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A GENERAL MODIFIED MANN'S ALGORITHM FOR k-STRICTLY PSEUDO CONTRACTIONS IN HILBERT SPACE†

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Abstract. In this paper, we propose a new algorithm to modify the standard Mann' process to have strong convergence for k-strictly pseudo-contractive non-self mapping in Hilbert spaces.

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1. Introduction

Let H be a real Hilbert space, K be a nonempty closed convex subset of H. $T : K \to H$ is a k-strictly pseudo contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in K. \tag{1}$$

Note that k-strictly pseudo contractions include nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. It is also said to be pseudo-contractive if $k = 1$. T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contractive. It is easy to see that k-strictly pseudo contractions are between nonexpansive mappings and pseudo contractions.

In 1953, W.R.Mann [1] introduced the standard Mann's iterative algorithm which generates a sequence $\{x_n\}$ by (2):

$$\forall x_0 \in K, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 0, \tag{2}$$

where the sequence $\{\alpha_n\}$ is in (0,1).

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Since then, construction algorithms to modify the standard Mann's iterative method for nonexpansive mappings and k-strict pseudo contractions have been extensively studied by many authors.

It is clear that if T is a nonexpansive mapping with fixed point and the sequence $\{\alpha_n\}$ is chosen so that $\alpha_n \in (0,1)$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (2) converges weakly to a fixed point of T. In 1967, Browder and Petryshyn [3] first extended the standard Mann's method to k-strictly pseudo-contractive self-mappings in real Hilbert spaces.

Since the sequence $\{x_n\}$ generated by the standard Mann's iterative method can only have weak convergence, modified Mann's iterations have recently been made to get strong convergence for nonexpansive mappings and k-strictly pseudo contractions; see, e.g., [2, 4, 6, 8, 9] and references therein.

T.H.Kim and H.K.Xu [4] introduced the following process:

$$\begin{cases} x_0 = x \in K, \quad \textit{arbitrarily chosen}, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{3}$$

where T is a nonexpansive self-mapping of K, $u \in K$ is a given point. Under some appropriate conditions on the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$, they obtained a strong convergence theorem.

Remark 1.1. A.Moudfi [5] introduced the standard viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ by:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. It is proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (4) converges strongly to the unique solution $x^* \in C$ of the variational inequality:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(T),$$

where $C = Fix(T)$.

Yao et al.[6] modified the Mann's iterative method by using the standard viscosity approximation method:

$$\begin{cases} x_0 = x \in K, \quad \textit{arbitrarily chosen}, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{5}$$

where T is a nonexpansive self-mapping of K, and $f : K \to K$ is a contraction. Under some mild conditions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$, they proved that the sequence $\{x_n\}$ defined by (5) converges strongly to a fixed point of T.

Remark 1.2. In 2006, G.Marino and H.K.Xu [7] proposed the following general iterative method:

$$x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall\, n \geq 0, \tag{6}$$

where T is a nonexpansive self-mapping of H, $f : H \to H$ is a contraction, and A is a strongly positive bounded linear operator on H. Under some appropriate conditions on $\{\alpha_n\}$, they proved that the sequence $\{x_n\}$ defined by (6) converges strongly to a fixed point x^* of T, which equivalently solves the variational inequality:

$$\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in Fix(T),$$

which is the optimality condition for the minimization problem:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf(i.e.,$h'(x) = \gamma f(x)\ for\ x \in H$).

Zhou [8] also modified the Mann's iterative process for non-self k-strict pseudo-contractions, and obtained strong convergence in Hilbert spaces.

Very recently, X.L.Qin et al.[9] modified the Mann's iterative method by using the following composite iteration scheme:

$$\begin{cases} x_1 = x \in K, \quad arbitrarily\ chosen, \\ y_n = P_K[\beta_n x_n + (1 - \beta_n)Tx_n], \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad \forall n \geq 1, \end{cases} \tag{7}$$

where $T : K \to H$ is k-strictly pseudo contractive mapping, $f : K \to K$ is a contraction, and A is a strongly positive bounded linear operator on K. Under some mild conditions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$, they proved that the sequence $\{x_n\}$ defined by (7) converges strongly to a fixed point of T.

