E^{\mathcal{F}}X_i^2)^{p/2}\right) \ a.s.\ for\ p > 2.$$

Using Lemma 2.1, we can obtain the following Hàjek-Rènyi-type inequality for $\mathcal{F} - NA$ random variables.

Theorem 2.1. Let $\{X_n|n \geq 1\}$ be a sequence of conditionally centered $\mathcal{F} - NA$ random variables with $E^{\mathcal{F}}|X_n|^p < \infty$ $a.s.$,$n \geq 1$ and $p \geq 1$ and let $\{b_n|n \geq 1\}$ be a sequence of positive nondecreasing real numbers. Then, for any $\mathcal{F}$-measurable variables $\varepsilon > 0$ $a.s.$, and positive integers m, n with $m \leq n$,

$$P\left(\max_{m\leq k\leq n}\left|\frac{S_k}{b_k}\right| \geq \varepsilon|\mathcal{F}\right) \leq c\varepsilon^{-2}\left(\sum_{i=1}^{m}\frac{E^{\mathcal{F}}|X_i|^p}{b_m^p} + \sum_{i=m+1}^{n}\frac{E^{\mathcal{F}}|X_i|^p}{b_i^p}\right) \ a.s.\ for\ 1 \leq p \leq 2$$

and

$$P\left(\max_{m\leq k\leq n}\left|\frac{S_k}{b_k}\right| \geq \varepsilon|\mathcal{F}\right) \leq c\varepsilon^{-2}\left(\sum_{i=1}^{m}\frac{E^{\mathcal{F}}|X_i|^p}{b_i^p} + (\sum_{i=1}^{m}\frac{E^{\mathcal{F}}X_i^2}{b_i^2})^{p/2}\right.$$
$$\left. + \sum_{i=m+1}^{n}\frac{E^{\mathcal{F}}|X_i|^p}{b_i^p} + (\sum_{i=m+1}^{n}\frac{E^{\mathcal{F}}X_i^2}{b_i^2})^{p/2}\right) \ a.s.\ for\ p > 2$$

Proof. Let $S_k = \sum_{i=1}^{k}X_i = \sum_{i=1}^{m}X_i + \sum_{i=m+1}^{k}X_i$, for any $1 \leq m \leq k$, and let $\varepsilon > 0$ $a.s.$ be an arbitrary $\mathcal{F}$-measurable variable. Without loss of generality, setting $b_0 = 0$, we have

$$\sum_{i=m+1}^{k}X_i = \sum_{i=m+1}^{k}\frac{b_iX_i}{b_i} = \sum_{i=m+1}^{k}\left(\sum_{j=1}^{i}(b_j - b_{j-1})\frac{X_i}{b_i}\right)$$

$$= \sum_{j=1}^{m}(b_j - b_{j-1}) \sum_{i=m+1}^{k} \frac{X_i}{b_i} + \sum_{j=m+1}^{k}(b_j - b_{j-1}) \sum_{i=j}^{k} \frac{X_i}{b_i}$$

$$= b_m \sum_{i=m+1}^{k} \frac{X_i}{b_i} + \sum_{j=m+1}^{k}(b_j - b_{j-1}) \sum_{i=j}^{k} \frac{X_i}{b_j},$$

and

$$\frac{S_k}{b_k} = \sum_{i=1}^{k} \frac{X_i}{b_k} = \sum_{i=1}^{m} \frac{X_i}{b_k} + \frac{b_m}{b_k} \sum_{i=m+1}^{k} \frac{X_i}{b_i} + \sum_{j=m+1}^{k} \frac{(b_j - b_{j-1})}{b_k} \sum_{i=j}^{k} \frac{X_i}{b_i}.$$

