J. Appl. Math. & Informatics Vol. **30**(2012), No. 3 - 4, pp. 445 - 454
Website: http://www.kcam.biz

MULTIVARIATE RIGHT FRACTIONAL OSTROWSKI INEQUALITIES

GEORGE A. ANASTASSIOU

ABSTRACT. Very general multivariate right Caputo fractional Ostrowski inequalities are presented. Some of them are proved to be sharp and attained. Estimates are with respect to $\|\cdot\|_\infty$.

AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Key words and phrases : multivariate Ostrowski inequality, right Caputo fractional derivative, radial derivative.

1. Introduction

In 1938, A. Ostrowski [7] proved the following important inequality:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty , \tag{1}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability. This paper is greatly motivated and inspired also by the following result.

Theorem 1.2 (see [1]). *Let $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$ and $x \in [a, b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then it holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right). \tag{2}$$

Received November 20, 2010. Revised July 19, 2011. Accepted September 1, 2011.

Inequality (2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, while when n is even the optimal function is

$$\overline{f}(y) := |y-x|^{n+\alpha} \cdot (b-a), \quad \alpha > 1.$$

Clearly inequality (2) generalizes inequality (1) for higher order derivatives of f.

Also in [2], see Chapters 24-26, we presented a complete theory of left fractional Ostrowski inequalities.

2. Main Results

We need

Remark 2.1. We define the ball $B(0,R) = \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, and the sphere

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ is the Lebesgue measure of the ball.

Following [5, pp. 149-150, exercise 6], and [6, pp.87-88, Theorem 5.2.2] we can write $F : \overline{B(0,R)} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr \right) d\omega; \tag{3}$$

we use this formula a lot.

Initially the function $f : \overline{B(0,R)} \to \mathbb{R}$ is radial; that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $\forall\ x \in \overline{B(0,R)}$. Here we assume that $g \in AC^m([0,R])$ (means $g^{(m-1)}$ is in $AC([0,R])$), $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceilling of the number), $\alpha > 0$, and $g^{(k)}(R) = 0$, $k = 1, ..., m-1$.

By [3] we get

$$g(s) - g(R) = \frac{1}{\Gamma(\alpha)} \int_s^R (J-s)^{\alpha-1} D_{R-}^{\alpha} g(J)\, dJ, \tag{4}$$

$\forall\ s \in [0,R]$, where $D_{R-}^{\alpha} g$ is the right Caputo derivative. Further assume that $D_{R-}^{\alpha} g \in L_\infty([0,R])$.

We obtain

$$\begin{aligned} |g(s)-g(R)| &\leq \frac{1}{\Gamma(\alpha)}\int_s^R (J-s)^{\alpha-1}\left|D_{R-}^{\alpha}g(J)\right|dJ \\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_s^R (J-s)^{\alpha-1}dJ\right)\left\|D_{R-}^{\alpha}g\right\|_{\infty,[0,R]} \\ &= \frac{(R-s)^{\alpha}}{\Gamma(\alpha+1)}\left\|D_{R-}^{\alpha}g\right\|_{\infty,[0,R]}. \end{aligned} \tag{5}$$

I.e.

$$|g(s)-g(R)| \leq \frac{\left\|D_{R-}^{\alpha}g\right\|_{\infty,[0,R]}}{\Gamma(\alpha+1)}(R-s)^{\alpha}, \tag{6}$$

$\forall\ s\in[0,R]$.

Next observe that

$$\begin{aligned} \left|f(R\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol(B(0,R))}\right| &= \left|g(R)-\frac{\int_{S^{N-1}}\left(\int_0^R g(s)\,s^{N-1}ds\right)d\omega}{\int_{S^{N-1}}\left(\int_0^R s^{N-1}ds\right)d\omega}\right| \\ &= \left|g(R)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right| \\ &= \frac{N}{R^N}\left|\int_0^R s^{N-1}(g(R)-g(s))\,ds\right| \\ &\leq \frac{N}{R^N}\int_0^R s^{N-1}|g(R)-g(s)|\,ds \\ &\leq \frac{N}{R^N}\frac{\|D_{R-}^{\alpha}g\|_{\infty,[0,R]}}{\Gamma(\alpha+1)}\int_0^R s^{N-1}(R-s)^{\alpha}\,ds \\ &= \frac{N}{R^N}\frac{\|D_{R-}^{\alpha}g\|_{\infty,[0,R]}}{\Gamma(\alpha+1)}\int_0^R (R-s)^{(\alpha+1)-1}s^{N-1}ds \\ &= \frac{N}{R^N}\frac{\|D_{R-}^{\alpha}g\|_{\infty,[0,R]}}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)(N-1)!}{\Gamma(\alpha+N+1)}R^{\alpha+N} \\ &= \|D_{R-}^{\alpha}g\|_{\infty,[0,R]}\frac{N!R^{\alpha}}{\Gamma(\alpha+N+1)}. \end{aligned} \tag{7}$$

