J. Appl. Math. & Informatics Vol. 30(2012), No. 3 - 4, pp. 517 - 529
Website: http://www.kcam.biz

THE EXISTENCE OF TWO POSITIVE SOLUTIONS FOR m-POINT BOUNDARY VALUE PROBLEM WITH SIGN CHANGING NONLINEARITY†

JIAN LIU

ABSTRACT. In this paper, the existence theorem of two positive solutions is established for nonlinear m-point boundary value problem by using an inequality for the following third-order differential equations

$$(\phi(u''))' + a(t)f(t,u(t)) = 0, \quad t \in (0,1),$$

$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i\phi(u''(\xi_i)), \quad u'(1) = 0, \quad u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. The nonlinear term f may change sign, as an application, an example to demonstrate our results is given.

AMS Mathematics Subject Classification : 34B15, 34B18.
Key words and phrases : m-point, Third-order, Positive solutions.

1. Introduction

In this paper, the existence of two positive solutions of the following third-order differential equations is obtained:

$$(\phi(u''))' + a(t)f(t,u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i\phi(u''(\xi_i)), \quad u'(1) = 0, \quad u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$.

A projection $\phi : R \longrightarrow R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:

Received August 9, 2011. Revised October 4, 2011. Accepted October 10, 2011.
†This work was supported by the research grant of the University Science and Technology Foundation of Shandong Provincial Education Department(J10LA62).

(i) if $x \le y$, then $\phi(x) \le \phi(y)$, $\forall x, y \in R$;
(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$, $\forall x, y \in R$.

We will assume that the following conditions are satisfied throughout this paper:

(H_1) $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, a_i, $b_i \in [0,1)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$;
(H_2) $a(t) \in C((0,1),[0,+\infty))$ and there exists $t_0 \in (\xi_{m-2}, 1)$, such that $a(t_0) > 0$;
(H_3) $f \in C([0,1] \times [0,+\infty), (-\infty,+\infty))$, $f(t,0) \ge 0$ *and* $f(t,0) \not\equiv 0$.

Recently, much attention has been paid to the existence of positive solutions for third-order multi-point nonlinear boundary value problems, see [1-7] and references therein.

However, to the best of our knowledge, there are not many results concerning the third-order differential equations of increasing homeomorphism and homomorphism.

In [1], Anderson considered the the following third-order nonlinear problem

$$x'''(t) = f(t, x(t)), \quad t_1 \le t \le t_3, \tag{1.3}$$

$$x(t_1) = x'(t_2) = 0, \quad \gamma x(t_3) + \delta x''(t_3) = 0. \tag{1.4}$$

He proved the existence of solutions to the nonlinear problem (1.3) and (1.4) by Using the Krasnoselskii and Leggett and Williams fixed point theorems.

In [2], Sun studied the following nonlinear singular third-order three-point boundary value problem

$$u'''(t) - \lambda a(t)F(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.5}$$

$$u(0) = u'(\eta) = u''(1) = 0. \tag{1.6}$$

He obtained various results on the existence of single and multiple positive solutions by using a fixed point theorem of cone expansion-compression type due to Krasnosel'skii.

In [3], Zhou and Ma studied the existence and iteration of positive solutions for the following third-order generalized right-focal boundary value problem with p-Laplacian operator:

$$(\phi_p(u''))'(t) = q(t)f(t, u(t)), \quad 0 \le t \le 1, \tag{1.7}$$

$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \quad u'(\eta) = 0, \quad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i). \tag{1.8}$$

They established a corresponding iterative scheme for (1.7) and (1.8) by using the monotone iterative technique.

In [4], Ji, Feng and Ge considered the existence of multiple positive solutions for the following BVP:

$$(\phi_p(u'))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.9}$$

$$u(0) = \sum_{i=1}^{m} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m} b_i u(\xi_i), \tag{1.10}$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, a_i, $b_i \in [0, +\infty)$ satisfy $0 < \sum\limits_{i=1}^{m-2} a_i, \sum\limits_{i=1}^{m-2} b_i < 1$. The nonlinearity f is allowed to change sign. Using a fixed point theorem for operators on a cone, they provided sufficient conditions for the existence of (1.9) and (1.10).