Since $\sum_{j=m+1}^{k}(b_j - b_{j-1}) = b_k - b_m < b_k$ and $\{b_n|n \geq 1\}$ is a positive nondecreasing real number sequence, we have

$$\left(|\frac{S_k}{b_k}| \geq \varepsilon\right) \subset \left(|\sum_{i=1}^{m} \frac{X_i}{b_k}| \geq \frac{\varepsilon}{2}\right) \cup \left(\max_{m+1 \leq j \leq k} |\sum_{i=j}^{k} \frac{X_i}{b_i}| \geq \frac{\varepsilon}{2}\right),$$

$$\begin{aligned}
&\left(\max_{m \leq k \leq n} |\frac{S_k}{b_k}| \geq \varepsilon\right) \\
&\subset \left(\max_{m \leq k \leq n} |\sum_{i=1}^{m} \frac{X_i}{b_k}| \geq \frac{\varepsilon}{2}\right) \cup \left(\max_{m \leq k \leq n} \max_{m+1 \leq j \leq k} |\sum_{i=j}^{k} \frac{X_i}{b_i}| \geq \frac{\varepsilon}{2}\right) \\
&\subset \left(|\sum_{i=1}^{m} \frac{X_i}{b_m}| \geq \frac{\varepsilon}{2}\right) \cup \left(\max_{m+1 \leq k \leq n} |\sum_{i=m+1}^{k} \frac{X_i}{b_i}| \geq \frac{\varepsilon}{4}\right)
\end{aligned}$$

Hence, by $\varepsilon > 0$ *a.s.* is an $\mathcal{F}$-measurable random variable,

$$\begin{aligned}
&P\left(\max_{m \leq k \leq n} |\frac{S_k}{b_k}| \geq \varepsilon|\mathcal{F}\right) \\
&\leq P\left(|\sum_{i=1}^{m} \frac{X_i}{b_m}| \geq \frac{\varepsilon}{2}|\mathcal{F}\right) + P\left(\max_{m+1 \leq k \leq n} |\sum_{i=m+1}^{k} \frac{X_i}{b_i}| \geq \frac{\varepsilon}{4}|\mathcal{F}\right) \\
&=: I_1 + I_2
\end{aligned}$$

Therefore, we have by Lemma 2.1,

$$\begin{aligned}
I_1 &= P\left(|\sum_{i=1}^{m} \frac{X_i}{b_m}| \geq \frac{\varepsilon}{2}|\mathcal{F}\right) \leq P\left(\frac{1}{b_m} \max_{1 \leq k \leq m} |\sum_{i=1}^{k} X_i| \geq \frac{\varepsilon}{2}|\mathcal{F}\right) \qquad (1) \\
&\leq c\varepsilon^{-2} \sum_{i=1}^{m} \frac{E^{\mathcal{F}}|X_i|^p}{b_m^p} \ a.s. \ for \ 1 \leq p \leq 2
\end{aligned}$$

and

$$I_1 \leq c\varepsilon^{-2}\left(\sum_{i=1}^{m} \frac{E^{\mathcal{F}}|X_i|^p}{b_i^p} + (\sum_{i=1}^{m} \frac{E^{\mathcal{F}} X_i^2}{b_i^2})^{p/2}\right) \ a.s. \ for \ p > 2 \qquad (2)$$

Next, as to I_2, we have

$$I_2 = P\Big(\max_{m+1 \le k \le n} | \sum_{i=m+1}^{k} \frac{X_i}{b_i} | \ge \frac{\varepsilon}{4} | \mathcal{F} \Big) \tag{3}$$

$$\le c\varepsilon^{-2} \sum_{i=m+1}^{n} \frac{E^{\mathcal{F}} |X_i|^p}{b_i^p} \ a.s. \ for \ 1 \le p \le 2$$

and

$$I_2 \le c\varepsilon^{-2} \Big(\sum_{i=m+1}^{n} \frac{E^{\mathcal{F}} |X_i|^p}{b_i^p} + (\sum_{i=m+1}^{n} \frac{E^{\mathcal{F}} X_i^2}{b_i^2})^{p/2} \Big) \ a.s. \ for \ p > 2 \tag{4}$$