So we have proved that

$$\left|f(R\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol(B(0,R))}\right| = \left|g(R)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right| \tag{8}$$

$$\leq \left\|D_{R-}^{\alpha}g\right\|_{\infty,[0,R]}\frac{N!R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{9}$$

The last inequality (9) is sharp, it is attained by $\overline{g}(r)=(R-r)^{\alpha}$, $\alpha>0$, $r\in[0,R]$. As in [4] we get

$$D_{R-}^{\alpha}\overline{g}(r)=\Gamma(\alpha+1),\quad \forall\ r\in[0,R].$$

Hence $\left\|D_{R-}^{\alpha}\overline{g}\right\|_{\infty,[0,R]}=\Gamma\left(\alpha+1\right)$. And $\overline{g}\left(R\right)=0$. Therefore

$$\begin{aligned} L.H.S.(9) &= \frac{N}{R^N}\int_0^R \left(R-s\right)^{\alpha} s^{N-1}ds \\ &= \frac{N}{R^N}\int_0^R \left(R-s\right)^{(\alpha+1)-1}\left(s-0\right)^{N-1}ds \\ &= \frac{N}{R^N}\frac{\Gamma\left(\alpha+1\right)\Gamma\left(N\right)}{\Gamma\left(\alpha+N+1\right)}R^{\alpha+N} = \frac{\Gamma\left(\alpha+1\right)N!}{\Gamma\left(\alpha+N+1\right)}R^{\alpha}. \end{aligned}$$

And

$$R.H.S.(9) = \frac{\Gamma\left(\alpha+1\right)N!R^{\alpha}}{\Gamma\left(\alpha+N+1\right)},$$

proving attainability of (9).

We have established the following multivariate Ostrowski inequality

Theorem 2.1. *Let $f:\overline{B\left(0,R\right)}\to\mathbb{R}$ which is radial, that is, there exists g such that $f\left(x\right)=g\left(r\right)$, $r=|x|$, $\forall\ x\in\overline{B\left(0,R\right)}$. Assume that $g\in AC^m\left(\left[0,R\right]\right)$, $m=\lceil\alpha\rceil$, $\alpha>0$, and $g^{(k)}\left(R\right)=0$, $k=1,...,m-1$, and $D_{R-}^{\alpha}g\in L_{\infty}\left(\left[0,R\right]\right)$. Then*

$$\begin{aligned} \left|f\left(R\omega\right)-\frac{\int_{B(0,R)}f\left(y\right)dy}{Vol\left(B\left(0,R\right)\right)}\right| &= \left|g\left(R\right)-\frac{N}{R^N}\int_0^R g\left(s\right)s^{N-1}ds\right| \\ &\leq \left\|D_{R-}^{\alpha}g\right\|_{\infty,[0,R]}\frac{N!R^{\alpha}}{\Gamma\left(\alpha+N+1\right)}. \end{aligned} \tag{10}$$

The last inequality is sharp, that is attained by $\overline{g}\left(r\right)=\left(R-r\right)^{\alpha}$, $\alpha>0$, $\forall$ $r\in\left[0,R\right]$.