In [7], Sang and Su established the key conditions in Theorem 3.1 and Theorem 3.2 by using a new inequality and obtained several existence theorems of positive solutions for nonlinear m-point boundary value problem for the following third-order differential equations

$$(\phi(u''))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1),$$

$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i \phi(u''(\xi_i)), \quad u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$.

Motivated by [7], the purpose of this paper is to show the existence of at least two positive solutions of the BVP (1.1) and (1.2). Compared with [7], the conditions and the methods in our paper are not the same.

In this paper, our work concentrates on the case when the nonlinear term may change sign, we will use the property of the solutions of the BVP and a new inequality to overcome the difficulty.

The rest of the paper is arranged as follows. We state or prove several preliminary results in Section 2, the Section 3 is devoted to the existence of two positive solutions of the BVP(1.1) and (1.2), at the end of the paper, an example is given to illustrate that the work is true. The main tool we use is the fixed-point index theorem in cone.

2. Preliminaries and some Lemmas

To prove the main results in this paper, we need several lemmas. These lemmas are based on the linear BVP

$$(\phi(u''))' + h(t) = 0, \quad t \in (0, 1), \tag{2.1}$$

$$\phi(u''(0)) = \sum_{i=1}^{m-2} a_i \phi(u''(\xi_i)), \quad u'(1) = 0, \quad u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{2.2}$$

Lemma 2.1. *If* $\sum_{i=1}^{m-2} a_i \neq 1$ *and* $\sum_{i=1}^{m-2} b_i \neq 1$*, then for* $h \in C[0,1]$*, the BVP (2.1) and (2.2) has the unique solution*

$$u(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds + Bt + C, \tag{2.3}$$

where

$$A = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)d\tau}{1 - \sum_{i=1}^{m-2} a_i},$$

$$B = -\int_0^1 \phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds,$$

$$C = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds - \sum_{i=1}^{m-2} b_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds}{1 - \sum_{i=1}^{m-2} b_i}.$$

Proof. Necessity. By taking the integral of the problem (2.1) on $(0,t)$, we have

$$\phi(u''(t)) = -\int_0^t h(\tau)d\tau + A, \tag{2.4}$$

then

$$u''(t) = \phi^{-1}\left(-\int_0^t h(\tau)d\tau + A\right). \tag{2.5}$$

By taking the integral of the (2.5) on $(0,t)$, we can get

$$u'(t) = \int_0^t \phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds + B. \tag{2.6}$$

By taking the integral of the (2.6) on $(0,t)$, we can get

$$u(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds + Bt + C. \tag{2.7}$$

Similarly, let $t = 0$ on (2.4), we have $\phi(u''(0)) = A$, let $t = \xi_i$ on (2.4), we have

$$\phi(u''(\xi_i)) = -\int_0^{\xi_i} h(\tau)d\tau + A.$$

Let $t = 1$ on (2.6), we have

$$u'(1) = \int_0^1 \phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds + B = 0.$$

Let $t = 0$ on (2.7), we have

$$u(0) = \int_0^0 (0 - s)\phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds + C.$$

Similarly, let $t = \xi_i$ on (2.7), we have

$$u(\xi_i) = \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds + B\xi_i + C.$$

By the boundary condition (2.2), we can get

$$B = -\int_0^1 \phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds,$$

By calculating, we can get

$$A = -\frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)d\tau}{1 - \sum\limits_{i=1}^{m-2} a_i},$$

$$C = \frac{\sum\limits_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)\phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds - \sum\limits_{i=1}^{m-2} b_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds}{1 - \sum\limits_{i=1}^{m-2} b_i}.$$

Sufficiency. Let u be as in (2.3). Taking the derivative of (2.3), we have

$$u'(t) = \int_0^t \phi^{-1}\left(-\int_0^s h(\tau)d\tau + A\right) ds + B,$$

moreover, we get

$$u''(t) = \phi^{-1}\left(-\int_0^t h(\tau)d\tau + A\right),$$

and

$$\phi(u'') = -\int_0^t h(\tau)d\tau + A,$$

taking the derivative of this expression yields $(\phi(u''))' = -h(t)$. And routine calculation verify that u satisfies the boundary value conditions in (2.2), so that u given in (2.3) is a solution of (2.1) and (2.2).

