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NUMERICAL SOLUTION OF A CLASS OF TWO-DIMENSIONAL NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract. In this work, we investigate solving two-dimensional nonlinear Volterra integral equations of the first kind (2DNVIEF). Here we convert 2DNVIEF to the two-dimensional linear Volterra integral equations of the first kind (2DLVIEF) and then we solve it by using operational approach of the Tau method. But for solving the 2DLVIEF we convert it to an equivalent equation of the second kind and then by giving some theorems we formulate the operational Tau method with standard base for solving the equation of the second kind. Finally, some numerical examples are given to clarify the efficiency and accuracy of presented method.

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1. Introduction

We consider two-dimensional nonlinear Volterra integral equations of the form

$$\int_c^t \int_a^x K(x,t,y,z)H(v(y,z))dydz = f(x,t), \quad (x,t) \in \Omega = [a,b] \times [c,d] \qquad (1)$$

where K and f are given analytic functions on $\Omega \times \Omega$ and Ω, respectively. Also, v is the unknown function to be find and H is a nonlinear function in v. Here, we assume that the function H has continuous inverse. Equations of the form (1) arises in applied sciences such as physics and mechanic engineering (see [1-3]).

For solving equation (1), we set $u(x,t) = H(v(x,t))$, to obtain the linear equation

$$\int_c^t \int_a^x K(x,t,y,z)u(y,z)dydz = f(x,t), \quad (x,t) \in \Omega = [a,b] \times [c,d] \qquad (2)$$

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with the new unknown function $u(x,t)$.
Obviously we require satisfying the condition

$$f(x,c) \equiv 0, \quad f(a,t) \equiv 0, \quad \forall (x,t) \in \Omega \tag{3}$$

on the function f. We also require smoothness of the functions $K(x,t,y,z)$ and $f(x,t)$ and the condition

$$K(x,t,x,t) \neq 0, \quad \forall (x,t) \in \Omega \tag{4}$$

to guarantee a unique solution for Eq.(2) (see [3]).

Numerical solution of equations of the form (2) has been investigated in literature. For example in [4], Bel'tyukov and Kuznechikhina proposed an explicit Rung-Kutta-type method of order three. An interesting proof for the existence and uniqueness of solution and an Euler-type method has been proposed for (2) in [3]. Authors of [5] applied two dimensional block-pulse functions for solving nonlinear Volterra integral equations of the first kind of the form (2).

In this paper, we are interested in solving differentiated form of equation (2) by the operational approach of the Tau method. The operational Tau method (see Ortiz [6], Ortiz and Samara [7]) is a well known method for solving functional equations. Up to now, this method has been developed for solving differential, integral and integro-differential equations. For example see $[6-13]$ for ordinary differential equations, $[14-16]$ for partial differential equations. Particularly, this method has been developed in $[17-19]$ for the numerical solution of one dimensional linear integral and integro-differential equations. Also it has been developed for solving systems of integro-differential equations in $[20-21]$. The authors of [22] proposed an extension of the Tau method for the numerical solution of nonlinear Volterra-Hammerstein integral equations. Also, the authors of [23] formulated the Tau method for solving the two-dimensional linear Fredholm integral equations of the second kind.

2. Preliminaries

In this section, we briefly introduce the operational Tau method. This method which was proposed by Ortiz and Samara [7], is based on the following three simple matrices

$$\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\iota = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We recall the following lemmas from ([7], [17]).

Lemma 2.1. *Let* $y_N(x) = a_N X$ *with* $a_N = (a_0, a_1, ..., a_N, 0, 0, ...)$ *and* $X = (1, x, x^2, ...)$ *be a polynomial, then*
a. *The effect of* r *repeated differentiation of* $y_N(x)$ *is equivalent to the post-multiplication of* a_N *by* η^r*, i.e.*

$$\frac{d^r}{dx^r} y_N(x) = a_N \eta^r X.$$

b. *The effect of* s *shifts on the coefficients of* $y_N(x)$ *is equivalent to the post-multiplication of* a_N *by* μ^s*, i.e.*

$$x^s y_N(x) = a_N \mu^s X.$$

c. *The effect of integration on* $y_N(x)$ *is equivalent to the post-multiplication of* a_N *by* ι*, i.e.*

$$\int y_N(x)dx = a_N \iota X.$$

Corollary 2.2. *With the assumptions of lemma 2.1, we have*

a. $x^i X = \mu^i X.$

b. $\displaystyle\int X dx = \iota X.$

Lemma 2.3. *If* $M = [m_{k,l}]$ *be an* $(N+1) \times (N+1)$ *matrix, then*

$$(\mu^j M \mu^i)_{k,l} = \begin{cases} m_{k+j,l-i}, & k = 1, 2, ..., N+1-j, \quad l = i+1, ..., N+1 \\ 0, & \textit{otherwise}. \end{cases} \tag{5}$$

