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INTERPOLATION APPROXIMATION OF $M/G/c/K$ RETRIAL QUEUE WITH ORDINARY QUEUES†

YANG WOO SHIN

Abstract. An approximation for the number of customers at service facility in $M/G/c/K$ retrial queue is provided with the help of the approximations of ordinary $M/G/c/K$ loss system and ordinary $M/G/c$ queue. The interpolation between two ordinary systems is used for the approximation.

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1. Introduction

Consider a queueing system with finite (possibly zero) queue capacity. In conventional loss system, the customers finding all servers busy and full buffer upon arrival leave the system forever without getting service and are lost. In many practical systems, some of the customers blocked may return later to try again to get service. The queueing system with returning customers is called the retrial queue and the customer who is waiting to try again is said to be in orbit. The time interval between two consecutive attempts of each customer in orbit is called a inter-retrial time. Figure 1 shows the schematic diagram for the retrial queue.

Retrial queue have been widely used for modelling many practical problems arising in computer and communication systems. Even for the Markovian retrial queues with multiple servers, the exact results have not been obtained except for some special cases. Instead, attempts to develop algorithmic or approximation methods have been made for long time. Truncation method are used for computing the stationary distribution of the number of customers in orbit and service facility in the Markovian retrial queue, e.g. see [1, 2]. Greenberg and

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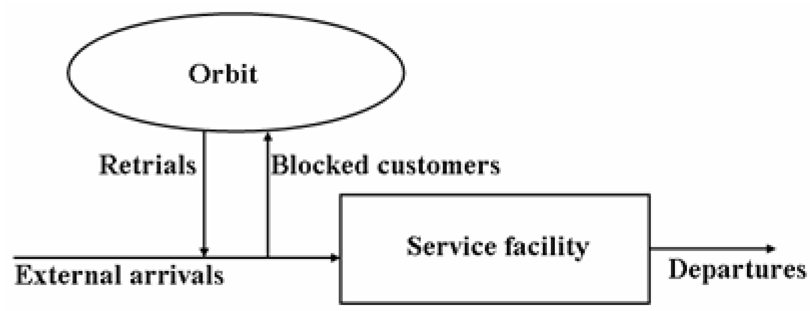

FIGURE 1. Schematic diagram of retrial queue

Wolff [3] present an approximation for the stationary distribution of the number of customers in service facility in the $M/M/c/K$ retrial queue with impatient customers under the assumption that retrials see time averages (RTA). The approximation using RTA assumption does not depend on the retrial rate and they show that the approximation works well for small value of retrial rate. Fredericks and Reisner [4] develop an approximation that may depend on the retrial rate for the number of customers at service facility in $M/M/c/c$ retrial queue with impatient customers. Riordan [5] suggested an approximation of $M/M/c/c$ retrial queues by interpolation between extreme cases that the retrial rate tends to zero and to infinity, see also [1, Section 3.4.5] and [2, Section 2.8]. For more literature of algorithms or approximation methods for retrial queues, we refer the monographs [1, 2] and references therein. However, there are few results for the case of the multiple-servers retrial queues with general service time.

Objective of this paper is to propose an approximation of the number of customers at service facility in $M/G/c/K$ retrial queue. Interpolation between two ordinary systems $M/G/c/K$ queue and $M/G/c$ queue is used for an approximation of the $M/G/c/K$ retrial queue. There are many approximations for the stationary queue length distribution in ordinary $M/G/c/K$ queue e.g. in [6, 7, 8, 9]. We adopt the approximations of the stationary distribution in the ordinary $M/G/c/K$ queue and $M/G/c$ queue given in Tijms [7, Sections 9.6 and 9.8].

This paper is organized as follows. The approximation using conventional loss system are considered in Section 2. Interpolation approximation is presented with some numerical examples in Section 3. Concluding remarks are given in Section 4.