Thus, combining (1), (2), (3) and (4), we obtain

$$P\Big(\max_{m \le k \le n} \Big| \frac{S_k}{b_k} \Big| \ge \varepsilon | \mathcal{F} \Big) \le c\varepsilon^{-2} \Big(\sum_{i=1}^{m} \frac{E^{\mathcal{F}} |X_i|^p}{b_m^p} + \sum_{i=m+1}^{n} \frac{E^{\mathcal{F}} |X_i|^p}{b_i^p} \Big) \ a.s. \ for \ 1 \le p \le 2$$

and

$$P\Big(\max_{m \le k \le n} \Big| \frac{S_k}{b_k} \Big| \ge \varepsilon | \mathcal{F} \Big) \le c\varepsilon^{-2} \Big(\sum_{i=1}^{m} \frac{E^{\mathcal{F}} |X_i|^p}{b_i^p} + (\sum_{i=1}^{m} \frac{E^{\mathcal{F}} X_i^2}{b_i^2})^{p/2}$$

$$+ \sum_{i=m+1}^{n} \frac{E^{\mathcal{F}} |X_i|^p}{b_i^p} + (\sum_{i=m+1}^{n} \frac{E^{\mathcal{F}} X_i^2}{b_i^2})^{p/2} \Big) \ a.s. \ for \ p > 2$$

The proof is complete. □

Corollary 2.1. Under the conditions of Theorem 2.1, taking $m = 1$, we obtain the following results;

$$P\Big(\max_{1 \le k \le n} \Big| \sum_{i=1}^{k} \frac{X_i}{b_k} \Big| \ge \varepsilon | \mathcal{F} \Big) \le c\varepsilon^{-2} \sum_{i=1}^{n} \frac{E^{\mathcal{F}} |X_i|^p}{b_i^p} \ a.s. \ for \ 1 \le p \le 2$$

and

$$P\Big(\max_{1 \le k \le n} \Big| \sum_{i=1}^{k} \frac{X_i}{b_k} \Big| \ge \varepsilon | \mathcal{F} \Big) \le c\varepsilon^{-2} \Big(\sum_{i=1}^{n} \frac{E^{\mathcal{F}} |X_i|^p}{b_i^p} + (\sum_{i=1}^{n} \frac{E^{\mathcal{F}} X_i^2}{b_i^2})^{p/2} \Big) \ a.s. \ for \ p > 2$$

Corollary 2.2. Under the conditions of Theorem 2.1, taking $p = 2$, we have

$$P\Big(\max_{1 \le k \le n} \Big| \sum_{i=1}^{k} \frac{X_i}{b_k} \Big| \ge \varepsilon | \mathcal{F} \Big) \le c\varepsilon^{-2} \sum_{i=1}^{n} \frac{E^{\mathcal{F}} X_i^2}{b_i^2} \ a.s.$$

Corollary 2.3. Under the condition of Theorem 2.1, taking $p = 2$ and $b_k = k$, $k = m + 1, \cdots, n$,

$$P\Big(\max_{m \le k \le n} \Big| \sum_{i=1}^{k} \frac{X_i}{k} \Big| \ge \varepsilon | \mathcal{F} \Big) \le \sum_{k=1}^{m} \frac{E^{\mathcal{F}} X_k^2}{m^2} + \sum_{k=m+1}^{n} \frac{E^{\mathcal{F}} X_k^2}{k^2} \ a.s.$$

3. Strong law of large numbers for $\mathcal{F}-NA$ random variables

In this section, we obtain the integrability of supremum and strong law of large numbers for $\mathcal{F}-NA$ random variables.

