J. Appl. Math. & Informatics Vol. **30**(2012), No. 3 - 4, pp. 541 - 553
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SOME LIMIT PROPERTIES OF RANDOM TRANSITION PROBABILITY FOR SECOND-ORDER NONHOMOGENEOUS MARKOV CHAINS ON GENERALIZED GAMBLING SYSTEM INDEXED BY A DOUBLE ROOTED TREE†

KANGKANG WANG* AND DECAI ZONG

Abstract. In this paper, we study some limit properties of the harmonic mean of random transition probability for a second-order nonhomogeneous Markov chain on the generalized gambling system indexed by a tree by constructing a nonnegative martingale. As corollary, we obtain the property of the harmonic mean and the arithmetic mean of random transition probability for a second-order nonhomogeneous Markov chain indexed by a double root tree.

AMS Mathematics Subject Classification : 60F15, 65F15.
Key words and phrases : double root tree, second-order nonhomogeneous Markov chains, random transition probability, harmonic mean, generalized gambling system.

1. Introduction

A tree is a graph $S = \{T, E\}$ which is connected and contains no circuits. Given any two vertices $\sigma, t(\ \sigma \neq t \in T)$, let $\overline{\sigma t}$ be the unique path connecting σ and t. Define the graph distance $d(\sigma, t)$ to be the number of edges contained in the path $\overline{\sigma t}$.

Let T_o be an arbitrary infinite tree that is partially finite (i.e. it has infinite vertices, and each vertex connects with finite vertices) and has a root o. Meanwhile, we consider another kind of double root tree T, that is, it is formed with the root o of T_o connecting with an arbitrary point denoted by the root -1. For a better explanation of the double root tree T, we take Cayley tree $T_{C,N}$ for example. It's a special case of the tree T_o, the root o of Cayley tree has N neighbors and all the other vertices of it have $N+1$ neighbors each. The double

Received December 30, 2010. Revised August 27, 2011. Accepted November 1, 2011.
*Corresponding author. †This work was supported by the research grant of Higher Schools' Natural Science Basic Research of Jiangsu Province of China (09KJD110002).

root tree $T'_{C,N}$ (see Fig.1) is formed with root o of tree $T_{C,N}$ connecting with another root -1.

Let σ, t be vertices of the double root tree T. Write $t \leq \sigma$ $(\sigma, t \neq -1)$ if t is on the unique path connecting o to σ, and $|\sigma|$ for the number of edges on this path. For any two vertices σ, t $(\sigma, t \neq -1)$ of the tree T, denote by $\sigma \wedge t$ the vertex farthest from o satisfying $\sigma \wedge t \leq \sigma$ and $\sigma \wedge t \leq t$.

The set of all vertices with distance n from root o is called the n-th generation of T, which is denoted by L_n. We say that L_n is the set of all vertices on level n and especially root-1 is on the -1st level on tree T. We denote by $T^{(n)}$ the subtree of the tree T containing the vertices from level -1 (the root -1) to level n and denote by $T_o^{(n)}$ the subtree of the tree T_o containing the vertices from level 0 (the root o) to level n. Let $t(\neq o, -1)$be a vertex of the tree T. We denote the first predecessor of t by 1_t, the second predecessor of t by 2_t, and denote by n_t the n-th predecessor of t. Let $X^A = \{X_t, t \in A\}$, and let x^A be a realization of X^A and denote by $|A|$ the number of vertices of A .

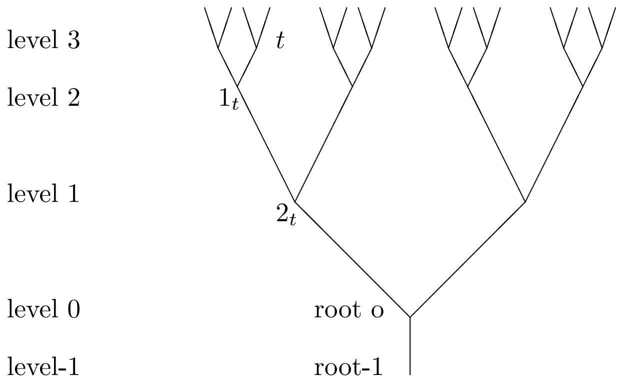

Fig.1 Double root tree $T'_{C,2}$

Definition 1. Let $S = \{s_1, s_2, \cdots, s_N\}$ and $P(z|y,x)$ be a nonnegative function on S^3. Let

$$P = ((P(z|y,x)), \quad P(z|y,x) \geq 0, x, y, z \in S.$$

If

$$\sum_{z \in S} P(z|y,x) = 1,$$

then P is called a second-order transition matrix.