In this paper, motivated by Kim and Xu [4], Moudafi [5], Yao et al. [6], G.Marino and Xu [7], Zhou [8], X.L.Qin et al.[9], we introduce a new composite algorithm:

$$\begin{cases} x_0 = x \in K, \\ y_n = P_K[\beta_n x_n + (1 - \beta_n)Tx_n] \\ x_{n+1} = [I - \alpha_n(\mu F - \gamma f)]y_n, \quad \forall n \geq 0. \end{cases} \tag{8}$$

where T is a k-strictly pseudo contraction from K onto H, f is a self-contraction on K such that $\|f(x)-f(y)\| \leq \alpha\|x-y\|$ for all $x, y \in K$ and F is a k-Lipschitzian and η-strongly monotone operator on K. $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Under some certain appropriate assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$, We obtain strong convergence theorems for the k-strictly pseudo contraction. Our results improve and extend the corresponding results.

2. Preliminaries

Lemma 2.1 ([2]). *Assume that $\{x_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n,\ \forall\ n \geq 0, \tag{9}$$

where $\{\gamma_n\}$ is a sequence $\in (0, 1)$, and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([3]). *Let $T : K \to H$ be a k-strictly pseudo-contraction. Define $S : K \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for all $x \in K$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 2.3 ([8]). *If T is a k-strictly pseudo contraction on a closed convex subset of K of a real Hilbert space H, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.4 ([8]). *Let $T : K \to H$ be a k-strictly pseudo contraction with $F(T) \neq \emptyset$, then $F(T) = F(P_K T)$.*

Lemma 2.5 ([10]). *Let λ be a number in $[0, 1]$ and $\mu > 0$. Let $F : H \to H$ be a κ-Lipschitzian continuous and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Associating with a nonexpansive mapping T on H, define a mapping $T^\lambda : H \to H$ by $T^\lambda x := Tx - \lambda\mu F(Tx)$, for all $x \in H$. Let $\tau = \mu(\eta - \frac{\mu\kappa^2}{2})$, thus $T^\lambda : H \to H$ is a contraction, that is:$\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\tau)\|x - y\|$.*

3. Main results

Theorem 3.1. *Let H be a real Hilbert space, let K be a nonempty closed convex subset of H such that $K \pm K \subset K$. Assume that $f : K \to K$ is a contraction with a coefficient $0 \le \alpha < 1$. Let $T : K \to H$ be a non-self k-strictly pseudo contraction such that $F(T) \neq \emptyset$. Let $F : K \to K$ be a κ-Lipschitzian continuous and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $\frac{\tau-1}{\alpha} < \gamma < \mu(\eta - \mu\kappa^2/2)/\alpha = \frac{\tau}{\alpha}$. Let the sequence $\{x_n\}$ generated by (1.3), where the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ and satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) $k \le \beta_n \le \lambda < 1$, $\lim_{n\to\infty} \beta_n = \lambda$ *for all* $n \ge 0$;
(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to $\tilde{x} \in F(T)$, which also solves the variational inequality:

$$\langle (\mu F - \gamma f)\tilde{x}, x - \tilde{x} \rangle \ge 0, \quad \forall x \in Fix(T). \tag{10}$$

Proof. First, we show that the sequences $\{x_n\}$ and $\{y_n\}$ are all bounded. Take any $p \in F(T)$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_K[\beta_n x_n + (1-\beta_n)Tx]_n - p\|^2 \\
&\le \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Tx_n - p\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 \\
&\le \|x_n - p\|^2 - (1-\beta_n)(\beta_n - k)\|Tx_n - x_n\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|[I-\alpha_n(\mu F-\gamma f)]y_n-p\| \\ &\le \|(I-\alpha_n\mu F)y_n-(I-\alpha_n\mu F)p\|+\alpha_n\|\gamma f(y_n)-\mu Fp\| \\ &\le (1-\alpha_n\tau)\|y_n-p\|+\alpha_n(\gamma\|f(y_n)-f(p)\|+\|\gamma f(p)-\mu Fp\|) \\ &\le [1-\alpha_n(\tau-\gamma\alpha)]\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Fp\|.\end{aligned}$$