2. Approximation with loss system

Consider an $M/G/c/K$ retrial queue in which customers arrive from outside according to a Poisson process with rate λ and there are c identical servers and $K - c$ waiting positions in service facility. When an arriving customer finds the service facility full, the customer joins orbit and repeats its request after an exponential amount of time with rate γ. Let $G(t)$ be the service time distribution of each server with mean $\frac{1}{\mu}$. For the stability of the system, we assume $\rho = \frac{\lambda}{c\mu} < 1$. By $\Sigma_R(\lambda)$ and $\Sigma_L(\lambda)$, denote the $M/G/c/K$ retrial queue and the conventional $M/G/c/K$ loss system with arrival rate λ, respectively.

Approximation procedures are as follows. Note that the input rate to the service facility in $\Sigma_R(\lambda)$ is the sum of the rate λ of external arrivals and the (long run) rate λ_R of returning customers from orbit. For an approximation of the number of customers in service facility, we assume that the flow of returning customers from orbit is Poisson and does not depend on the external arrivals. Then the superposition of external arrivals and retrials forms a Poisson with rate $\Lambda = \lambda + \lambda_R$. Once λ_R is determined, $\Sigma_R(\lambda)$ is approximated by $\Sigma_L(\Lambda)$.

Now we approximate λ_R iteratively as follows. Note that the customers blocked in $\Sigma_L(\lambda)$ is lost, but the customers blocked in $\Sigma_R(\lambda)$ return to the service facility and eventually get served. As an initial approximation of λ_R, we use the loss rate λB_0 in $\Sigma_L(\lambda)$, where B_0 is the blocking probability in $\Sigma_L(\lambda)$. Then approximate $\Sigma_R(\lambda)$ with $\Sigma_L(\lambda_1)$, where $\lambda_1 = \lambda + \lambda B_0$. Note that the blocked customers in $\Sigma_L(\lambda_1)$ become the retrial customers in retrial queue. In the second step, modify the approximating system by $\Sigma_L(\lambda_2)$ with $\lambda_2 = \lambda + \lambda_1 B_1$, where B_1 is the blocking probability in $\Sigma_L(\lambda_1)$. Repeating this procedure, we have a sequence of loss systems $\Sigma_L(\lambda_k)$ with arrival rate

$$\lambda_k = \lambda + \lambda_{k-1} B_{k-1}, \quad k = 1, 2, \cdots, \tag{1}$$

where $\lambda_0 = \lambda$ and B_k is the blocking probability in $\Sigma_L(\lambda_k)$, $k = 0, 1, 2, \cdots$. The schematic diagrams for $\Sigma_L(\lambda_k)$, $k = 0, 1, 2$ ($\lambda_0 = \lambda$) are presented in Figure 2. For given tolerance $\epsilon > 0$, we choose an integer N such that $|\lambda_{N-1} - \lambda_N| < \epsilon$ and use $\Sigma_L(\lambda_N)$ for approximation of $\Sigma_R(\lambda)$.

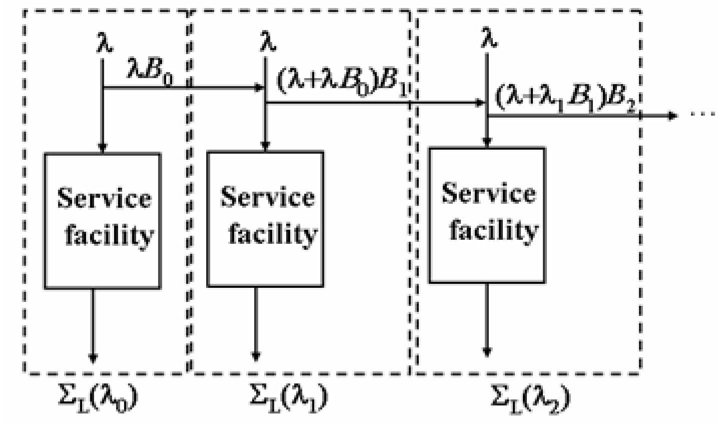

FIGURE 2. Schematic diagram of $\Sigma_L(\lambda_k)$, $k = 0, 1, 2$.