Theorem 3.1. Let $\{b_n|n \geq 1\}$ be a sequence of positive nondecreasing real numbers and let $\{X_n|n \geq 1\}$ be a sequence of conditionally centered $\mathcal{F}-NA$ random variables such that $\sum_{n=1}^{\infty} \frac{E^{\mathcal{F}}|X_n|^2}{b_n^2} < \infty$ *a.s.*, $n \geq 1$, then for $0 < r < 2$ and conditionally on $\mathcal{F}$,

(a) $E^{\mathcal{F}} \sup_{n\geq 1} \left(\frac{|\sum_{i=1}^{n} X_i|}{b_n} \right)^r < \infty$ *a.s.*

(b) If $0 < b_n \to \infty$, then $\sum_{i=1}^{n} \frac{X_i}{b_n} \to 0$ *a.s. as* $n \to \infty$.

Proof of (a). Note that

$$E^{\mathcal{F}} \sup_{n\geq 1} \left(\frac{|\sum_{i=1}^{n} X_i|}{b_n} \right)^r < \infty \ a.s. \ \Leftrightarrow \int_1^{\infty} P\left(\sup_{n\geq 1} \frac{|\sum_{i=1}^{n} X_i|}{b_n} \geq t^{1/r} |\mathcal{F} \right) dt < \infty \ a.s.$$

Taking an enough large natural number L, we obtain from Theorem 2.1 that

$$\begin{aligned}
&\int_1^{\infty} P\left(\sup_{n\geq 1} \frac{|\sum_{i=1}^{n} X_i|}{b_n} \geq t^{1/r} |\mathcal{F} \right) dt \\
&\leq \int_1^{\infty} P\left(\max_{1\leq n<L} \frac{|\sum_{i=1}^{n} X_i|}{b_n} \geq t^{1/r} |\mathcal{F} \right) dt \\
&+ \int_1^{\infty} P\left(\bigcup_{k=1}^{\infty} (\max_{kL\leq n<(k+1)L} \frac{|\sum_{i=1}^{n} X_i|}{b_n} \geq t^{1/r} |\mathcal{F} \right) dt \\
&\leq c \sum_{i=1}^{L} \frac{E^{\mathcal{F}}|X_i|^2}{b_i^2} \int_1^{\infty} t^{-2/t} dt + \sum_{k=1}^{\infty} \int_1^{\infty} P\left(\max_{kL\leq n<(k+1)L} \frac{|\sum_{i=1}^{n} X_i|}{b_n} \geq t^{1/r} |\mathcal{F} \right) dt \\
&\leq c + c \sum_{k=1}^{\infty} \left(\frac{1}{b_{kM}^2} \sum_{i=1}^{kL} E^{\mathcal{F}} X_i^2 + \sum_{i=kL}^{(k+1)L-1} \frac{E^{\mathcal{F}} X_i^2}{b_i^2} \right) \int_1^{\infty} t^{-2/t} dt \\
&\leq c + c \sum_{i=1}^{\infty} E^{\mathcal{F}} X_i^2 \sum_{kL\geq i} \frac{1}{b_{kL}^2} + c \sum_{k=1}^{\infty} \sum_{i=kL}^{(k+1)L-1} \frac{E^{\mathcal{F}} X_i^2}{b_i^2}
\end{aligned}$$

$$\leq c + c\sum_{i=1}^{\infty}\frac{E^{\mathcal{F}}X_i^2}{b_i^2} < \infty \ a.s.$$

The proof is complete.