Lemma 2.4. *For* $(N+1) \times (N+1)$ *matrix* ι*, we have*

$$(\mu^j \iota \mu^i)_{k,k+i+j+1} = \begin{cases} \frac{1}{k+j}, & i, j, k = 1, 2, ..., N+1, \quad k+i+j+1 \leq N+1 \\ 0, & \textit{otherwise}. \end{cases} \tag{6}$$

3. Main results

In this section, we give some theorems and lemmas to write a matrix representation for the integral equation

$$u(x,t) = \int_c^t \int_a^x K_1(x,t,y,z)u(y,z)dydz + \int_c^t K_2(x,t,z)u(x,z)dz + \int_a^x K_3(x,t,y)u(y,t)dy + F(x,t) \tag{7}$$

where

$$K_1(x,t,y,z) = -\frac{\partial^2 K}{\partial x \partial t}(x,t,y,z)/K(x,t,x,t)$$
$$K_2(x,t,z) = -\frac{\partial K}{\partial t}(x,t,x,z)/K(x,t,x,t)$$

$$K_3(x,t,y) = -\frac{\partial K}{\partial x}(x,t,y,t)/K(x,t,x,t)$$
$$F(x,t) = \frac{\partial^2 f}{\partial x \partial t}(x,t)/K(x,t,x,t)$$

which is obtained by differentiating from both sides of equation (2). Since integral equations of the first kind are inherently ill-posed problems, meaning that an small change on the function $f(x,t)$ make a very large change on the solution of (2) (see [24], we transform equation (2) to the equation (7). To show the preference of this transformation, we will also formulate the operational Tau method directly on some examples of the first kind integral equations. To covert equation (7) to the corresponding system of linear equations by the operational approach of the Tau method, it is necessary the functions $K(x,t,y,z)$ and $f(x,t)$ to be polynomials, otherwise, we should approximate these functions by polynomials of suitable degree.

Now let us consider the approximate solution of equation (7) of the form

$$u_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} C_{ij}x^i t^j = \underline{X}^T \underline{CT} \tag{8}$$

which is a partial sum of the series solution

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij}x^i t^j = X^T CT \tag{9}$$

of equation (7), where $X = (1, x, x^2, ..., x^N, ...)^T$, $T = (1, t, t^2, ..., t^N, ...)^T$ and $\underline{X} = (1, x, x^2, ..., x^N)^T$, $\underline{T} = (1, t, t^2, ..., t^N)^T$ and

$$C = \begin{pmatrix} C_{00} & C_{01} & \cdots & C_{0N} & \cdots \\ C_{10} & C_{11} & \cdots & C_{1N} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ C_{N0} & C_{N1} & \cdots & C_{NN} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and $\underline{C}$ is a matrix including first $N+1$ rows and columns of C.

Now we state the following fundamental theorem of [25].

Theorem 3.1.

(a) *Let* $K_1(x,t,y,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} k_{ijmn}^{(1)} x^i t^j y^m z^n$, *then*

$$\int_c^t \int_a^x K_1(x,t,y,z)u(y,z)dydz = \underline{X}^T \Pi_1 \underline{T} \tag{10}$$

where

$$\Pi_1 = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} k_{ijmn}^{(1)} P_{ijmn}^{(1)} \tag{11}$$

with

$$P^{(1)}_{ijmn} = \left[(\mu^m \iota \mu^i)^T - e_{i+1}\underline{\xi}^{(m)T}(a)\right] C \left[\mu^n \iota \mu^j - \underline{\xi}^{(n)}(c) e^T_{j+1}\right] \tag{12}$$

and

$$\underline{\xi}^{(m)}(x) = \mu^m \iota \underline{X}, \quad \underline{\xi}^{(n)}(x) = \mu^n \iota \underline{X}.$$