The stationary distribution $\{p_j, 0 \le j \le K\}$ of the system $\Sigma_L(\lambda)$ is computed by the following approximation formula [7, Section 9.8]:

$$p_j = \begin{cases} \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j p_0, & 0 \le j \le c-1, \\ \frac{1}{1-\lambda b_0}\left(\lambda p_{c-1} a_{j-c} + \lambda \sum_{k=c}^{j-1} p_k b_{j-k}\right), & c \le j \le K-1, \\ \rho p_{c-1} - (1-\rho)\sum_{k=c}^{K-1} p_k, & j = K, \end{cases} \quad (2)$$

where

$$a_n = \int_0^\infty (1-G_e(t))^{c-1}(1-G(t))e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,dt, \; n = 0,1,2,\cdots, K-1-c,$$
$$b_n = \int_0^\infty (1-G(ct))e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,dt, \; n = 0,1,2,\cdots, K-1-c$$

and $G_e(t) = \mu \int_0^t (1-G(x))\,dx$, $t \ge 0$ is the equilibrium excess distribution function. Note that the approximation (2) is exact for $M/M/c/K$ queue, $M/G/1/K$ queue and $M/G/c/c$ queue. By the PASTA (Poisson Arrivals See Time Average) property, the blocking probability B_0 in $\Sigma_L(\lambda)$ is given by

$$B_0 = P_K. \quad (3)$$

Now we show that the sequence $\{\lambda_k\}$ is convergent for the service time distribution is exponential or $K = c$, that is, $M/M/c/K$ queue or $M/G/c/c$ queue. Let $B(x)$ be the blocking probability in $\Sigma_L(x)$ and $f(x) = \lambda + xB(x)$. One can easily see from the formula of $B(x)$ that $B(x)$ is an increasing function of x and $\lim_{x\to\infty} x(1-B(x)) = c\mu$ for $M/M/c/K$ queue and $M/G/c/c$ queue and hence $f(x)$ is an increasing function of x. Thus $\{\lambda_k\}$ is an increasing sequence. Let $g(x) = f(x) - x$. Since $g(0) = \lambda > 0$ and $\lim_{x\to\infty} g(x) = \lambda - c\mu < 0$, there exists an $\lambda^* > 0$ satisfying $g(\lambda^*) = 0$, that is, $f(\lambda^*) = \lambda^*$. Note that $\lambda_1 = f(\lambda_0) > \lambda_0$. By mathematical induction, we have $\lambda_k \le \lambda^*$ for all $k \ge 1$. Thus $\{\lambda_k\}$ is convergent for $M/M/c/K$ retrial queue or $M/G/c/c$ retrial queue.

Although the convergence of the iteration scheme in (1) is not proved generally, extensive numerical experiments show the convergence of the sequence $\{\lambda_n\}_{n=0}^\infty$. The behaviors of $\{\lambda_k\}$ in (1) and the blocking probabilities $\{B_k\}$ are depicted in Figure 3 for $M/H_2/3/5$ queue with $\mu = 1.0$, the squared coefficient of variation $C_s^2 = 2.0$ and $\rho = 0.75$, where the service time distribution is hyperexponential distribution of order 2 (H_2) with probability density function $g(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}$, $t \ge 0$, and the parameters are determined as follows [1, p.56] $A = C_s^2 + \sqrt{C_s^4 - 1}$, $p = 1/(A+1)$, $\mu_1 = 2\mu/(A+1)$, $\mu_2 = A\mu_1$.