By induction, we have

$$\|x_n-p\|\le \max\{\|x_0-p\|,\frac{\|\gamma f(p)-\mu F(p)\|}{\tau-\gamma\alpha}\},\quad \forall n\ge 0,$$

and $\{x_n\}$ is bounded, so is $\{y_n\}$.

Next we show that $\|x_{n+1}-x_n\|\to 0$.

Consider a mapping T_n on K define by

$$T_nx=P_K[\beta_nx+(1-\beta_n)Tx],\quad x\in K. \tag{11}$$

It is easy to see that T_n is nonexpansive. Indeed, for all $x,y\in K$, we have

$$\begin{aligned}\|T_nx-T_ny\|^2 &= \|P_K[\beta_nx+(1-\beta_n)Tx]-P_K[\beta_ny+(1-\beta_n)Ty\|^2 \\ &\le \|[\beta_nx+(1-\beta_n)Tx]-[\beta_ny+(1-\beta_n)Ty\|^2 \\ &= \beta_n\|x-y\|^2+(1-\beta_n)\|Tx-Ty\|^2 \\ &\quad -\beta_n(1-\beta_n)\|(I-T)x-(I-T)y\|^2 \\ &\le \beta_n\|x-y\|^2+(1-\beta_n)[\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2] \\ &\quad -\beta_n(1-\beta_n)\|(I-T)x-(I-T)y\|^2 \\ &\le \|x-y\|^2,\end{aligned}$$

which implies that T_n is nonexpansive. Since $\{x_n\}$ bounded, so $\{T_nx_n\}$ and $\{FT_nx_n\}$ are also bounded. For simplicity, we rewrite (3) by: $x_{n+1}=[I-\alpha_n(\mu F-\gamma f)]T_nx_n$. It follows that

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\| \\ &= \|[I-\alpha_{n+1}(\mu F-\gamma f)]T_{n+1}x_{n+1} \\ &\quad -[I-\alpha_n(\mu F-\gamma f)]T_nx_n\| \\ &\le \|(I-\alpha_{n+1}\mu F)T_{n+1}x_{n+1}-(I-\alpha_{n+1}\mu F)T_nx_n\| \\ &\quad +|\alpha_n-\alpha_{n+1}|\|\mu FT_nx_n\|+\gamma[\alpha_{n+1}\|f(T_{n+1}x_{n+1}) \\ &\quad -f(T_nx_n)\|+|\alpha_{n+1}-\alpha_n|\|f(T_nx_n)\|] \\ &\le (1-\alpha_{n+1}\tau)(\|x_{n+1}-x_n\|+\|T_{n+1}x_n-T_nx_n\|) \\ &\quad +|\alpha_n-\alpha_{n+1}|\|\mu FT_nx_n\|+\gamma[\alpha_{n+1}\alpha\|T_{n+1}x_{n+1}-T_nx_n\| \\ &\quad +|\alpha_{n+1}-\alpha_n|\|f(T_nx_n)\|].\end{aligned} \tag{12}$$

From (11), we have that

$$\begin{aligned}
&\|T_{n+1}x_n - T_n x_n\| \\
&= \|P_K[\beta_{n+1}x_n + (1-\beta_{n+1})Tx_n] \\
&\quad - P_K[\beta_n x_n + (1-\beta_n)Tx_n]\| \\
&\le \|[\beta_{n+1}x_n + (1-\beta_{n+1})Tx_n - [\beta_n x_n + (1-\beta_n)Tx_n]\| \\
&\le |\beta_n - \beta_{n+1}|\|x_n - Tx_n\|.
\end{aligned} \tag{13}$$