Definition 2. Let T be a double root tree and $S = \{s_1, s_2, \cdots, s_N\}$ be a finite state space, and $\{X_t, t \in T\}$ be a collection of S-valued random variables defined on the probability space (Ω, F, P). Let

$$P = (p(x,y)), \quad x, y \in S \tag{1}$$

be a distribution on S^2, and

$$P_t = (P_t(z|y,x)), \quad x,y,z \in S, t \in T\backslash\{o\}\{-1\} \tag{2}$$

be a collection of second-order transition matrices. For any vertex t ($t \neq o, -1$), if

$$\begin{aligned} &P(X_t = z|X_{1_t} = y, X_{2_t} = x, \text{and } X_\sigma \text{ for } \sigma \wedge t \leq 1_t) \\ = &P(X_t = z|X_{1_t} = y, X_{2_t} = x) = P_t(z|y,x) \;\; \forall x,y,z \in S \end{aligned} \tag{3}$$

and

$$P(X_{-1} = x, X_o = y) = p(x,y), \quad x,y \in S, \tag{4}$$

then $\{X_t, t \in T\}$ is called a S-valued second-order nonhomogeneous Markov chain indexed by a tree T with the initial distribution (1) and second-order transition matrices (2), or called a T-indexed second-order nonhomogeneous Markov chain.

Remark 1. Benjamini and Peres [1] have given the definition of the tree-indexed homogeneous Markov chains. Here we improve their definition and give the definition of the tree-indexed second-order nonhomogeneous Markov chains in a similar way.

There have been some works on limit theorems for tree-indexed stochastic processes. Benjamini and Peres [1] have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [2] have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger (see [13],[14]), by using Pemantle's result [8] and a combinatorial approach, have studied the Shannon-McMillan theorem with convergence in probability for a PPS-invariant and ergodic random field on a homogeneous tree. Yang and Liu [11] have studied a strong law of large numbers for the frequency of occurrence of states for Markov chains field on a homogeneous tree (a particular case of tree-indexed Markov chains field and PPS-invariant random fields). Yang (see [10]) has studied the strong law of large numbers for frequency of occurrence of state and Shannon-McMillan theorem for homogeneous Markov chains indexed by a homogeneous tree. Recently, Yang (see [12]) has studied the strong law of large numbers and Shannon-McMillan theorem for nonhomogeneous Markov chains indexed by a homogeneous tree. Huang and Yang (see [3]) have also studied the strong law of large numbers for Markov chains indexed by an infinite tree with uniformly bounded degree. Shi and Yang (see [9]) have discussed the limit properties for the harmonic mean of second-order nonhomogeneous Markov chain indexed by a tree. Liu (see[5],[6]) have studied some limit properties for the transition probabilities of second-order nonhomogeneous Markov chains.

Definition 3. Let $\{f_n(x_1, \cdots, x_n), n \geq 1\}$ be a sequence of real-valued functions defined on $S^n (n = 1, 2, \cdots)$, which will be called the generalized selection functions if $\{f_n, n \geq 1\}$ take values in a set A of positive real numbers. We let

$$Y_0 = y \text{ } (y \textit{ is an arbitrary real number}),$$

$$Y_t = f_{|t|}(X_{1_t}, X_{2_t}, \cdots, X_0), \quad |t| \geq 1, \tag{5}$$

where $|t|$ stands for the number of the edges on the path from the root o to t. Then $\{Y_t,\ t \in T^{(n)}\}$ is called the generalized gambling system or the generalized random selection system indexed by an infinite tree with uniformly bounded degree. The traditional random selection system $\{Y_n,\ n \geq 0\}$[4] takes values in the set of $\{0, 1\}$.