Remark. Assuming λ_n and B_n converge to say, λ^* and B^* as n tends to infinity, the equation (1) becomes

$$\lambda^* = \lambda + \lambda^* B^*. \quad (4)$$

Setting $\lambda^* = \lambda + \lambda_R$, one can easily see that the equation (4) is the same as equation (5) in [3] for RTA approximation. Thus our approximation can be

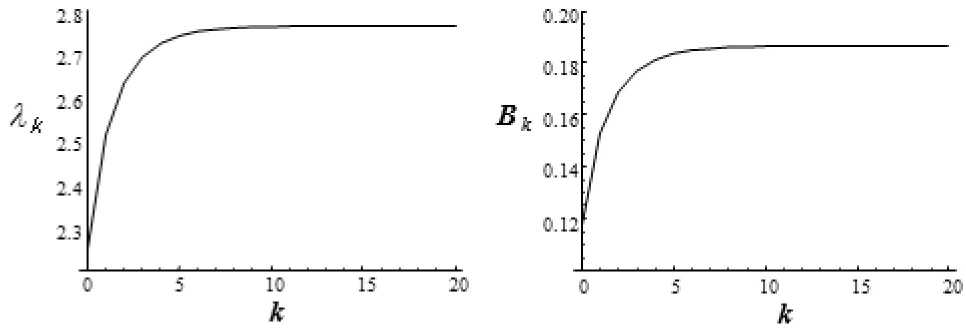

FIGURE 3. Behaviors of λ_k and B_k

considered as an extension of RTA approximation to the system with general distribution of service time.

Now we compare the approximation results with the exact one. Let X_0 denote the number of customers in service facility of the approximating system, that is $\Sigma_L(\lambda^*)$. In Table 1, the approximation $B^* = P(X_0 = K)$ of blocking probability B in $M/G/3/K$ retrial queue with $\mu = 1.0$ is compared with the exact one. Three cases of the squared coefficient of variation $C_s^2 = 0.5, 1.0, 2.0$ of the service time are considered. The service time distributions used in Table 1 are Erlang of order 2 distribution (E_2) for $C_s^2 = 0.5$, exponential distribution (M) for $C_s^2 = 1.0$ and hyperexponential distribution of order 2 (H_2) for $C_s^2 = 2.0$. Numerical results in Table 1 exhibit that the approximations work well for small value of retrial rate γ and deteriorate as γ increases. This result can be expected from the one for $M/M/c/c$ retrial queue [2, Section 2.8].

3. Interpolation approximation

It is known that the stationary joint distribution of the number of busy servers and the number of customers in orbit of the $M/M/c/c$ retrial queue converges to the corresponding distribution of ordinary $M/M/c$ queue as γ tends to infinity [2, Lemma 2.1]. Shin [10] showed that the number of customers in $A/G/c$ retrial queue where the arrival process is general point process stochastically decreases as γ increases. We can expect that the performance measures in the $M/G/c/K$ retrial queues approach to the ordinary $M/G/c$ queue as γ tends to infinity. For an approximation of the number of customers in service facility of the $M/G/c/K$ retrial queue with intermediate value of γ, we interpolate the ordinary loss system $\Sigma_L(\lambda^*)$ and the ordinary $M/G/c$ queue.

Table 1. Approximations and exact results for B

		$K=3$			$K=5$		
ρ	γ	E_2	M	H_2	E_2	M	H_2
0.4	0^*	0.112	0.112	0.112	0.011	0.014	0.019
	0.05	0.112	0.113	0.113	0.011	0.015	0.019
	0.1	0.113	0.113	0.115	0.011	0.015	0.019
	0.5	0.116	0.118	0.121	0.012	0.016	0.021
	1.0	0.120	0.122	0.126	0.012	0.017	0.023
	5.0	0.130	0.132	0.136	0.015	0.020	0.027
0.8	0^*	0.556	0.556	0.556	0.183	0.209	0.238
	0.05	0.558	0.558	0.559	0.182	0.216	0.243
	0.1	0.559	0.560	0.562	0.187	0.223	0.252
	0.5	0.569	0.573	0.580	0.219	0.261	0.301
	1.0	0.578	0.584	0.593	0.246	0.290	0.333
	5.0	0.609	0.617	0.626	0.317	0.361	0.405

* The results for $\gamma = 0$ correspond to the B^*.