(b) *If* $K_2(x,t,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{n=0}^{N} k^{(2)}_{ijn} x^i t^j z^n$, *then*

$$\int_c^t K_2(x,t,z)u(x,z)dz = \underline{X}^T \Pi_2 \underline{T} \tag{13}$$

where

$$\Pi_2 = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{n=0}^{N} k^{(2)}_{ijn} P^{(2)}_{ijn} \tag{14}$$

with

$$P^{(2)}_{ijn} = (\mu^i)^T C \left[\mu^n \iota \mu^j - \underline{\xi}^{(n)}(c) e^T_{j+1}\right]. \tag{15}$$

(c) *For* $K_3(x,t,y) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N} k^{(3)}_{ijm} x^i t^j y^m$,, *we have*

$$\int_a^x K_3(x,t,y)u(y,t)dy = \underline{X}^T \Pi_3 \underline{T} \tag{16}$$

where

$$\Pi_3 = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N} k^{(3)}_{ijm} P^{(3)}_{ijm} \tag{17}$$

with

$$P^{(3)}_{ijm} = \left[(\mu^m \iota \mu^i)^T - e_{i+1}\underline{\xi}^{(m)T}(a)\right] C \mu^j. \tag{18}$$

Proof. See [25]. □

Lemma 3.2. *The entries of* $\underline{\xi}^{(m)}(x) = \left(\xi^{(m)}_1(x), ..., \xi^{(m)}_{N+1}(x)\right)$ *are computed as follows :*

$$\xi^{(m)}_k(x) = \begin{cases} \frac{x^{k+m}}{k+m}, & k = 1,2,...,N-m \\ 0, & otherwise. \end{cases} \tag{19}$$

A similar result can be obtained for $\underline{\xi}^{(n)}(x)$ *by replacing* m *by* n.

Proof. See [25]. □

By the following theorem we determine structure of the matrices $P^{(1)}_{ijmn}, P^{(2)}_{ijn}$ and $P^{(3)}_{ijm}$ which help us to obtain the matrices Π_1, Π_2 and Π_3.

Theorem 3.3.
(a) *The matrix $P^{(1)}_{ijmn}$ in theorem 3.1(a) has the following structure :*

$$\begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & p_{i+1,j+1} & 0 & \cdots & 0 & p_{i+1,n+j+2} & \cdots & p_{i+1,N+1} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & p_{m+i+2,j+1} & 0 & \cdots & 0 & p_{m+i+2,n+j+2} & \cdots & p_{m+i+2,N+1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & p_{N+1,j+1} & 0 & \cdots & 0 & p_{N+1,n+j+2} & \cdots & p_{N+1,N+1}
\end{pmatrix}$$

where the nonzero entries are computed by the following formulas

$$p_{i+1,j+1} = \sum_{r=1}^{N-n} \sum_{k=1}^{N-m} (\frac{c^{r+n}}{r+n})(\frac{a^{k+m}}{k+m})C_{k-1,r-1}.$$

$$p_{i+1,(n+j+1)+r} = -\frac{1}{r+n} \sum_{k=1}^{N-m} \frac{a^{k+m}}{k+m} C_{k-1,r-1}, \quad r = 1, 2, \cdots, N-n-j.$$

$$p_{(m+i+1)+k,j+1} = -\frac{1}{k+m} \sum_{r=1}^{N-n} \frac{c^{r+n}}{r+n} C_{k-1,r-1}, \quad k = 1, 2, \cdots, N-m-i.$$

$$p_{(m+i+1)+k,(n+j+1)+r} = \frac{1}{(k+m)(r+n)} C_{k-1,r-1}, \quad k = 1, \cdots, N-m-i,$$

$$r = 1, \cdots, N-n-j.$$

(b) *The form of matrix $P^{(2)}_{ijn}$ in theorem 3.1(b) is :*

$$\begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & p_{i+1,j+1} & 0 & \cdots & 0 & p_{i+1,n+j+2} & \cdots & p_{i+1,N+1} \\
0 & \cdots & 0 & p_{i+2,j+1} & 0 & \cdots & 0 & p_{i+2,n+j+2} & \cdots & p_{i+2,N+1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & p_{N+1,j+1} & 0 & \cdots & 0 & p_{N+1,n+j+2} & \cdots & p_{N+1,N+1}
\end{pmatrix}$$

where the nonzero entries are computed by the formulas

$$p_{i+r,j+1} = -\sum_{k=1}^{N-n} \frac{c^{k+n}}{k+n} C_{r-1,k-1}, \quad r = 1, 2, \cdots, N-i+1,$$