Proof of (b). For $\varepsilon > 0$ $a.s.$ is an $\mathcal{F}$-measurable random variable, we have by Theorem 2.1,

$$P\left(\max_{m\leq k\leq n}\left|\frac{\sum_{i=1}^k X_i}{b_k}\right| \geq \varepsilon|\mathcal{F}\right) \leq c\varepsilon^{-2}\left(\sum_{i=1}^{m}\frac{E^{\mathcal{F}}|X_i|^2}{b_m^2} + \sum_{i=m+1}^{n}\frac{E^{\mathcal{F}}|X_i|^2}{b_i^2}\right)$$

But

$$P\left(\sup_{k\geq m}\frac{|\sum_{i=1}^{k}X_i|}{b_k} > \varepsilon|\mathcal{F}\right) = \lim_{n\to\infty}P\left(\frac{\max_{m\leq k\leq n}\left|\sum_{i=1}^{k}X_i\right|}{b_k} \geq \varepsilon|\mathcal{F}\right) \tag{5}$$

$$\leq c\varepsilon^{-2}\left(\sum_{i=1}^{m}\frac{E^{\mathcal{F}}|X_i|^2}{b_m^2} + \sum_{i=m+1}^{n}\frac{E^{\mathcal{F}}|X_i|^2}{b_i^2}\right)$$

By Kronecker Lemma and $\displaystyle\sum_{n=1}^{\infty}\frac{E^{\mathcal{F}}|X_n|^2}{b_n^2} < \infty$ $a.s.$, we obtain

$$\sum_{i=1}^{m}\frac{E^{\mathcal{F}}|X_i|^2}{b_m^2} \to 0 \ as \ m \to \infty \tag{6}$$

Hence, by (5) and (6), we have

$$\lim_{n\to\infty}P\left(\sup_{k\geq n}\frac{|\sum_{i=1}^{k}X_i|}{b_k} > \varepsilon|\mathcal{F}\right) = 0, \ i.e. \ \sum_{i=1}^{n}\frac{X_i}{b_n} \to 0 \ a.s. \ as \ n \to \infty.$$

The proof is complete. □

To prove Theorem 3.2, we need the following conditional version of Borel-cantelli lemma which is proved by Majerak et al.[10].

Lemma 3.1 ([10]). Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. Then the following results hold.
(i) Let $\{A_n|n \geq 1\}$ be a sequence of events such that $\sum_{n=1}^{\infty}P(A_n) < \infty$.
Then $\sum_{n=1}^{\infty}P(A_n|\mathcal{F}) < \infty$ $a.s.$
(ii) Let $\{A_n|n \geq 1\}$ be a sequence of events and let $A = \{\omega|\sum_{n=1}^{\infty}P(A_n|\mathcal{F}) < \infty\}$ with $P(A) < 1$. Then, only finitely many events from the sequence $\{A_n \cap A, n \geq 1\}$ hold with probability one, namely $P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}(A_k \bigcap A)\right) = 0$.

Theorem 3.2. Let $\{X_n|n \geq 1\}$ be a sequence of conditionally centered $\mathcal{F}-NA$ random variables such that $E^{\mathcal{F}}|X|^{2/\alpha} < \infty$ $a.s.$ for some $0 < \alpha \leq 1$ and let

$|a_{ni}| \le n^{-\alpha/2-\delta}$, $0 < \delta < \alpha/2$ and $\sum_{i=1}^{n} a_{ni}^2 \le cn^{-\theta}, \theta > 0$. If $P(|X_n| > x|\mathcal{F}) \le cP(|X| > x|\mathcal{F})$ for all $n, x \ge 0$, then

$$\sum_{i=1}^{n} a_{ni}X_i \to 0 \ a.s. \ as \ n \to \infty \tag{7}$$

Proof. Note that $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, where $a_{ni}^{+} = \max(a_{ni}, 0)$, $a_{ni}^{-} = \max(-a_{ni}, 0)$. Thus, to prove (7), it suffices to show that

$$\sum_{i=1}^{n} a_{ni}^{+}X_i \to 0 \ a.s. \ as \ n \to \infty \tag{8}$$

$$\sum_{i=1}^{n} a_{ni}^{-}X_i \to 0 \ a.s. \ as \ n \to \infty \tag{9}$$

Note $\{a_{ni}^{+}X_i : 1 \le i \le n, n \ge 1\}$ and $\{a_{ni}^{-}X_i : 1 \le i \le n, n \ge 1\}$ are still an $\mathcal{F} - NA$ random variables, we prove only (8), the proof of (9) is analogous. So, without loss of generality, we assume $a_{ni} > 0$, and let $a_{ni}Y_i = n^{-\delta/2}I(a_{ni}X_i > n^{-\delta/2}) + a_{ni}X_iI(|a_{ni}X_i| \le n^{-\delta/2}) - n^{-\delta/2}I(a_{ni}X_i < -n^{-\delta/2})$.