Let Y be the number of customers in the ordinary $M/G/c$ queue and define the probability distribution $P_{\infty,j}$, $0 \le j \le K$ by

$$P_{\infty,j} = \begin{cases} P(Y=j), & j = 0,1,2,\cdots,K-1, \\ P(Y \ge K), & j \ge K. \end{cases}$$

The distribution $P_j = P(X=j)$, $0 \le j \le K$ of the number X of customers in service facility of the $M/G/c/K$ retrial queue with retrial rate γ is approximated by

$$\hat{P}_j = \frac{\mu}{\mu+\gamma} P_{0j} + \frac{\gamma}{\mu+\gamma} P_{\infty,j}, \ j = 0,1,\cdots,K, \tag{5}$$

where $P_{0j} = P(X_0 = j)$, $j = 0,1,\cdots,K$. We have from (5) that the approximating formulae $\hat{L}$ and $\hat{V}$ for mean $L = \mathbb{E}(X)$ and variance $V = \mathrm{Var}(X)$ of X, respectively as follows

$$\hat{L} = \frac{\mu}{\mu+\gamma} L_0 + \frac{\gamma}{\mu+\gamma} L_\infty, \tag{6}$$

$$\hat{V} = \frac{\mu}{\mu+\gamma} V_0 + \frac{\gamma}{\mu+\gamma} V_\infty + \frac{\mu}{\mu+\gamma}\frac{\gamma}{\mu+\gamma}(L_0 - L_\infty)^2, \tag{7}$$

where L_0, L_∞ and V_0, V_∞ are the means and variances of $\{P_{0j}\}$ and $\{P_{\infty,j}\}$, respectively. The approximation results with the exact ones for L, V, P_0 and P_K for $M/M/3/K$ retrial queue are presented in Table 2. The relative percent errors

$$\mathrm{Err}(\%) = \frac{|\mathrm{Exact} - \mathrm{Approximation}|}{\mathrm{Exact}} \times 100$$

for the systems with service time distributions M, E_2 and H_2 with $\mu = 1.0$ are listed in Tables 3 – 5. By Little's law, the mean number L of busy servers in $M/G/c/c$ retrial queue is $L = \frac{\lambda}{\mu}$ and does not depend on the retrial rate γ. Approximations of L provide the exact ones for $K = 3$ and the numerical