We also make

Remark 2.2. Let the spherical shell $A:=B\left(0,R_2\right)-\overline{B\left(0,R_1\right)}$, $0<R_1<R_2$, $A\subseteq\mathbb{R}^N$, $N\geq 2$, $x\in\overline{A}$. Consider again that $f:\overline{A}\to\mathbb{R}$ is radial, that is, there exists g such that $f\left(x\right)=g\left(r\right)$, $r=|x|$, $r\in\left[R_1,R_2\right]$, $\forall\ x\in\overline{A}$. Here again x can be written uniquely as $x=r\omega$, where $r=|x|>0$, and $\omega=\frac{x}{r}\in S^{N-1}$, $|\omega|=1$. We can write for $F:\overline{A}\to\mathbb{R}$ a Lebesgue integrable function that

$$\int_A F\left(x\right)dx = \int_{S^{N-1}}\left(\int_{R_1}^{R_2} F\left(r\omega\right)r^{N-1}dr\right)d\omega. \tag{11}$$

Here $Vol\left(A\right)=\frac{\omega_N\left(R_2^N-R_1^N\right)}{N}$, and we assume that $g\in AC^m\left(\left[R_1,R_2\right]\right)$, $m=\lceil\alpha\rceil$, $\alpha>0$, and $g^{(k)}\left(R_2\right)=0$, $k=1,...,m-1$. We get (see [3])

$$g\left(s\right)-g\left(R_2\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_s^{R_2}\left(J-s\right)^{\alpha-1}D_{R_2-}^{\alpha}g\left(J\right)dJ, \tag{12}$$

$\forall\ s \in [R_1, R_2]$, where $D^{\alpha}_{R_2-}g$ is the right Caputo fractional derivative. Further assume that $D^{\alpha}_{R_2-}g \in L_{\infty}([R_1, R_2])$. Hence

$$\begin{aligned} |g(s) - g(R_2)| &\leq \frac{1}{\Gamma(\alpha)} \int_s^{R_2} (J-s)^{\alpha-1} \left| D^{\alpha}_{R_2-}g(J) \right| dJ \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_s^{R_2} (J-s)^{\alpha-1} dJ \right) \left\| D^{\alpha}_{R_2-}g \right\|_{\infty,[R_1,R_2]} \\ &= \frac{1}{\Gamma(\alpha)} \frac{(R_2-s)^{\alpha}}{\alpha} \left\| D^{\alpha}_{R_2-}g \right\|_{\infty,[R_1,R_2]}. \end{aligned} \tag{13}$$

Therefore

$$|g(s) - g(R_2)| \leq \frac{\left\| D^{\alpha}_{R_2-}g \right\|_{\infty,[R_1,R_2]}}{\Gamma(\alpha+1)} (R_2-s)^{\alpha}, \tag{14}$$

$\forall\ s \in [R_1, R_2]$.

Next we observe that

$$\begin{aligned} \left| f(R_2\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| &= \left| g(R_2) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right| \\ &= \left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(R_2) - g(s))\, s^{N-1} ds \right| \\ &\leq \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |g(R_2) - g(s)|\, s^{N-1} ds \\ &\leq \left(\frac{N}{R_2^N - R_1^N} \right) \frac{\left\| D^{\alpha}_{R_2-}g \right\|_{\infty,[R_1,R_2]}}{\Gamma(\alpha+1)} \int_{R_1}^{R_2} (R_2-s)^{\alpha} s^{N-1} ds \\ &=: (*). \end{aligned} \tag{15}$$

We evaluate

$$\begin{aligned} &\int_{R_1}^{R_2} (R_2-s)^{\alpha} s^{N-1} ds \\ &= \int_{R_1}^{R_2} (R_2-s)^{\alpha} ((s-R_1)+R_1)^{N-1} ds \\ &= \int_{R_1}^{R_2} (R_2-s)^{\alpha} \left(\sum_{k=0}^{N-1} \binom{N-1}{k} (s-R_1)^k R_1^{N-1-k} \right) ds \\ &= \sum_{k=0}^{N-1} \binom{N-1}{k} R_1^{N-1-k} \int_{R_1}^{R_2} (R_2-s)^{(\alpha+1)-1} (s-R_1)^{(k+1)-1} ds \\ &= \sum_{k=0}^{N-1} \binom{N-1}{k} R_1^{N-1-k} \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+2)} (R_2-R_1)^{\alpha+k+1} \\ &= \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} R_1^{N-1-k} \frac{\Gamma(\alpha+1)\,k!}{\Gamma(\alpha+k+2)} (R_2-R_1)^{\alpha+k+1}. \end{aligned} \tag{16}$$