Applying (13) to (12), we deduce that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| \le {} & [1-\alpha_{n+1}(\tau-\gamma\alpha)]\|x_{n+1} - x_n\| \\
& + M(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|),
\end{aligned} \tag{14}$$

where M is an appropriate constant such that

$$M \ge [1-\alpha_{n+1}(\tau-\gamma\alpha)]\|x_n - Tx_n\| + \gamma\|f(T_n x_n)\| + \mu\|FT_n x_n\|.$$

From (C1), (C3) and by Lemma 2.1, we have

$$\|x_{n+1} - x_n\| \to 0. \tag{15}$$

Next we show $\|x_n - Tx_n\| \to 0$.
From $x_{n+1} = [I - \alpha_n(\mu F - \gamma f)]T_n x_n$, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|(\mu F - \gamma f)T_n x_n\|.
\end{aligned}$$

By (C1) and (15), it shows that

$$\|x_n - Tx_n\| \to 0. \tag{16}$$

On the other hand, conditions (C2) and (C3) imply that $\beta_n \to \lambda$ as $n \to \infty$, where $\lambda \in [k, 1)$. Define a mapping $S : K \to H$ by $Sx = \lambda x + (1-\lambda)Tx$. Then, by Lemma 2.2, S is nonexpansive mapping with $F(S) = F(T)$. It follows from Lemma 2.4, that $F(P_K S) = F(S) = F(T)$. We calculate that

$$\begin{aligned}
\|P_K S x_n - x_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - P_K S x_n\| \\
&\le \|x_n - T_n x_n\| + \|\beta_n x_n + (1-\beta_n)Tx_n \\
&\quad - [\lambda x_n + (1-\lambda)Tx_n]\| \\
&\le \|x_n - T_n x_n\| + |\beta_n - \lambda|\|x_n - Tx_n\|.
\end{aligned}$$

From (16), we have

$$\lim_{n\to\infty} \|P_K S x_n - x_n\| = 0. \tag{17}$$

For each $t \in (0, 1)$, we consider a mapping G_t on K defined by

$$G_t x = [I - t(\mu F - \gamma f)]P_K S x.$$

Indeed, by Lemma 2.5, we have

$$\|G_t x - G_t y\| \le [1 - t(\tau - \gamma\alpha)]\|x - y\|,$$

which implies that the mapping G_t is a contraction from K to K. Using the Banach contraction principle, there exists a unique point, denoted by x_t, which uniquely solves the fixed point equation:$x = [I - t(\mu F - \gamma f)]P_K Sx$.

Now, we show that

$$\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle \leq 0,$$

where $\tilde{x} = \lim_{t\to 0^+} x_t$, and x_t is the fixed point of the contraction $x \mapsto [I - t(\mu F - \gamma f)]P_K Sx$. Thus, by Lemma 2.5, it follows that

$$\begin{aligned}
&\|x_t - x_n\|^2\\
&= \|(I - t\mu F)P_K Sx_t - (I - t\mu F)x_n + t(\gamma f(P_K Sx_t) - \mu F x_n)\|^2\\
&\leq (1 - t\tau)^2\|P_K Sx_t - x_n\|^2 + 2t\langle \gamma f(P_K Sx_t) - \mu F x_n, x_t - x_n\rangle\\
&\leq (1 - t\tau)^2\|P_K Sx_t - P_K Sx_n + P_K Sx_n - x_n\|^2\\
&\quad + 2t\langle \gamma f(P_K Sx_t) - \mu F x_n, x_t - x_n\rangle\\
&\leq (1 - t\tau)^2[\|P_K Sx_t - P_K Sx_n\|^2 + \|P_K Sx_n - x_n\|^2\\
&\quad + 2\|P_K Sx_t - P_K Sx_n\|\|P_K Sx_n - x_n\|]\\
&\quad + 2t\langle \gamma f(P_K Sx_t) - \mu F x_n, x_t - x_n\rangle\\
&\leq (1 - t\tau)^2\|x_t - x_n\|^2 + p_n(t) + 2t\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle\\
&\quad + 2t\langle \gamma f(P_K Sx_t) - \mu F x_t, x_t - x_n\rangle,
\end{aligned} \tag{18}$$