$$p_{i+r,(n+j+1)+k} = \frac{1}{k+n} C_{r-1,k-1}, \quad r = 1, 2, \cdots, N-i+1,$$

$$k = 1, \cdots, N-n-j.$$

(c) *The structure of the matrix $P^{(3)}_{ijm}$ in theorem 3.1(c) is :*

$$\begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & p_{i+1,j+1} & p_{i+1,j+2} & \cdots & p_{i+1,N+1} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & p_{m+i+2,j+1} & p_{m+i+2,j+2} & \cdots & p_{m+i+2,N+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & p_{N+1,j+1} & p_{N+1,j+2} & \cdots & p_{N+1,N+1}
\end{pmatrix}$$

where the nonzero entries are computed by the formulas

$$p_{i+1,j+k} = -\sum_{r=1}^{N-m} \frac{a^{r+m}}{r+m} C_{r-1,k-1}, \quad k = 1, 2, \cdots, N-j+1,$$

$$p_{(m+i+1)+r,j+k} = \frac{1}{r+m} C_{r-1,k-1}, \quad r = 1, 2, \cdots, N-m-i,$$

$$k = 1, 2, \cdots, N-j+1.$$

Proof. See [25]. □

Now by using theorem 3.1, we convert the equation (7) to a linear system of algebraic equations. To this end, note that our assumption on the functions let us to write the function $F(x,t)$ of the form :

$$F(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} F_{ij} x^i t^j = \underline{X}^T \underline{FT} \tag{20}$$

where $\underline{F} = [F_{ij}]$ is an $(N+1) \times (N+1)$ matrix.
Consequently substituting from relations (8), (10), (13), (16) and (20) in equation (7) leads to the equation

$$\underline{X}^T \underline{CT} - \underline{X}^T \Pi_1 \underline{T} - \underline{X}^T \Pi_2 \underline{T} - \underline{X}^T \Pi_3 \underline{T} = \underline{X}^T \underline{FT}$$

or equivalently to the matrix equation

$$\underline{C} - \Pi_1 - \Pi_2 - \Pi_3 = \underline{F} \tag{21}$$

since $\underline{X}$ and $\underline{T}$ are the basis vectors.
Equation (21) is the matrix representation of equation (7) and it is solved for the unknown coefficients C_{ij}. Therefore the approximate solution $u_N(x,t)$ of equation (7) is obtained in the form (8), and so the approximate solution of equation (1 is obtained as

$$v_N(x,t) = H^{-1}(u_N(x,t))$$

Definition 3.1. The polynomial $u_N(x,t) = \underline{X}^T \underline{CT}$ is called a Tau method approximate solution of equation (7), if the matrix $\underline{C}$ is the solution of the system (21).

Remark 3.1. From the above definition and structure of the Tau method, it is clear that if the solution $u(x,t)$ of equation (7) is a polynomial of degree (n_1, n_2), then any Tau method approximate solution of degree (N_1, N_2) with $N_1 \geq n_1$ and $N_1 \geq n_1$ will be exact (see [17]).

4. Error bound and convergence

In this section, we obtain an error bound for the approximate solution. We also prove convergence of the presented method. To this end, we define the error function

$$e_u(x,t) = u(x,t) - u_N(x,t) \tag{22}$$

where $u(x,t)$ and $u_N(x,t)$ are the exact and approximate solutions of equation (7).

Theorem 4.1. *The error function $e_u(x,t)$ in (22) has a bound of the form*

$$|e_u(x,t)| = |u(x,t) - u_N(x,t)| \leq \frac{M_1|x|^{N+1} + M_2|t|^{N+1}}{(N+1)!} \tag{23}$$

where $u_N(x,t)$ is approximation of $u(x,t)$ as introduced in (8) and M_1 and M_2 are nonnegative constants such that

$$\left| \frac{\partial^{N+1} u(x,t)}{\partial x^{N+1}} \right| \leq M_1, \qquad \left| \frac{\partial^{N+1} u(x,t)}{\partial t^{N+1}} \right| \leq M_2 \tag{24}$$