We first explain the conception of the traditional random selection, which is the crucial part of the gambling system. We give a set of real-valued functions $f_n(x_1, \cdots, x_n)$ defined on $S^n (n = 1, 2, \cdots)$, which will be called the random selection function if they take values in a two-valued set $\{0, 1\}$. Then let

$$Y_1 = y(y\ \textit{is an arbitrary real number}),$$

$$Y_{n+1} = f_n(X_1, \cdots, X_n), \quad n \geq 1. \tag{6}$$

where $\{Y_n, n \geq 1\}$ be called the gambling system (the random selection system).

In order to explain the real meaning of the notion of the random selection, we consider the traditional gambling model. Let $\{X_n, n \geq 0\}$ be a second-order nonhomogeneous Markov chain, and $\{g_n(x, y, z), n \geq 2\}$ be a real-valued function sequence defined on S^3. Interpret X_n as the result of the nth trial, the type of which may change at each step. Let $\mu_n = Y_n g_n(X_{n-2}, X_{n-1}, X_n)$ denote the gain of the bettor at the nth trial, where Y_n represents the bet size, $g_n(X_{n-2}, X_{n-1}, X_n)$ is determined by the gambling rules, and $\{Y_n, n \geq 0\}$ is called a gambling system or a random selection system. The bettor's strategy is to determine $\{Y_n, n \geq 1\}$ by the results of the last two trials. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n depends on X_{n-1} and X_{n-2} as $n \geq 2$, and b_2 is a constant. Thus $\sum_{k=2}^n Y_k g_k(X_{k-2}, X_{k-1}, X_k)$ represents the total gain in the first n trials, $\sum_{k=2}^n b_k$ the accumulated entrance fees, and $\sum_{k=2}^n [Y_k g_k(X_{k-2}, X_{k-1}, X_k) - b_k]$ the accumulated net gain. Motivated by the classical definition of "fairness" of game of chance (see Kolmogorov[4]), we introduce the following definition:

Definition 4. The game is said to be fair, if for almost all $\omega \in \{\omega : \sum_{k=2}^\infty Y_k = \infty\}$, the accumulated net gain in the first n trial is to be of smaller order of magnitude than the accumulated stake $\sum_{k=2}^n Y_k$ as n tends to infinity, that is

$$\lim_{n\to\infty} \frac{1}{\sum_{k=2}^n Y_k} \sum_{k=2}^n [Y_k g_k(X_{k-2}, X_{k-1}, X_k) - b_k] = 0\ a.s.\ on\ \{\omega : \sum_{k=2}^\infty Y_k = \infty\}.$$

Let $P_t(x_t \mid x_{1_t}, x_{2_t}) = P_t(X_t = x_t \mid X_{1_t} = x_{1_t}, X_{2_t} = x_{2_t})$. Then $P_t(X_t \mid X_{1_t}, X_{2_t})$ is called the random transition probability of a T-indexed second-order nonhomogeneous Markov chain. Liu [5] has studied a strong limit theorem for the harmonic mean of the random transition probability of finite nonhomogeneous Markov chains. In this paper, we study some limit properties of the harmonic mean of random transition probability for a second-order nonhomogeneous Markov chain in the generalized gambling system indexed by a tree by constructing a nonnegative martingale. As corollaries, we obtain some limit properties for a second-order nonhomogeneous Markov chain indexed by a tree and general second-order nonhomogeneous Markov chain. The results of [5] and [6] have been generalized.

2. Main results

In this section, we generalize the traditional gambling system to the case of the second-order nonhomogeneous Markov chain indexed by the a tree and investigate some limit properties of the harmonic mean of random transition probability for a second-order nonhomogeneous Markov chain on the generalized gambling system indexed by a tree. Let us give the following conclusion:

Theorem 1. *Let $\{X_t, t \in T\}$ be a T-indexed second-order nonhomogeneous Markov chain with state space S defined as in Definition 2, and its initial distribution and probability transition collection satisfying*

$$P(X_{-1} = x_{-1}, X_o = x_o) = P(x, y) > 0, \quad \forall x, y \in S, \tag{7}$$

and

$$P_t(z \mid y, x) > 0, \quad \forall x, y, z \in S, t \in T\backslash\{o\}\{-1\}, \tag{8}$$

respectively. $\{Y_t, t \in T\}$ is defined as in Definition 3. Denote

$$\alpha_t = \min\{P_t(z \mid y, x), \quad x, y, z \in S\}, \quad t \in T \setminus \{o\}\{-1\}. \tag{9}$$

Take an $s_0 > 1$, denote

$$D(\omega) = \{\omega : \lim_n a_n(\omega) = \infty, \ \limsup_{n\to\infty} \frac{1}{a_n(\omega)} \sum_{t\in T^{(n)}\backslash\{o\}\{-1\}} Y_t s_0^{Y_t/\alpha_t} = M < \infty\}. \tag{10}$$

Then the following holds

$$\lim_{n\to\infty} \frac{1}{a_n(\omega)} \sum_{t\in T^{(n)}\backslash\{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N] = 0. \ a.s. \ \omega \in D(\omega). \tag{11}$$

Remark 2. In Theorem 1, there is no direct relations between the sum equation $\sum_{t\in T^{(n)}\backslash\{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N]$ and $a_n(\omega)$. The aim of the theorem is to show that $\sum_{t\in T^{(n)}\backslash\{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N]$ is to be of smaller order of magnitude than $a_n(\omega)$ as n tends to infinity.

Proof. Obviously, when $n \geq 1$, we have

$$P(x^{T^{(n)}}) = P(X^{T^{(n)}} = x^{T^{(n)}}) = P(X_{-1} = x_{-1}, X_o = x_o) \prod_{t\in T^{(n)}\backslash\{o\}\{-1\}} P_t(x_t|x_{1_t}, x_{2_t}). \tag{12}$$

Hence

$$P(X^{L_n} = x^{L_n}|X^{T^{(n-1)}} = x^{T^{(n-1)}}) = \frac{P(x^{T^{(n)}})}{P(x^{T^{(n-1)}})} = \prod_{t\in L_n} P_t(x_t|x_{1_t}, x_{2_t}). \tag{13}$$

Let us denote

$$\begin{aligned} &M_t(s;x_{1_t},x_{2_t}) = E[s^{Y_tP_t(X_t|X_{1_t},X_{2_t})^{-1}}|X_{1_t}=x_{1_t},X_{2_t}=x_{2_t}] \\ &= \sum_{x_t\in S} s^{Y_tP_t(x_t|x_{1_t},x_{2_t})^{-1}} P_t(x_t|x_{1_t},x_{2_t}),\ t\in T\backslash\{o\}\{-1\}. \end{aligned} \tag{14}$$

$M_t(s;X_{1_t},X_{2_t})$ is called the conditional generating function of $Y_tP_t(X_t|X_{1_t},X_{2_t})^{-1}$ given $X_{1_t}=x_{1_t},X_{2_t}=x_{2_t}$. Denote $s\in(\frac{1}{s_0},s_0)$,

$$U_n(s,\omega) = \frac{s^{\sum_{t\in T^{(n)}\backslash\{o\}\{-1\}} Y_tP_t(X_t|X_{1_t},X_{2_t})^{-1}}}{\prod_{t\in T^{(n)}\backslash\{o\}\{-1\}} M_t(s;X_{1_t},X_{2_t})}. \tag{15}$$

In view of the (13), (14), (15) and Markov's property, $F_n=\sigma(X^{T^{(n)}})$, we can conclude that