It is easy to see that BVP $(\phi(u''))' = 0$, $\phi(u''(0)) = \sum\limits_{i=1}^{m-2} a_i\phi(u''(\xi_i))$, $u'(1) = 0$, $u(0) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i)$ has only the trivial solution. Thus u in (2.3) is the unique solution of (2.1) and (2.2). The proof is complete. □

By the same method of [7], we can get the following Lemma 2.2.

Lemma 2.2. *Assume that* (H_1) *holds for* $h \in C[0,1]$ *and* $h \geq 0$, *then the unique solution* u *of (2.1)and (2.2) satisfies* $u(t) \geq 0$, *for* $t \in [0,1]$.

Let the norm on $C[0,1]$ *be the maximum norm. Then the* $C[0,1]$ *is a Banach space.*

We define two cones by

$$K = \{u | u \in C[0,1], u(t) \geq 0, \forall\ t \in [0,1]\},$$

and

$$P' = \{u | u \in C[0,1], u \textit{ is nonnegative, concave and nondecreasing on } [0,1]\}.$$

Lemma 2.3. *If* $u \in P'$ *and satisfies (2.2), then*

$$\inf_{t\in[0,1]} u(t) \geq \gamma \parallel u \parallel,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} b_i \xi_i}{1 - \sum_{i=1}^{m-2} b_i(1-\xi_i)}, \qquad \|u\| = \max_{t\in[0,1]} |u(t)|.$$

Proof. Firstly, according to the concavity of u, we get

$$\frac{u(1)-u(0)}{1-0} \leq \frac{u(\xi_i)-u(0)}{\xi_i - 0}. \tag{2.8}$$

or

$$u(0)(1-\xi_i) \leq u(\xi_i) - \xi_i u(1). \tag{2.9}$$

By the condition

$$u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

we get

$$\sum_{i=1}^{m-2} b_i u(0)(1-\xi_i) \leq u(0) - \sum_{i=1}^{m-2} b_i \xi_i u(1),$$

thus

$$u(0) \geq \frac{\sum_{i=1}^{m-2} b_i}{1 - \sum_{i=1}^{m-2} b_i(1-\xi_i)} \xi_i u(1),$$

by (2.5),we know that the graph of u is concave down on (0,1), that is $u'(t)$ is nonincreasing on (0,1), noticing the condition $u'(1) = 0$ implies that $u'(t) \geq 0$, then

$$\inf_{t\in[0,1]} u(t) \geq \frac{\sum_{i=1}^{m-2} b_i\xi_i}{1 - \sum_{i=1}^{m-2} b_i(1-\xi_i)} \parallel u \parallel .$$

We define a cone by

$$K' = \{u|u \in C[0,1], u \textit{ is nonnegative, and nondecreasing on } [0,1], \inf_{t\in[0,1]} u(t) \geq \gamma \parallel u \parallel\},$$

where γ is the same as in Lemma 2.3.