Therefore we get

$$\int_{R_1}^{R_2} (R_2 - s)^{\alpha} s^{N-1} ds = (N-1)!\Gamma(\alpha+1) \sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)}. \quad (17)$$

Consequently we find

$$(*) = \left(\frac{N!}{R_2^N - R_1^N} \right) \| D_{R_2-}^{\alpha} g \|_{\infty,[R_1,R_2]} \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)} \right). \quad (18)$$

So we have proved that

$$\begin{aligned} & \left| f(R_2\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| \\ & = \left| g(R_2) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right| \\ & \leq \left(\frac{N!}{R_2^N - R_1^N} \right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)} \right) \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,[R_1,R_2]}. \end{aligned} \quad (19)$$

The last inequality (19) is sharp, that is attained by $AC^m([R_1, R_2]) \ni \overline{g}(r) = (R_2 - r)^{\alpha}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $r \in [R_1, R_2]$. Indeed

$$D_{R_2-}^{\alpha} \overline{g}(r) = \Gamma(\alpha+1), \quad \forall\, r \in [R_1, R_2]$$

and

$$\left\| D_{R_2-}^{\alpha} \overline{g} \right\|_{\infty,[R_1,R_2]} = \Gamma(\alpha+1). \quad (20)$$

Also we have $\overline{g}^{(k)}(R_2) = 0$, $k = 0, 1, ..., m-1$, and $D_{R_2-}^{\alpha} \overline{g} \in L_{\infty}([R_1, R_2])$. So $\overline{g}$ fulfills all the assumptions here.

We observe that

$$\begin{aligned} & L.H.S.(19) \\ & = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} (R_2 - s)^{\alpha} s^{N-1} ds \frac{N!\Gamma(\alpha+1)}{R_2^N - R_1^N} \sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)} \\ & = \left(\frac{N!}{R_2^N - R_1^N} \right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)!\Gamma(\alpha+2+k)} \right) \| D_{R_2-}^{\alpha} \overline{g} \|_{\infty,[R_1,R_2]} \\ & = R.H.S.(19), \end{aligned} \quad (21)$$

proving the optimality of (19).

We have established the Ostrowski inequality

Theorem 2.2. *Let $f : \overline{A} \to \mathbb{R}$ be radial; that is there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\, x \in \overline{A}$; $\omega \in S^{N-1}$. Assume $g \in AC^m([R_1, R_2])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $g^{(k)}(R_2) = 0$, $k = 1, ..., m-1$, and $D_{R_2-}^{\alpha} g \in L_{\infty}([R_1, R_2])$. Then*

$$
\begin{aligned}
& \left| f\left(R_2\omega\right) - \frac{\int_A f\left(y\right)dy}{Vol\left(A\right)} \right| \\
= & \left| g\left(R_2\right) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g\left(s\right) s^{N-1} ds \right| \\
\leq & \left(\frac{N!}{R_2^N - R_1^N}\right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k}\left(R_2 - R_1\right)^{\alpha+1+k}}{\left(N-1-k\right)!\Gamma\left(\alpha+2+k\right)}\right) \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,\left[R_1,R_2\right]}.
\end{aligned} \tag{22}
$$

The last inequality (22) is sharp, that is attained by

$$g\left(s\right) = \left(R_2 - s\right)^{\alpha}, \quad \alpha > 0,\ s \in \left[R_1, R_2\right]. \tag{23}$$

We need

Definition 2.1. Let $F : \overline{A} \to \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$ such that $F\left(\cdot\omega\right) \in AC^m\left(\left[R_1, R_2\right]\right)$, for all $\omega \in S^{N-1}$. We call the Caputo right radial fractional derivative the following function

$$\frac{\partial_{R_2-}^{\alpha} F\left(x\right)}{\partial r^{\alpha}} = \frac{\left(-1\right)^m}{\Gamma\left(m-\alpha\right)} \int_r^{R_2} \left(t-r\right)^{m-\alpha-1} \frac{\partial^m F\left(t\omega\right)}{\partial r^m} dt, \tag{24}$$

where $x \in \overline{A}$; that is, $x = r\omega$, $r \in \left[R_1, R_2\right]$, $\omega \in S^{N-1}$.