$$\begin{aligned} & E[U_n(s,\omega)|F_{n-1}] \\ = & E\left[\prod_{t\in T^{(n)}\backslash\{o\}\{-1\}} \frac{s^{Y_tP_t(X_t|X_{1_t},X_{2_t})^{-1}}}{M_t(s;X_{1_t},X_{2_t})}|F_{n-1}\right] \\ = & U_{n-1}(s,\omega)E\left[\prod_{t\in L_n} \frac{s^{Y_tP_t(X_t|X_{1_t},X_{2_t})^{-1}}}{M_t(s;X_{1_t},X_{2_t})}|F_{n-1}\right] \\ = & U_{n-1}(s,\omega)\sum_{x^{L_n}\in S^{L_n}}\prod_{t\in L_n} \frac{s^{Y_tP_t(x_t|X_{1_t},X_{2_t})^{-1}}}{M_t(s;X_{1_t},X_{2_t})}\cdot P(X^{L_n}=x^{L_n}|X^{T^{(n-1)}}) \\ = & U_{n-1}(s,\omega)\sum_{x^{L_n}\in S^{L_n}}\prod_{t\in L_n} \frac{s^{Y_tP_t(x_t|X_{1_t},X_{2_t})^{-1}}}{M_t(s;X_{1_t},X_{2_t})}P_t(x_t|X_{1_t},X_{2_t}) \\ = & U_{n-1}(s,\omega)\prod_{t\in L_n}\sum_{x_t\in S} \frac{s^{Y_tP_t(x_t|X_{1_t},X_{2_t})^{-1}}}{M_t(s;X_{1_t},X_{2_t})}P_t(x_t|X_{1_t},X_{2_t}) \\ = & U_{n-1}(s,\omega)\prod_{t\in L_n}\frac{M_t(s;X_{1_t},X_{2_t})}{M_t(s;X_{1_t},X_{2_t})} = U_{n-1}(s,\omega). \end{aligned} \tag{16}$$

In equation (16), since $S=\{s_1,s_2,\cdots,s_N\}$ is a finite alphabet-set, $\sum$ and $\prod$ can be changed in the derivation of eq(16).

Then $\{U_n(s,\omega),F_n,n\ge 1\}$ is a nonnegative martingale. According to Doob martingale convergence theorem, we have

$$\lim_{n\to\infty} U_n(s,\omega) = U(s,\omega) < \infty \quad a.s. \quad . \tag{17}$$

Thus, by (10) and (17), we get

$$\limsup_{n\to\infty}\frac{1}{a_n}\ln U_n(s,\omega)\le 0 \quad a.s.\ \omega\in D(\omega). \tag{18}$$

It follows from (15) and (18) that

$$\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}[Y_tP_t(X_t|X_{1_t},X_{2_t})^{-1}\ln s-\ln M_k(s;X_{1_t},X_{2_t})]\leq 0. \tag{19}$$

$$a.s.\ \omega\in D(\omega).$$

On the other hand, by (9), (19) and the inequalities $\ln x\leq x-1 (x>0)$, and $0\leq s^x-1-x\ln s\leq\frac{1}{2}(x\ln s)^2e^{|x\ln s|}$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}Y_t\ln s[P_t(X_t|X_{1_t},X_{2_t})^{-1}-N]\\
\leq\ &\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}[\ln M_t(s;X_{1_t},X_{2_t})-Y_tN\ln s]\\
\leq\ &\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}[M_t(s;X_{1_t},X_{2_t})-1-Y_tN\ln s]\\
=\ &\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\sum_{x_t\in S}P_t(x_t|X_{1_t},X_{2_t})\\
&\cdot[s^{Y_tP_t(x_t|X_{1_t},X_{2_t})^{-1}}-1-Y_tP_t(x_t|X_{1_t},X_{2_t})^{-1}\ln s]\\
\leq\ &\frac{(\ln s)^2}{2}\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\sum_{x_t\in S}P_t(x_t|X_{1_t},X_{2_t})^{-1}Y_t^2e^{|Y_tP_t(x_t|X_{1_t},X_{2_t})^{-1}\ln s|}\\
\leq\ &\frac{(\ln s)^2}{2}\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\sum_{x_t\in S}\alpha_t^{-1}Y_t^2e^{|Y_t\alpha_t^{-1}\ln s|}\\
\leq\ &\frac{N(\ln s)^2}{2}\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\frac{1}{\alpha_t}Y_t^2\exp\{Y_t\alpha_t^{-1}|\ln s|\}.
\end{aligned}$$

$$a.s.\qquad \omega\in D(\omega). \tag{20}$$

It is easy to see that

$$\max_{0<\lambda<1}\{x\lambda^x,x>0\}=-\frac{e^{-1}}{\ln\lambda}\quad. \tag{21}$$