Let $X = C[0,1]$, define the operator $A, T, T^{'}$ as follow.

$$Au(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s a(\tau)f(\tau,u(\tau))d\tau + \hat{A}\right)ds + \hat{B}t + \hat{C},$$

$$Tu(t) = (Au(t))^+,$$

where

$$\hat{A} = \frac{-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))d\tau}{1-\sum_{i=1}^{m-2} a_i},$$

$$\hat{B} = -\int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f(\tau,u(\tau))d\tau + \hat{A}\right)ds,$$

$$\hat{C} = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\phi^{-1}\left(-\int_0^s a(\tau)f(\tau,u(\tau))d\tau + \hat{A}\right)ds}{1-\sum_{i=1}^{m-2} b_i}$$

$$-\frac{\sum_{i=1}^{m-2} b_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f(\tau,u(\tau))d\tau + \hat{A}\right)ds}{1-\sum_{i=1}^{m-2} b_i},$$

$$(T^{'}u)(t) = \int_0^t (t-s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau)))d\tau + \hat{A}^+\right)ds + \hat{B}^+t + \hat{C}^+,$$

where

$$f^+(t,u) = \max\{f(t,u),0\},$$

$$\hat{A}^+ = \frac{-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f^+(\tau,u(\tau))d\tau}{1-\sum_{i=1}^{m-2} a_i},$$

$$\hat{B}^+ = -\int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds,$$

$$\hat{C}^+ = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds}{1-\sum_{i=1}^{m-2} b_i}$$
$$-\frac{\sum_{i=1}^{m-2} b_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds}{1-\sum_{i=1}^{m-2} b_i}.$$

$$(D)^+ = \max\{D, 0\}.$$

The proof of completely continuity of T and T' is the same, by the method of [7], we can prove the following Lemma 2.4, here, we omit it. □

Lemma 2.4. *$T' : K' \to K'$ is completely continuous.*

Our main tool of this paper is the following fixed-point index theory.

Lemma 2.5 (see ([8]). *Let $\mathbf{B}$ be a Banach space, and let $K \subset \mathbf{B}$ be a cone in $\mathbf{B}$. Assume $r > 0$ and that $T : K_r \longrightarrow K$ is completely continuous operator such that $Tx \neq x$ for $x \in \partial K_r := \{x \in K : \|x\| = r\}$. Then the following assertions hold:*
(1) If $\|x\| \le \|Tx\|$, for all $x \in \partial K_r$, then $i(T, K_r, K) = 0$.
(2) If $\|x\| \ge \|Tx\|$, for all $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

3. Main results

For the convenience, we introduce the following notations.

$$M = \int_0^1 \phi^{-1}\left(\int_0^s a(\tau)d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)d\tau}{1-\sum_{i=1}^{m-2} a_i}\right)ds \times \left(\frac{1-\sum_{i=1}^{m-2} b_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} b_i}\right)$$

$$N = \int_0^1 s\phi^{-1}\left(\int_0^s a(\tau)d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)d\tau}{1-\sum_{i=1}^{m-2} a_i}\right)ds \times \left(\frac{1-\sum_{i=1}^{m-2} b_i + \sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}\right)$$

Theorem 3.1. *Assume $(H_1), (H_2)$ and (H_3) hold, there exist $c, b, d > 0$ such that $0 < \frac{d}{\gamma} < c < \gamma b < b$. And suppose f satisfies the following conditions:*
(H_4) $f(t, u) \ge 0, \quad (t, u) \in [0, 1] \times [d, b]$;

(H_5) $f(t,u) < \phi(\frac{c}{M}), \quad (t,u) \in [0,1] \times [0,c]$;

(H_6) $f(t,u) > \phi(\frac{b}{N}), \quad (t,u) \in [0,1] \times [\gamma b, b]$.

Then (1.1), (1.2) has at least two positive solutions u_1 *and* u_2.

Proof. Firstly, we prove that T has a fixed point $u_1 \in K$ with $0 < \|u_1\| < c$. For all $u \in \partial K_c$, from (H_5), we have

$$\begin{aligned}
&\int_0^s a(\tau) f(\tau, u(\tau)))d\tau - \hat{A} \\
&= \int_0^s a(\tau) f(\tau, u(\tau)))d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) f(\tau, u(\tau)))d\tau}{1 - \sum_{i=1}^{m-2} a_i} \\
&< \phi(\frac{c}{M}) \left[\int_0^s a(\tau)d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)d\tau}{1 - \sum_{i=1}^{m-2} a_i} \right],
\end{aligned}$$