Clearly

$$\frac{\partial_{R_2-}^{0} F\left(x\right)}{\partial r^{0}} = F\left(x\right), \tag{25}$$

$$\frac{\partial_{R_2-}^{\alpha} F\left(x\right)}{\partial r^{\alpha}} = \frac{\partial^{\alpha} F\left(x\right)}{\partial r^{\alpha}}, \quad \text{if } \alpha \in \mathbb{N}. \tag{26}$$

The above defined function exists almost everywhere for $x \in \overline{A}$. We justify this next.

Note 2.1. *Call*

$$\Lambda_1 := \left\{ r \in \left[R_1, R_2\right] : \frac{\partial_{R_2-}^{\alpha} F\left(x\right)}{\partial r^{\alpha}} \textit{ does not exist} \right\}.$$

We have that Lebesgue measure $\lambda_{\mathbb{R}}\left(\Lambda_1\right) = 0$. Call $\Lambda_N := \Lambda_1 \times S^{N-1}$. So there exists a Borel set $\Lambda_1^* \subset \left[R_1, R_2\right]$, such that $\Lambda_1 \subset \Lambda_1^*$, $\lambda_{\mathbb{R}}\left(\Lambda_1^*\right) = \lambda_{\mathbb{R}}\left(\Lambda_1\right) = 0$; thus $R_N\left(\Lambda_1^*\right) = 0$, see [2], *pp. 419-422.*

Consider now $\Lambda_N^* := \Lambda_1^* \times S^{N-1} \subset \overline{A}$, which is a Borel set of $\mathbb{R}^N - \{0\}$. Clearly then by Theorem 16.59, p. 420, [2], *$\lambda_{\mathbb{R}^N}\left(\Lambda_N^*\right) = 0$, but $\Lambda_N \subset \Lambda_N^*$, implying $\lambda_{\mathbb{R}^N}\left(\Lambda_N\right) = 0$. Consequently the above radial derivative exists a.e. in x w.r.t. $\lambda_{\mathbb{R}^N}$ on $\overline{A}$.*

We make

Remark 2.3. We treat here the general, not necessarily radial, case of f. We apply last Theorem 2.2 to $f(r\omega)$, ω is fixed, $r \in [R_1, R_2]$, under the following assumptions: $f(\cdot\omega) \in AC^m([R_1, R_2])$, for all $\omega \in S^{N-1}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, where $f : \overline{A} \to \mathbb{R}$ is Lebesgue integrable; $\frac{\partial^k f}{\partial r^k}$, $k = 1, ..., m-1$ vanish on $\partial B(0, R_2)$, and $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \in B(\overline{A})$, along with $D^\alpha_{R_2-} f(\cdot\omega) \in L_\infty([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$.

So we have

$$\left| f(R_2\omega) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right|$$
$$\leq \left(\frac{N!}{R_2^N - R_1^N} \right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)! \Gamma(\alpha+2+k)} \right) \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A}} \tag{27}$$
$$=: \lambda_1.$$

Consequently it holds

$$\left| \frac{\int_{S^{N-1}} f(R_2\omega) d\omega}{\omega_N} - \frac{N}{(R_2^N - R_1^N)\omega_N} \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right) d\omega \right| \leq \lambda_1. \tag{28}$$

That is

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega) d\omega - \frac{\int_A f(x) dx}{Vol(A)} \right| \leq \lambda_1. \tag{29}$$

Therefore, it holds for $x \in \overline{A}$, that

$$\left| f(x) - \frac{\int_A f(x) dx}{Vol(A)} \right|$$
$$= \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega) d\omega + \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega) d\omega - \frac{\int_A f(x) dx}{Vol(A)} \right| \tag{30}$$
$$\leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega) d\omega \right| + \lambda_1.$$

We have proved

Theorem 2.3. *Let $f : \overline{A} \to \mathbb{R}$ be Lebesgue integrable with $f(\cdot\omega) \in AC^m([R_1, R_2])$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $\forall\ \omega \in S^{N-1}$; $\frac{\partial^k f}{\partial r^k}$, $k = 1, ..., m-1$ vanish on $\partial B(0, R_2)$; $\partial^\alpha_{R_2-} f(\cdot\omega) \in L_\infty([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$; and $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \in B(\overline{A})$ (bounded functions on $\overline{A}$). Then, for $x \in \overline{A}$, we have*

$$\left| f(x) - \frac{\int_A f(x) dx}{Vol(A)} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R_2\omega) d\omega \right|$$
$$+ \left(\frac{N!}{R_2^N - R_1^N} \right) \left(\sum_{k=0}^{N-1} \frac{R_1^{N-1-k} (R_2 - R_1)^{\alpha+1+k}}{(N-1-k)! \Gamma(\alpha+2+k)} \right) \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A}}. \tag{31}$$