where $p_n(t) = (1 - t\tau)^2(2\|x_t - x_n\| + \|x_n - P_K Sx_n\|)\|x_n - P_K Sx_n\|$, and from (17), it follows that $p_n(t) \to 0$. From (18), and recall that F is η-strongly monotone and also from the Theorem 3.1, we find that $\mu\eta \geq \tau$. Thus, we have

$$\begin{aligned}
2t\langle \mu F x_t - \gamma f(P_K Sx_t), x_t - x_n\rangle &\leq (t^2\tau^2 - 2t\tau)\|x_t - x_n\|^2 + p_n(t)\\
&\quad + 2t\mu\langle F x_t - F x_n, x_t - x_n\rangle\\
&\leq (t^2\tau - 2t)\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle + p_n(t)\\
&\quad + 2t\mu\langle F x_t - F x_n, x_t - x_n\rangle\\
&\leq t^2\tau\mu\langle F x_t - F x_n, x_t - x_n\rangle + p_n(t).
\end{aligned}$$

It follows that

$$\langle \mu F x_t - \gamma f(P_K Sx_t), x_t - x_n\rangle \leq \frac{t\tau\mu}{2}\langle F x_t - F x_n, x_t - x_n\rangle + \frac{1}{2t}p_n(t). \tag{19}$$

Let $n \to \infty$, and recall that $p_n(t) \to 0$, we have

$$\limsup_{n\to\infty}\langle \mu F x_t - \gamma f(P_K Sx_t), x_t - x_n\rangle \leq \frac{t\tau\mu\kappa}{2}\langle F x_t - F x_n, x_t - x_n\rangle, \quad \forall t \in (0, 1).$$

Let $t \to 0$, it follows that

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle \mu F x_t - \gamma f(P_K Sx_t), x_t - x_n\rangle \leq 0. \tag{20}$$

It is obvious that

$$\begin{aligned}
&\langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle \\
&\quad = \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle - \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - x_t\rangle \\
&\qquad + \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - x_t\rangle - \langle \gamma f(\tilde{x}) - \mu F x_t, x_n - x_t\rangle \\
&\qquad + \langle \gamma f(\tilde{x}) - \mu F x_t, x_n - x_t\rangle - \langle \gamma f(P_K S x_t) - \mu F x_t, x_n - x_t\rangle \\
&\qquad + \langle \gamma f(P_K S x_t) - \mu F x_t, x_n - x_t\rangle \\
&\quad = \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_t - \tilde{x}\rangle + \langle \mu F x_t - \mu F\tilde{x}, x_n - x_t\rangle \\
&\qquad + \langle \gamma f(\tilde{x}) - \gamma f(P_K S x_t), x_n - x_t\rangle + \langle \gamma f(P_K S x_t) - \mu F x_t, x_n - x_t\rangle.
\end{aligned}$$

Let $n \to \infty$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle \le{}& \|\gamma f(\tilde{x}) - \mu F\tilde{x}\| \|x_t - \tilde{x}\| \\
&+ \mu\kappa \|x_t - \tilde{x}\| \lim_{n\to\infty} \|x_n - x_t\| \\
&+ \gamma\alpha \|P_K S x_t - \tilde{x}\| \lim_{n\to\infty} \|x_n - x_t\| \\
&+ \limsup_{n\to\infty} \langle \gamma f(P_K S x_t) - \mu F x_t, x_n - x_t\rangle,
\end{aligned}$$