Proof. Writing the bivariate Taylor expansion of $u(x,t)$ around $(0,0)$ we have

$$u(x,t) = \sum_{k=0}^{N} \left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \right)^k u(0,0) + \frac{1}{(N+1)!} \left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \right)^{N+1} u(\xi_1, \xi_2)$$

where $\xi_1 \in (0, x)$ and $\xi_2 \in (0, t)$. Since the function $u_N(x, t)$ in (8) include all terms with degree less than $N + 1$, we have

$$u(x,t) - u_N(x,t) \approx \frac{1}{(N+1)!}\left[\left(\frac{\partial^{N+1}}{\partial x^{N+1}}u(\xi_1,\xi_2)\right)x^{N+1} + \left(\frac{\partial^{N+1}}{\partial t^{N+1}}u(\xi_1,\xi_2)\right)t^{N+1}\right]$$

then using (24) completes the proof. □

Corollary 4.2. *By the conditions of theorem 4.1, we have*

$$\lim_{n\to\infty} u_n(x,t) = u(x,t)$$

Corollary 4.3. *Note that the error bound (23) confirm the remark 3.1, since the conditions of this remark imply $M_1 = 0$ and $M_2 = 0$.*

Theorem 4.4. *According to the conditions of the theorem 4.1, if the function H has continuous inverse, then we have*

$$\lim_{n\to\infty} v_n(x,t) = v(x,t)$$

Proof. Since $u(x, t) = H(v(x, t))$ and H^{-1} is continuous, we have

$$\lim_{n\to\infty} v_n(x,t) = \lim_{n\to\infty} H^{-1}(u_n(x,t)) = H^{-1}(u(x,t)) = v(x,t)$$

This shows that the presented method is convergent. □

5. Numerical Results

In this section, we give some examples to demonstrate accuracy and efficiency of the presented method. Notice that all non-polynomial terms in these examples approximated by the Taylor polynomial of degree N and all computations have been done by programming in Maple.

Example 1 (See [5], example 2)**.**

$$\int_0^t\int_0^x v^2(y,z)dydz = \frac{1}{45}xt(9x^4+10x^2t^2+9t^4), \quad (x,t)\in[0,1]\times[0,1]. \quad (25)$$

The exact solution is $v(x, t) = x^2 + t^2$. To solve this equation, we substitute $u(x, t) = v^2(x, t)$ to obtain the linear equation

$$\int_0^t\int_0^x u(y,z)dydz = \frac{1}{45}xt(9x^4+10x^2t^2+9t^4), \quad (x,t)\in[0,1]\times[0,1], \quad (26)$$

then we transformed this equation to the equation of second kind similar to (7) and we apply the Tau method on it and obtained the solution $u(x, t) = (x^2+t^2)^2$ for (26) which implies the solution $v(x, t) = x^2 + t^2$ for the equation (25) and it is the exact solution. This confirm the remark 3.1. To compare the result, we report the result of [5] in Table 1.

Table 1 : Results of [5] for the example.1

$(x,t) = 2^{-l}$	$m = 16$	$m = 32$	$m = 64$
$l = 1$	$0.6300e-1$	$0.3100e-1$	$0.1600e-1$
$l = 2$	$0.3200e-1$	$0.1600e-1$	$0.7800e-2$
$l = 3$	$0.1600e-1$	$0.8000e-2$	$0.3900e-2$
$l = 4$	$0.8500e-2$	$0.4100e-2$	$0.2000e-2$
$l = 5$	$0.2500e-3$	$0.2100e-2$	$0.1000e-2$
$l = 6$	$0.1200e-2$	$0.6400e-4$	$0.5300e-4$

Note that applying the Tau method directly to equation (26) leads to a linear system of algebraic equations that its coefficients matrix is singular.

Example 2 (See [5], example 3)**.**

$$\int_0^t \int_0^x e^{x+t} v^3(y,z)dydz = f(x,t), \quad (x,t) \in [0,1] \times [0,1] \tag{27}$$

where $f(x,t)$ is selected in such a way that $v(x,t) = e^{x+2t}$ to be the exact solution. Solving this example is similar to the previous example. We substitute $u(x,t) = v^3(x,t)$ to get a linear equation , then we approximated $K_1(x,t,y,z)$, $K_2(x,t,z)$, $K_3(x,t,y)$ and $F(x,t)$ by polynomials of degree N by using truncated Taylor series around the point $(x_0,t_0,y_0,z_0) = (0,0,0,0)$ and finally we apply the method to obtain the results of Tables 2 for $N = 14$, $N = 16$ and $N = 18$ which is represented for the absolute errors $|e_v(x,t)| = |v(x,t) - v_N(x,t)|$ at some selected points of $[0,1] \times [0,1]$.