Table 2. Approximations of $M/M/3/K$ retrial queue

K	ρ		γ							
			0^*	0.05	0.1	0.5	1.0	5.0	10.0	$\infty^\dagger$
3	0.4	V		0.9315	0.9348	0.9542	0.9690	1.0054	1.0159	1.0306
		$\hat{V}$	0.9279	0.9328	0.9373	0.9622	0.9793	1.0135	1.0213	1.0306
		P_0		0.2731	0.2739	0.2789	0.2825	0.2902	0.2920	0.2941
		$\hat{P}_0$	0.2721	0.2731	0.2741	0.2794	0.2831	0.2904	0.2921	0.2941
		P_K		0.1127	0.1135	0.1182	0.1220	0.1325	0.1359	0.1412
		$\hat{P}_K$	0.1119	0.1133	0.1145	0.1216	0.1265	0.1363	0.1385	0.1412
	0.8	V		0.6005	0.6061	0.6428	0.6746	0.7671	0.7983	0.8467
		$\hat{V}$	0.5947	0.6067	0.6176	0.6787	0.7207	0.8047	0.8238	0.8467
		P_0		0.0220	0.0228	0.0285	0.0334	0.0469	0.0509	0.0562
		$\hat{P}_0$	0.0211	0.0228	0.0243	0.0328	0.0386	0.0503	0.0530	0.0562
		P_K		0.5583	0.5602	0.5729	0.5838	0.6167	0.6282	0.6472
		$\hat{P}_K$	0.5563	0.5606	0.5645	0.5866	0.6017	0.6320	0.6389	0.6472
5	0.4	L		1.2650	1.2654	1.2682	1.2704	1.2757	1.2772	1.2791
		$\hat{L}$	1.2645	1.2652	1.2658	1.2693	1.2718	1.2766	1.2777	1.2791
		V		1.3431	1.3464	1.3657	1.3800	1.4129	1.4216	1.4332
		$\hat{V}$	1.3396	1.3440	1.3481	1.3708	1.3864	1.4176	1.4247	1.4332
		P_0	0.2914	0.2917	0.2919	0.2929	0.2934	0.2941	0.2941	0.2941
		$\hat{P}_0$	0.2914	0.2916	0.2917	0.2923	0.2928	0.2937	0.2939	0.2941
		P_K		0.0146	0.0148	0.0160	0.0170	0.0200	0.0210	0.0226
		$\hat{P}_K$	0.0144	0.0148	0.0152	0.0172	0.0185	0.0212	0.0218	0.0226
	0.8	L		3.0386	3.0503	3.1170	3.1645	3.2683	3.2967	3.3320
		$\hat{L}$	3.0258	3.0404	3.0536	3.1278	3.1789	3.2809	3.3041	3.3320
		V		2.2060	2.2352	2.3944	2.5015	2.7240	2.7805	2.8569
		$\hat{V}$	2.1734	2.2102	2.2433	2.4221	2.5386	2.7560	2.8025	2.8569
		P_0	0.0439	0.0451	0.0461	0.0506	0.0529	0.0559	0.0561	0.0562
		$\hat{P}_0$	0.0439	0.0444	0.0450	0.0480	0.0500	0.0541	0.0551	0.0562
		P_K		0.2162	0.2226	0.2607	0.2899	0.3612	0.3829	0.4142
		$\hat{P}_K$	0.2094	0.2192	0.2280	0.2777	0.3118	0.3801	0.3956	0.4142

*These are the results for P_{0j}. †These are the results for $P_{\infty j}$.

results for $\hat{L}$ are omitted in Tables 2–4. The approximation results for $\gamma = \infty$ are calculated by the formula [7, Theorem 9.6.1].

Some relative errors Err(%) for P_0 and P_K in Tables 3 – 5 look large, but the absolute error is not so large and this difference is not significant for application area. For example, in $M/M/3/3$ retrial queue with $\gamma = 1.0$, $\rho = 0.8$, Err(%) for P_0 is 15.6% but the absolute error is 0.0052. Considering this situation, we can see from the numerical results that the interpolation between the corresponding $M/G/c/K$ loss system and $M/G/c/\infty$ system provides useful method for moderate size of parameters.

Table 3. Err(%) of approximations of $M/M/3/K$ retrial queue

			γ						
K	ρ		0.05	0.1	0.5	1.0	5.0	10.0	∞
3	0.4	V	0.14	0.27	0.84	1.06	0.81	0.53	0.0
		P_0	0.0	0.07	0.18	0.21	0.07	0.03	0.0
		P_K	0.53	0.88	2.88	3.69	2.87	1.91	0.0
	0.8	V	1.03	1.90	5.58	6.83	4.90	3.19	0.0
		P_0	3.63	6.58	15.1	15.6	7.25	4.13	0.0
		P_K	0.41	0.77	2.39	3.07	2.48	1.70	0.0
5	0.4	L	0.02	0.03	0.09	0.11	0.07	0.04	0.0
		V	0.07	0.13	0.37	0.46	0.33	0.22	0.0
		P_0	0.03	0.07	0.20	0.20	0.14	0.07	0.0
		P_K	1.37	2.70	7.50	8.82	6.00	3.81	0.0
	0.8	L	0.06	0.11	0.35	0.46	0.39	0.22	0.0
		V	0.19	0.36	1.16	1.48	1.17	0.79	0.0
		P_0	1.55	2.39	5.14	5.48	3.22	1.78	0.0
		P_K	1.39	2.43	6.52	7.55	5.23	3.32	0.0