so that

$$\begin{aligned}
&\phi^{-1}\left(\int_0^s a(\tau) f(\tau, u(\tau)))d\tau - \hat{A}\right) \\
&< \frac{c}{M}\phi^{-1}\left(\int_0^s a(\tau)d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)d\tau}{1 - \sum_{i=1}^{m-2} a_i} \right).
\end{aligned}$$

$$\begin{aligned}
&\|Tu\| \\
&= \max_{t\in[0,1]} \{\textstyle\int_0^t (t-s)\phi^{-1}\left(-\int_0^s a(\tau) f(\tau, u(\tau))d\tau + \hat{A}\right) ds + \hat{B}t + \hat{C}\}^+ \\
&< \frac{c}{M} \int_0^1 \phi^{-1}\left(\int_0^s a(\tau)d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)d\tau}{1 - \sum_{i=1}^{m-2} a_i} \right) ds \times \left(1 + \frac{\sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}\right) \\
&= \frac{c}{M} \int_0^1 \phi^{-1}\left(\int_0^s a(\tau)d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)d\tau}{1 - \sum_{i=1}^{m-2} a_i} \right) ds \times \left(\frac{1-\sum_{i=1}^{m-2} b_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} b_i}\right) \\
&= c = \|u\|.
\end{aligned}$$

By Lemma 2.5, we have

$$i(T, K_c, K) = 1.$$

Therefore, T has a fixed point u_1 in K_c. Noticing $f(t,0)\neq 0$ in (H_3), we can know that u_1 is also a fixed point of A in K_c. Thus, u_1 is a solution (1.1) and (1.2).

We now show that A has the other fixed point u_2 such that $c<\|u_2\|\leq b$. For $u\in\partial K'_c,\ i.e., \|u\|=c$. From (H_5), we have

$$\begin{aligned}&\|T'u\|\\&<\frac{c}{M}\phi^{-1}\left(\int_0^1 a(\tau)d\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)d\tau}{1-\sum_{i=1}^{m-2}a_i}\right)\times\left(\frac{1-\sum_{i=1}^{m-2}b_i(1-\xi_i)}{1-\sum_{i=1}^{m-2}b_i}\right)\\&=c=\|u\|.\end{aligned}$$

For $u\in\partial K'_b,\ i.e., \|u\|=b$. For t $[0,T]$, we have $\gamma b\leq u(t)\leq b$. From (H_6), we can get

$$\begin{aligned}&\int_0^s a(\tau)f^+(\tau,u(\tau)))d\tau-\hat{A}^+\\&=\int_0^s a(\tau)f^+(\tau,u(\tau)))d\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)f^+(\tau,u(\tau)))d\tau}{1-\sum_{i=1}^{m-2}a_i}\\&>\phi(\frac{b}{N})\left(\int_0^s a(\tau)d\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)d\tau}{1-\sum_{i=1}^{m-2}a_i}\right),\end{aligned}$$

so that

$$\begin{aligned}&\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau)))d\tau+\hat{A}^+\right)\\&<\frac{b}{N}\phi^{-1}\left(-\int_0^s a(\tau)d\tau-\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)d\tau}{1-\sum_{i=1}^{m-2}a_i}\right).\end{aligned}$$

Let

$$\varphi(s)=\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau+\hat{A}^+\right)ds.$$

For $\xi_i(i=1,\cdots,m-2)$, we can get

$$\int_0^{\xi_i}(\xi_i-s)\varphi(s)ds\geq\xi_i\int_0^1(1-s)\varphi(s)ds.$$

In fact, noticing that $\varphi(s) \le 0$, for $\forall\ t \in (0,1]$, we can get

$$\left(\frac{\int_0^t (t-s)\varphi(s)ds}{t}\right)' = \frac{t\int_0^t \varphi(s)ds - \int_0^t (t-s)\varphi(s)ds}{t^2} \le 0.$$