Let $1<s<s_0$, by (9), (10), (20) and (21), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}Y_t[P_t(X_t|X_{1_t},X_{2_t})^{-1}-N]\\
\leq\ &\frac{N\ln s}{2}\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\frac{Y_t^2}{\alpha_t}\exp\{Y_t\alpha_t^{-1}\ln s\}\\
=\ &\frac{N\ln s}{2}\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}}\frac{Y_t^2}{\alpha_t}s^{Y_t/\alpha_t}
\end{aligned}$$

$$\begin{aligned}
&= \frac{N \ln s}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \frac{Y_t}{\alpha_t}\left(\frac{s}{s_0}\right)^{Y_t/\alpha_t} Y_t s_0^{Y_t/\alpha_t} \\
&\le \frac{N \ln s}{2} \limsup_{n\to\infty} \frac{-1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} e^{-1}\left(\ln \frac{s}{s_0}\right)^{-1} Y_t s_0^{Y_t/\alpha_t} \\
&= \frac{N e^{-1} \ln s^{-1}}{2}\left(\ln \frac{s}{s_0}\right)^{-1} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} Y_t s_0^{Y_t/\alpha_t} \\
&\le \frac{N M e^{-1} \ln s^{-1}}{2}\left(\ln \frac{s}{s_0}\right)^{-1}. \qquad a.s. \quad \omega \in D(\omega).
\end{aligned} \tag{22}$$

Letting $s \to 1^+$, by (22), we have

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N] \le 0. \ a.s. \ \omega \in D(\omega). \tag{23}$$

Let $1/s_0 < s < 1$, by (10), (20) and (21) we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N] \\
&\ge \frac{N \ln s}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \frac{Y_t^2}{\alpha_t} \exp\{-Y_t \alpha_t^{-1} \ln s\} \\
&= \frac{N \ln s}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \frac{Y_t^2}{\alpha_t} s^{-Y_t/\alpha_t} \\
&= \frac{N \ln s}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} \frac{Y_t}{\alpha_t}\left(\frac{1}{s_0 s}\right)^{Y_t/\alpha_t} Y_t s_0^{Y_t/\alpha_t} \\
&\ge \frac{N \ln s}{2} \limsup_{n\to\infty} \frac{-1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} e^{-1}\left(\ln \frac{1}{s_0 s}\right)^{-1} Y_t s_0^{Y_t/\alpha_t} \\
&\ge \frac{N M e^{-1} \ln s^{-1}}{2}\left(\ln \frac{1}{s_0 s}\right)^{-1}. \qquad a.s. \quad \omega \in D(\omega).
\end{aligned} \tag{24}$$

Letting $s \to 1^-$, by (24), we get

$$\liminf_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N] \ge 0. \ a.s. \ \omega \in D(\omega). \tag{25}$$

Combining (23) and (25), we obtain (11) directly. □

3. Some Corollaries

Corollary 1. *Let $\{X_t, t \in T\}$ be a T-indexed second-order nonhomogeneous Markov chain with state space S defined as in Theorem 1, denote*

$$\alpha_t = \min\{P_t(z \mid y, x), \quad x, y, z \in S\}, \quad t \in T \setminus \{o\}\{-1\}. \tag{26}$$

Take an $s_0 > 1$, denote

$$D_0(\omega) = \{\omega : \lim_n \sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t = \infty,$$

$$\limsup_{n\to\infty} \frac{1}{\sum\limits_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t} \sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t s_0^{Y_t/\alpha_t} = M < \infty\}. \tag{27}$$

Then the harmonic mean of the random conditional probability $\{P_t(X_t \mid X_{1_t}, X_{2_t})$, $t \in T^{(n)} \setminus \{o\}\{-1\}\}$ in the generalized gambling system converges to $\frac{1}{N}$ a.s., that is

$$\lim_n \frac{\sum\limits_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t}{\sum\limits_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t P_t(X_t|X_{1_t}, X_{2_t})^{-1}} = \frac{1}{N}. \text{ a.s. } \omega \in D_0(\omega). \tag{28}$$