Table 4. Err(%) of approximations of $M/E_2/3/K$ retrial queue

			γ						
K	ρ		0.05	0.1	0.5	1.0	5.0	10.0	∞
3	0.4	V	0.27	0.49	1.59	2.05	1.80	1.38	0.57
		P_0	0.15	0.26	0.83	1.03	0.76	0.58	0.34
		P_K	0.80	1.42	4.47	5.77	5.17	3.90	1.36
	0.8	V	1.30	2.44	7.59	9.63	7.80	5.57	1.30
		P_0	4.61	8.97	22.8	24.9	14.3	9.50	2.93
		P_K	0.52	0.97	3.13	4.12	3.73	2.77	0.62
5	0.4	L	0.10	0.12	0.20	0.22	0.19	0.17	0.13
		V	0.96	1.04	1.39	1.51	1.36	1.25	1.05
		P_0	0.34	0.34	0.27	0.20	0.27	0.27	0.34
		P_K	1.82	3.60	10.2	12.0	10.3	7.79	2.99
	0.8	L	0.47	0.58	0.98	1.10	0.82	0.63	0.34
		V	2.24	2.58	3.78	3.98	2.80	2.14	0.97
		P_0	2.68	1.97	0.81	1.74	0.0	1.10	2.93
		P_K	5.78	7.50	13.2	14.0	8.73	5.98	1.42

4. Conclusions

We have presented an approximation of the stationary distribution of the number of customers in service facility in $M/G/c/K$ retrial queue by using the ordinary loss system and the ordinary multi-server queue with infinite waiting space. The approximation of $M/G/c/K$ retrial queue is few and our study may provide the first step for further research about the multi-server retrial queue with general distribution of service time. There are a number of further research issues that remain to be addressed such as an approximation method for the number of customers in orbit and waiting time.

Table 5. Err(%) of approximations of $M/H_2/3/K$ retrial queue with $C_s^2 = 2.0$

			γ						
K	ρ		0.05	0.1	0.5	1.0	5.0	10.0	∞
3	0.4	V	0.16	0.26	0.44	0.39	0.39	0.51	0.80
		P_0	0.22	0.40	0.92	0.94	0.79	0.68	0.54
		P_K	0.0	0.09	0.25	0.64	0.15	0.50	1.81
	0.8	V	0.45	0.85	2.46	2.99	1.56	0.41	1.83
		P_0	1.33	2.10	3.80	3.49	0.59	2.40	4.26
		P_K	0.20	0.36	1.19	1.54	1.04	0.44	0.83
5	0.4	L	0.08	0.08	0.10	0.11	0.17	0.20	0.24
		V	0.85	0.92	1.11	1.14	1.28	1.39	1.59
		P_0	0.58	0.68	0.88	0.88	0.71	0.64	0.54
		P_K	3.23	4.23	4.74	4.85	1.87	0.0	3.34
	0.8	L	0.23	0.33	0.54	0.52	0.46	0.49	0.60
		V	0.82	1.11	1.52	1.28	0.90	1.01	1.40
		P_0	7.58	9.03	11.9	11.5	7.72	6.31	4.26
		P_K	2.14	1.94	1.93	2.16	0.94	0.14	2.32

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Yang Woo Shin received M.Sc. from Kyungpook National University, and Ph.D at KAIST. He is currently a professor at Changwon National University since 1991. His research interests include queueing theory and its applications.

Department of Statistics, Changwon National University, Changwon 641-773, Korea.
e-mail: ywshin@changwon.ac.kr