therefore, let $t \to 0$, and combined with (20), we get

$$\begin{aligned}
&\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle \\
&\quad = \limsup_{t\to 0} \limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle \\
&\quad \le \limsup_{t\to 0} \|\gamma f(\tilde{x}) - \mu F\tilde{x}\| \|x_t - \tilde{x}\| \\
&\qquad + \limsup_{t\to 0} \mu\kappa \|x_t - \tilde{x}\| \limsup_{n\to\infty} \|x_n - x_t\| \\
&\qquad + \limsup_{t\to 0} \gamma\alpha \|P_K S x_t - \tilde{x}\| \lim_{n\to\infty} \|x_n - x_t\| \\
&\qquad + \limsup_{t\to 0} \limsup_{n\to\infty} \langle \gamma f(P_K S x_t) - \mu F x_t, x_n - x_t\rangle \\
&\quad \le 0.
\end{aligned}$$

So, we conclude that $\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle \le 0$.

Finally, we prove $x_n \to \tilde{x}$. To this end, we calculate

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
&= \|\alpha_n(\gamma f(y_n) - \mu F(\tilde{x})) + (I - \alpha_n \mu F) y_n - (I - \alpha_n \mu F)\tilde{x}\|^2 \\
&= \|(I - \alpha_n \mu F) y_n - (I - \alpha_n \mu F)\tilde{x}\|^2 + 2\alpha_n \langle \gamma f(y_n) - \mu F(\tilde{x}), x_{n+1} - \tilde{x}\rangle \\
&\le (1 - \alpha_n \tau)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \langle \gamma f(y_n) - \gamma f(\tilde{x}) + \gamma f(\tilde{x}) - \mu F(\tilde{x}), x_{n+1} - \tilde{x}\rangle \\
&\le (1 - \alpha_n \tau)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \gamma \langle f(y_n) - f(\tilde{x}), x_{n+1} - \tilde{x}\rangle \\
&\quad + 2\alpha_n \langle \gamma f(\tilde{x}) - \mu F(\tilde{x}), x_{n+1} - \tilde{x}\rangle \\
&\le (1 - \alpha_n \tau)^2 \|x_n - \tilde{x}\|^2 + \alpha_n \gamma \alpha (\|x_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2) \\
&\quad + 2\alpha_n \langle \gamma f(\tilde{x}) - \mu F(\tilde{x}), x_{n+1} - \tilde{x}\rangle.
\end{aligned}$$

Since $\{x_n\}$ is bounded, we take a constant $L > 0$ such that $L \geq \|x_n - \tilde{x}\|^2$. In fact, we have

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
&\leq \frac{(1-\alpha_n\tau)+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n - \tilde{x}\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(\tilde{x}) - \mu F(\tilde{x}), x_{n+1} - \tilde{x}\rangle \\
&\leq [1 - \frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}]\|x_n - \tilde{x}\|^2 \\
&\quad + \frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}[\frac{1}{\tau-\gamma\alpha}\langle \gamma f(\tilde{x}) - \mu F(\tilde{x}), x_{n+1} - \tilde{x}\rangle + \frac{\alpha_n\tau^2}{2(\tau-\gamma\alpha)}L] \\
&= (1-\bar{\alpha_n})\|x_n - \tilde{x}\|^2 + \bar{\alpha_n}\bar{\beta_n},
\end{aligned} \tag{21}$$

where

$$\bar{\alpha_n} = \frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha},$$

$$\bar{\beta_n} = \frac{1}{\tau-\gamma\alpha}\langle \gamma f(\tilde{x}) - \mu F(\tilde{x}), x_{n+1} - \tilde{x}\rangle + \frac{\alpha_n\tau^2}{2(\tau-\gamma\alpha)}L.$$

It is easily seen that $\lim_{n\to\infty}\bar{\alpha_n} = 0$, $\sum_{n=0}^{\infty}\bar{\alpha_n} = \infty$, and $\limsup_{n\to\infty}\bar{\beta_n} \leq 0$. By Lemma 2.1, we conclude that $x_n \to \tilde{x}$. □

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