Then $\{a_{ni}Y_i|1 \le i \le n, n \ge 1\}$ are $\mathcal{F} - NA$ random variables by definition of $\mathcal{F} - NA$ random variables, and

$$\begin{aligned}\sum_{i=1}^{n} a_{ni}X_i &= \sum_{i=1}^{n} a_{ni}(X_i - Y_i) + \sum_{i=1}^{n} a_{ni}E^{\mathcal{F}}Y_i + \sum_{i=1}^{n} a_{ni}(Y_i - E^{\mathcal{F}}Y_i) \\ &= : I_3 + I_4 + I_5\end{aligned}$$

First, we prove that $I_3 \to 0$ $a.s.$ as $n \to \infty$. Thus, we have for $\varepsilon > 0$ $a.s.$ is an $\mathcal{F}$-measurable random variable,

$$\begin{aligned}&P(\sum_{i=1}^{\infty} a_{ni}(X_i - Y_i) > \varepsilon/2|\mathcal{F}) \\ &\le P(\bigcup_{i=1}^{\infty} X_i \ne Y_i|\mathcal{F}) \\ &\le \sum_{i=1}^{\infty} P(|a_{ni}X_i| > n^{-\delta/2}|\mathcal{F}) \\ &\le C\sum_{i=1}^{\infty} n^{-1-\delta/\alpha}E^{\mathcal{F}}|X|^{2/\alpha} \\ &\le C\sum_{i=1}^{\infty} n^{-1-\delta/\alpha} < \infty \ a.s.\end{aligned}$$

by Borel-Cantelli Lemma, we have that

$$I_3 \to 0 \ a.s. \ as \ n \to \infty. \tag{10}$$

Secondly, we prove that $I_4 \to 0$ $a.s.$ as $n \to \infty$.

Note that by $E^{\mathcal{F}} X_n = 0$, we have

$$\begin{aligned} I_4 &= \sum_{i=1}^{n} E^{\mathcal{F}} |a_{ni} Y_i| \\ &\leq \sum_{i=1}^{n} E^{\mathcal{F}} |a_{ni} X_i| I(|a_{ni} X_i| \leq n^{-\delta/2}) + \sum_{i=1}^{n} n^{-\delta/2} P(|a_{ni} X_i| > n^{-\delta/2} | \mathcal{F}) \\ &\leq \sum_{i=1}^{n} E^{\mathcal{F}} |a_{ni} X| I(|a_{ni} X| > n^{-\delta/2}) + \sum_{i=1}^{n} n^{-\delta/2} P(|a_{ni} X| > n^{-\delta/2} | \mathcal{F}) \\ &\leq C n^{-\delta/2 - \delta/\alpha} E^{\mathcal{F}} |X|^{2/\alpha} \\ &\leq C n^{-\delta/2 - \delta/\alpha} \to 0 \ as \ n \to \infty, \end{aligned} \tag{11}$$

since $|a_{ni}X| = |a_{ni}X|^{2/\alpha} |a_{ni}X|^{1-2/\alpha}, |a_{ni}|^{2/\alpha} |a_{ni}X|^{1-2/\alpha} \leq n^{-1-\delta/2-\delta/\alpha}$.