Table 2 : Absolute errors of example 2

(x,t)	$N = 14$	$N = 16$	$N = 18$
$(\frac{1}{16},\frac{1}{16})$	$0.3000e-18$	$0.2000e-18$	$0.1000e-18$
$(\frac{1}{8},\frac{1}{8})$	$0.2456e-14$	$0.5200e-17$	$0.2000e-18$
$(\frac{1}{4},\frac{1}{4})$	$0.5815e-10$	$0.4756e-12$	$0.3101e-14$
$(\frac{1}{2},\frac{1}{2})$	$0.1001e-5$	$0.3234e-7$	$0.8348e-9$
$(0.75,0.5)$	$0.1320e-5$	$0.4215e-7$	$0.1081e-8$
$(1,0.75)$	$0.3018e-3$	$0.2149e-4$	$0.1230e-5$
$(1,1)$	$0.9384e-2$	$0.1171e-2$	$0.1178e-3$

For comparing the results of table 2, we report the result of [5] in Table 3.

Table 3 : Results of [5] for example.2

$(x,t)=2^{-l}$	$m=16$	$m=32$	$m=64$
$l=1$	0.4200	0.2100	0.1000
$l=2$	0.1900	$0.9300e-1$	$0.4600e-1$
$l=3$	0.1200	$0.6000e-1$	$0.2900e-1$
$l=4$	0.1400	$0.4700e-1$	$0.2300e-1$
$l=5$	$0.2400e-1$	$0.6300e-1$	$0.2000e-1$
$l=6$	$0.2600e-1$	$0.1200e-1$	$0.3100e-1$

Similar to example.1 using the Tau method directly to linear form of equation (27) leads to a singular linear system of algebraic equations.

Example 3 (Constructed)**.** The equation

$$\int_0^t \int_0^x cos(y-z)e^{v(y,z)}dydz = f(x,t), \quad x,t \in [0,2] \tag{28}$$

with

$$f(x,t) = 2 + \frac{1}{4}(x+t) - 2(cosx + cost) - \frac{1}{4}(xcos2t + tcos2x) + 2cos(x-t)$$

has the exact solution as $v(x,t) = \ln(sin(x+t)+2)$. We proceed as before by setting $u(x,t) = e^{v(x,t)}$ and obtain the results of Table 4 for $|e_v(x,t)| = |v(x,t) - v_N(x,t)|$.

Table 4 : Results of example.3 by the Tau method

(x,t)	$N=10$	$N=12$	$N=13$
$(0.5,0.5)$	$0.7372e-11$	$0.1186e-13$	$0.2505e-15$
$(1,0.5)$	$0.6958e-8$	$0.4497e-10$	$0.2050e-11$
$(1,1)$	$0.8045e-8$	$0.5276e-10$	$0.6891e-11$
$(1.5,1.5)$	$0.1082e-6$	$0.1048e-8$	$0.3113e-8$
$(2,1)$	$0.9328e-5$	$0.2531e-6$	$0.7888e-7$
$(2,2)$	$0.4572e-4$	$0.1135e-5$	$0.2539e-6$

The system obtained of using the Tau method directly to linear form of equation (28), also is singular.

Example 4 (Constructed)**.** Consider the equation

$$\int_0^t \int_0^x e^{2y-z}\frac{1}{v(y,z)}dydz = f(x,t), \quad x,t\epsilon[0,2] \tag{29}$$

where $f(x,t)$ is selected in such a way that $v(x,t) = \frac{1}{x^2+t^2+1}$ to be the exact solution.
By setting $u(x,t) = \frac{1}{v(y,z)}$ we obtain the linear equation

$$\int_0^t \int_0^x e^{2y-z} u(y,z)dydz = f(x,t), \quad x, t\epsilon[0,2] \tag{30}$$

and applying the proceed gives $u(x,t) = x^2 + t^2 + 1$ and $v(x,t) = \frac{1}{x^2+t^2+1}$ which is the exact of equation (29). But using the Tau method directly to linear equation (30) leads to an inconsistent system.

6. Conclusion

In this paper, we designed a simple high accurate method for solving nonlinear two-dimensional Volterra integral equations of the first kind by using a simple transform. As the examples show, the presented method has high accuracy. Also, it will be possible to investigate the numerical solution of the two dimensional Fredholm integral equations of the first kind.

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