Proof. Let $a_n(\omega) = \sum\limits_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t$, by (10) we obtain $D(\omega) = D_0(\omega)$ and

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{a_n} \sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N] \\
= \; &\lim_{n\to\infty} \frac{1}{\sum\limits_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t} \sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t[P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N] \\
= \; &\lim_{n\to\infty} \frac{1}{\sum\limits_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t} \sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} Y_t P_t(X_t|X_{1_t}, X_{2_t})^{-1} - N = 0. \quad (29)
\end{aligned}$$

Therefore, (28) follows from (29) immediately. □

Corollary 2. *Let $\{X_t, t \in T\}$ be a T-indexed second-order nonhomogeneous Markov chain with state space S defined as in Theorem 1, denote*

$$\alpha_t = \min\{P_t(z \mid y, x), \quad x, y, z \in S\}, \quad t \in T \setminus \{o\}\{-1\}. \tag{30}$$

If there exists $a(> 0)$ such that

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} e^{a/\alpha_t} = M < \infty. \tag{31}$$

Then

$$\lim_{n\to\infty} \frac{|T^{(n)}|}{\sum_{t \in T^{(n)} \setminus \{o\}\{-1\}} P_t(X_t \mid X_{1_t}, X_{2_t})^{-1}} = \frac{1}{N}, \quad a.s. \tag{32}$$

where $|T^{(n)}|$ represents the number of all the vertices from Level -1 to Level n.

Proof. Let $Y_t \equiv 1,\ t \in T^{(n)}$, $s_0 = e^a$, by (27) we obtain

$$\lim_n \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} Y_t = \lim_n(|T^{(n)}| - 2) = \infty,$$

and

$$\begin{aligned} & \limsup_{n\to\infty} \frac{1}{\sum\limits_{t\in T^{(n)}\setminus\{o\}\{-1\}} Y_t} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} Y_t s_0^{Y_t/\alpha_t} \\ = & \limsup_{n\to\infty} \frac{1}{|T^{(n)}| - 2} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} s_0^{1/\alpha_t} \\ = & \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} e^{a/\alpha_t} = M < \infty. \end{aligned} \tag{33}$$

Therefore, it is easy yo see that $D_0(\omega) = \Omega$ and (32) follows from (28) directly. □

If the successor of each vertex of the tree T_o has only one vertex, the second-order nonhomogeneous Markov chains on the double-rooted tree T degenerate into the general second-order nonhomogeneous Markov chains. Thus we obtain the following results:

Corollary 3 (see [5],[6])**.** *Let $\{X_n, n \geq 0\}$ be a second-order nonhomogeneous Markov chain with state space S, and its initial distribution and probability transition sequence satisfying*

$$p(i,j) > 0, \quad i,j \in S \quad , \tag{34}$$

and

$$P_k(h|i,j) > 0,\ i,j,h \in S,\ k = 1,2,\cdots \quad . \tag{35}$$

respectively. Denote

$$a_k = \min\{P_k(h|i,j), i,j,h \in S\}, \quad k = 1,2,\cdots \quad , \tag{36}$$

If there exists $s_0(>1)$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} s_0^{\frac{1}{a_k}} = M < \infty \quad , \tag{37}$$

then

$$\lim_{n\to\infty} \frac{n}{\sum_{k=1}^{n} P_k(X_k \mid X_{k-1}, X_{k-2})^{-1}} = \frac{1}{N} \quad a.s. \quad . \tag{38}$$

Proof. When the successor of each vertex of the tree T has only one vertex, the nonhomogeneous Markov chains on the tree T degenerate into the general nonhomogeneous Markov chains, $|T^{(n)}| = n + 2$, the corollary follows directly from Corollary 2. □

4. Limit Property for Arithmetic Mean of Transition Probability of Second-Order Nonhomogeneous Markov Chain Indexed by a Tree

Taking into account the theoretical and practical importance of transition probability of second-order nonhomogeneous Markov chain, in this section we will make an estimation for the arithmetic mean of $P_t(X_t \mid X_{1_t}, X_{2_t})$. For this purpose, we introduce the following theorem.