For $\forall\ t \in (0,1]$,

$$\frac{\int_0^t (t-s)\varphi(s)ds}{t} \ge \frac{\int_0^1 (1-s)\varphi(s)ds}{1}.$$

For $\xi_i (i = 1, \cdots, m-2)$, we have

$$\int_0^{\xi_i} (\xi_i - s)\varphi(s)ds \ge \xi_i \int_0^1 (1-s)\varphi(s)ds. \tag{3.1}$$

Applying (3.1), we can get

$$\begin{aligned}
&\|T'u\| = \max_{t\in[0,1]}\{\int_0^t (t-s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds + \hat{B}^+ t + \hat{C}^+\}\\
&= \max_{t\in[0,1]}\{\int_0^t (t-s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds\\
&-t\int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds +\\
&\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds - \sum_{i=1}^{m-2} b_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds}{1-\sum_{i=1}^{m-2} b_i}\}\\
&= T'u(1)\\
&= \int_0^1 (1-s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds\\
&-\int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds +\\
&\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(\xi_i - s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds - \sum_{i=1}^{m-2} b_i\xi_i \int_0^1 \phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds}{1-\sum_{i=1}^{m-2} b_i}\\
&\ge \int_0^1 (-s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds + \frac{\sum_{i=1}^{m-2} b_i\xi_i \int_0^1 (-s)\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds}{1-\sum_{i=1}^{m-2} b_i}\\
&= \frac{\int_0^1 s\phi^{-1}\left(-\int_0^s a(\tau)f^+(\tau,u(\tau))d\tau + \hat{A}^+\right)ds(-1+\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i)}{1-\sum_{i=1}^{m-2} b_i}\\
&\ge \frac{b}{N}\int_0^1 s\phi^{-1}\left(\int_0^s a(\tau)d\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)d\tau}{1-\sum_{i=1}^{m-2} a_i}\right)ds \times \left(\frac{1-\sum_{i=1}^{m-2} b_i + \sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i}\right)\\
&= b = \|u\|.
\end{aligned}$$

By Lemma 2.5, we can obtain that

$$i(T', K'_c, K') = 1, \quad i(T', K'_b, K') = 0.$$

Hence,

$$i(T', K'_b \backslash \overline{K'_c}, K') = 0 - 1 = -1,$$

therefore, T' has a fixed point in $K'_b \backslash \overline{K'_c}$, and $c < \|u_2\| \le b$.

Finally, we show that u_2 is also the fixed point of A in $K'_b \backslash \overline{K'_c}$. We only need prove that $Au = T'u,\ \forall u \in (K'_b \backslash \overline{K'_c}) \bigcap \{u|T'u = u\}$.

In fact, $\forall u_2 \in (K'_b \backslash \overline{K'_c}) \bigcap \{u|T'u = u\}$, we have $u_2(1) = \|u_2\| > c$, it follows from Lemma 2.3 that

$$\inf_{t\in[0,1]} u_2(t) \ge \gamma \parallel u_2 \parallel > \gamma c > d.$$

Thus, $d \le u_2(t) \le b$ for $t \in [0, 1]$. From condition (H_4), we know that $f^+(t, u_2(t)) = f(t, u_2(t))$, which implies $Au_2 = T'u_2 = u_2,\ i.e., u_2$ is a positive solution and $c < \|u_2\| \le b$. The proof is complete. □

4. Example

In the section, as an application, an example to demonstrate our results is given.