Next, to prove $I_5 \to 0$ $a.s.$ as $n \to \infty$, it suffices to show that for an arbitrary $\mathcal{F}$-measurable variables $\varepsilon > 0$ $a.s.$,

$$I_5^* = P\left(|\sum_{i=1}^{n} a_{ni} (Y_i - E^{\mathcal{F}} Y_i)| > \varepsilon/2 | \mathcal{F} \right) < \infty \ a.s. \tag{12}$$

In fact, from the definition of $\mathcal{F} - NA$ random variables, we know that $\{a_{ni}(Y_i - E^{\mathcal{F}} Y_i)| \ 1 \leq i \leq n, n \geq 1\}$ is still an $\mathcal{F} - NA$ random variables. Hence, by Theorem 3.1, taking $b_n = 1$ and $q > 2$, according to Lemma 2.1, we obtain

$$I_5^* \leq \sum_{i=1}^{n} E^{\mathcal{F}} |a_{ni} Y_i|^q + (\sum_{i=1}^{n} E^{\mathcal{F}} |a_{ni} Y_i|^2)^{q/2} =: I_6 + I_7$$

First, we prove that $I_6 \to 0$ as $n \to \infty$. By assumptions, we have that

$$\begin{aligned} I_6 &= \sum_{i=1}^{n} E^{\mathcal{F}} |a_{ni} Y_i|^q \\ &\leq c \sum_{i=1}^{n} \left(E^{\mathcal{F}} |a_{ni} X_i|^q I(|a_{ni} X_i| \leq n^{-\delta/2}) + n^{-\delta q/2} P(|a_{ni} X_i| > n^{-\delta/2} | \mathcal{F}) \right) \\ &\leq c \sum_{i=1}^{n} \left(E^{\mathcal{F}} |a_{ni} X|^q I(|a_{ni} X| \leq n^{-\delta/2}) + n^{-\delta q/2} P(|a_{ni} X| > n^{-\delta/2} | \mathcal{F}) \right) \\ &\leq c \sum_{i=1}^{n} n^{-1-\delta/\alpha - \delta q/2} E^{\mathcal{F}} |X|^{2/\alpha} \\ &\leq c n^{-(\delta q/2 + \delta/\alpha)} \to 0 \ as \ n \to \infty, \end{aligned}$$

since $|a_{ni}X|^q = |a_{ni}X|^{2/\alpha}|a_{ni}X|^{q-2/\alpha} = |a_{ni}|^{2/\alpha}|X|^{2/\alpha}|a_{ni}X|^{q-2/\alpha},\ |a_{ni}|^{2/\alpha}\cdot$ $|a_{ni}X|^{q-2/\alpha} \leq n^{-1-\delta q/2-\delta/\alpha}$.

Finally, note that $E^{\mathcal{F}}|X|^{2/\alpha} < \infty\ a.s.$, we have by taking large $q > 2$,

$$\begin{aligned}
I_7 &= \sum_{i=1}^{n}(E^{\mathcal{F}}|a_{ni}Y_i|^2)^{q/2} \\
&\leq c\sum_{i=1}^{n}\left(E^{\mathcal{F}}|a_{ni}X_i|^2 I(|a_{ni}X_i| \leq n^{-\delta/2}) + n^{-\delta}P(|a_{ni}X_i| > n^{-\delta/2}|\mathcal{F})\right)^{q/2} \\
&\leq c\sum_{i=1}^{n} n^{-\delta}\int_0^{n^{-\delta}} P(|a_{ni}X|^2 > x|\mathcal{F})dx \\
&\leq c\sum_{i=1}^{n} n^{-\delta}E^{\mathcal{F}}|a_{ni}X|^{2/\alpha}\int_0^{n^{-\delta}} x^{-1/\alpha}dx \\
&\leq cn^{-(1/2+\theta+\delta)q/2} \to 0\ as\ n \to \infty.
\end{aligned}$$

Thus, combining (10),(11)and (12) we obtain

$$\sum_{i=1}^{n} a_{ni}X_i \to 0\ a.s.\ as\ n \to 0.$$

The proof is complete. □

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