Theorem 2. *Let $\{X_t, t \in T\}$ be a T-indexed second-order nonhomogeneous Markov chain with state space S defined as in Theorem 1, if $\alpha_t \geq \alpha > 0$, $t \in T^{(n)} \backslash \{o\}\{-1\}$, then*

$$\frac{1}{N} \leq \limsup_{n} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \backslash \{o\}\{-1\}} P_t(X_t|X_{1_t}, X_{2_t}) \leq \frac{[1-(N-2)\alpha]^2}{4\alpha[1-(N-1)\alpha]N}. \ a.s. \quad (39)$$

Proof. In view of $\alpha_t \geq \alpha > 0,\ t \in T^{(n)} \backslash \{o\}\{-1\}$ and (31), we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \backslash \{o\}\{-1\}} e^{a/\alpha_t} \\ \leq\ &\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \backslash \{o\}\{-1\}} e^{a/\alpha} = \limsup_{n\to\infty} \frac{|T^{(n)}|-2}{|T^{(n)}|} e^{a/\alpha} = e^{a/\alpha} < \infty \quad (40) \end{aligned}$$

Hence (31) holds naturally. Let us denote $\beta_t = \max\{p_t(z|x,y), x, y, z \in S\}$, it is easy to show

$$\alpha_t \leq 1/N \leq \beta_t \leq 1-(N-1)\alpha_t. \quad (41)$$

By $\alpha_t \geq \alpha > 0,\ t \in T^{(n)} \backslash \{o\}\{-1\}$ and (41), we get

$$\alpha \leq P_t(X_t|X_{1_t}, X_{2_t}) \leq 1-(N-1)\alpha. \quad (42)$$

According to Schweitzer inequality, we obtain

$$1 = \frac{1}{n^2}\left(\sum_{m=1}^{n} \sqrt{d_m \cdot \frac{1}{d_m}}\right)^2 \leq (\frac{1}{n}\sum_{m=1}^{n} d_m)(\frac{1}{n}\sum_{m=1}^{n} \frac{1}{d_m}) \leq \frac{(A+B)^2}{4AB}, \quad (43)$$

where $0 < A \leq d_m \leq B,\ m = 1, 2, \cdots$. Therefore, we can calculate

$$\begin{aligned} &\left(\frac{|T^{(n)}|-2}{|T^{(n)}|}\right)^2 \\ =\ &\frac{1}{|T^{(n)}|^2}\left[\sum_{t \in T^{(n)} \backslash \{o\}\{-1\}} \sqrt{P_t(X_t|X_{1_t}, X_{2_t}) P_t(X_t|X_{1_t}, X_{2_t})^{-1}}\right]^2 \end{aligned}$$

$$
\begin{aligned}
&\le \left[\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} P_t(X_t|X_{1_t},X_{2_t})\right] \\
&\qquad\cdot\left[\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} P_t(X_t|X_{1_t},X_{2_t})^{-1}\right] \\
&\le \frac{[1-(N-2)\alpha]^2}{4\alpha[1-(N-1)\alpha]} \qquad (44)
\end{aligned}
$$

By the superior limit property and (32), we can write

$$
\begin{aligned}
1 &= \limsup_{n\to\infty}\left(\frac{|T^{(n)}|-2}{|T^{(n)}|}\right)^2 \\
&\le \limsup_{n\to\infty}\frac{N}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}\{-1\}} P_t(X_t|X_{1_t},X_{2_t}) \le \frac{[1-(N-2)\alpha]^2}{4\alpha[1-(N-1)\alpha]}. \quad (45)
\end{aligned}
$$

(39) follows from (45) immediately. □

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Kangkang Wang received M.Sc. from Jiangsu University and Ph.D at Nanjing University of Aeronautics and Astronautics. Since 2005 he has been at Jiangsu University of Science and Technology. His research interests include limit theory in probability theory and information theory.

School of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, China.
e-mail: wkk.cn@126.com

Decai Zong received M.Sc. from Jiangsu University. Since 2005 he has been at Changshu Institute of Technology. His research interest includes intelligent algorithm.

College of Computer Science and Engineering, Changshu Institute of Technology, Changshu 215500, China.
e-mail: zongdecaixhl@163.com