Consider the following BVP

$$(\phi(u''))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1), \tag{4.1}$$

$$\phi(u''(0)) = \frac{1}{2}\phi(u''(\frac{1}{3})), \quad u'(1) = 0, \quad u(0) = \frac{1}{4}u(\frac{1}{3}), \tag{4.2}$$

where

$$\phi(u) = u,$$

$$f(t,u) = \begin{cases} 2 - u^2, \quad (t,u) \in [0,1] \times [0,1], \\ 1 + \frac{76}{5\times 14^2+1}(u-1)^2, \quad (t,u) \in [0,1] \times [1,15], \\ 1 + \frac{76}{5\times 14^2+1} \times 14^2 + (33 \times 8 \times \frac{9}{5} - \frac{76}{5\times 14^2+1} \times 14^2)(u-15)^2, \quad (t,u) \in [0,1] \times [15, 33\times 8], \\ 1 + \frac{76}{5\times 14^2+1} \times 14^2 + (33 \times 8 \times \frac{9}{5} - \frac{76}{5\times 14^2+1} \times 14^2)(33\times 8 - 15)^2 \\ -(33\times 8 - 15)^2(u - 33\times 8)^2, \quad (t,u) \in [0,1] \times [33\times 8, +\infty). \end{cases}$$

It is easy to check that $f : [0, 1] \times [0, +\infty) \longrightarrow (-\infty, +\infty)$ is continuous. In this case, $a(t) \equiv 1,\ a_1 = \frac{1}{2},\ b_1 = \frac{1}{4},\ \xi_1 = \frac{1}{3}$, it follows from a direct calculation that

$$M = \frac{25}{27}, N = \frac{5}{9}, \gamma = \frac{2}{33}.$$

Choose $d = 1,\ c = 15,\ b = 33\times 8$, it is easy to check that $0 < \dfrac{d}{\gamma} < c < \gamma b < b$, and f satisfies the following conditions:

(i) $f(t, u) \ge 0, \quad (t, u) \in [0, 1] \times [1, 33 \times 8]$;

(ii) $f(t, u) \le \frac{76}{5\times 14^2+1}(15 - 1)^2 < \frac{81}{5} = \phi(\dfrac{c}{M}), \quad (t, u) \in [0, 1] \times [1, 15]$;

(iii) $f(t, u) \ge 1 + \frac{76}{5\times 14^2+1} \times 14^2 + (33 \times 8 \times \frac{9}{5} - \frac{76}{5\times 14^2+1} \times 14^2)(16 - 15)^2 \ge 33 \times 8 \times \frac{9}{5}, \quad (t, u) \in [0, 1] \times [16, 33 \times 8)$.

Therefore, the conditions of Theorem 3.1 are satisfied, then BVP (4.1) and (4.2) has at least two positive solutions.

References

1. D.R.Anderson, Green's function for a third-order generalized right focal problem,J.Math. Anal. Appl. 288 (2003) 1-14.
2. Y.P.Sun, Positive solutions of singular third-order three-point boundary value problem, J. Math. Anal. Appl. 306 (2005) 589-603.
3. C.L.Zhou, D.X.Ma, Existence and iteration of positive solutions for a generalized right-focal boundary value problem with p-Laplacian operator, J. Math. Anal. Appl. 324 (2006) 409-424.
4. D.Ji, M.Feng, W.Ge, Multiple positive solutions for multipoint boundary value problems with sign changing nonlnearity, Appl. Math. Comput. 196 (2008) 515-520.
5. C.Bai, J. Fang, Existence of multiple positive solutions for nonlinear m-point boundary value problems, J.Math.Appl.Appl. 81 (2003) 76-85.
6. R.Ma, Positive solutions of nonlinear m-point boundary value problems, Comput. Math. Appl.42 (2001) 755-765.
7. Y.Sang, H.Su, Positive solutions of nonlinear third-order m-point BVP for an increasing homeomorphism and homomorphism with sign-changing nonlinearity, J.Comput.Appl.Math. 225 (2009) 288-300.
8. D.Guo, V.Lakshmikanthan, Nonlinear problems in Abstract Cones, Academic Press, San Diego, 1988.

Jian Liu received master from Qufu Normal University. He works in School of Statistics and Mathematics Science, Shandong University of Finance and Economics. His research interests focus on nonlinear analysis.

1. School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong, 273165, China.

2. School of Statistics and Mathematics Science, Shandong University of Finance and Economics, Jinan, Shandong, 250014, China.

e-mail: kkword@163.com