We also make

Remark 2.4. Let $f : \overline{B(0,R)} \to \mathbb{R}$ be a Lebesgue integrable function, that is not necessarily a radial function. Assume $f(\cdot\omega) \in AC^1([0,R])$, $\forall\ \omega \in S^{N-1}$; $0<\alpha<1$, and $D^{\alpha}_{R-}f(\cdot\omega) \in L_{\infty}([0,R])$, $\forall\ \omega \in S^{N-1}$. Clearly here we obtain

$$f(s\omega) - f(R\omega) = \frac{1}{\Gamma(\alpha)} \int_s^R (J-s)^{\alpha-1} D^{\alpha}_{R-} f(J\omega)\, dJ, \tag{32}$$

$\forall\ \omega \in S^{N-1}$, $\forall\ s \in [0,R]$.

We further assume that

$$\left\| D^{\alpha}_{R-} f(J\omega) \right\|_{\infty,(J\in[0,R])} \le K, \quad \forall\ \omega \in S^{N-1},$$

where $K>0$.

Applying the earlier Theorem 2.1 we get

$$\begin{aligned} \left| f(R\omega) - \frac{N}{R^N} \int_0^R f(s\omega)\, s^{N-1} ds \right| &\le \left(\left\| D^{\alpha}_{R-} f(t\omega) \right\|_{\infty,(t\in[0,R])} \right) \frac{N! R^{\alpha}}{\Gamma(\alpha+N+1)} \\ &\le \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \end{aligned} \tag{33}$$

Consequently we get

$$\left| \frac{\int_{S^{N-1}} f(R\omega)\, d\omega}{\omega_N} - \frac{N}{R^N \omega_N} \int_{S^{N-1}} \left(\int_0^R f(s\omega)\, s^{N-1} ds \right) d\omega \right| \le \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{34}$$

Hence

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega)\, d\omega - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))} \right| \le \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{35}$$

Consequently it holds

$$\begin{aligned} &\left| f(R\omega) - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))} \right| \\ &= \left| f(R\omega) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega)\, d\omega + \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega)\, d\omega - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))} \right| \\ &\le \left| f(R\omega) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega)\, d\omega \right| + \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \end{aligned} \tag{36}$$

So we have proved the Ostrowski inequality

Theorem 2.4. *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be a Lebesgue integrable function, not necessarily radial. Assume $f(\cdot\omega) \in AC^1([0,R])$, $R>0$, $\forall\ \omega \in S^{N-1}$; $0<\alpha<1$, and $D^{\alpha}_{R-} f(\cdot\omega) \in L_{\infty}([0,R])$, $\forall\ \omega \in S^{N-1}$.*
Suppose also that $\left\| D^{\alpha}_{R-} f(t\omega) \right\|_{\infty,(t\in[0,R])} \le K$, $\forall\ \omega \in S^{N-1}$, where $K>0$. Then

$$\left| f(R\omega) - \frac{\int_{B(0,R)} f(x)\, dx}{Vol(B(0,R))} \right| \le \left| f(R\omega) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(R\omega)\, d\omega \right| + \frac{K N! R^{\alpha}}{\Gamma(\alpha+N+1)}. \tag{37}$$

References

1. G.A. Anastassiou, *Ostrowski type inequalities*, Proc. AMS 123 (1995), 3775-3781.
2. G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.
3. G.A. Anastassiou, *On Right Fractional Calculus*, Chaos, Solitons and Fractals, 42 (2009), 365-376.
4. G.A. Anastassiou, *Univariate right fractional Ostrowski inequalities*, submitted, 2010.
5. W. Rudin, *Real and Complex Analysis*, International Student edition, Mc Graw Hill, London, New York, 1970.
6. D. Stroock, *A Concise Introduction to the Theory of Integration*, Third Edition, Birkhaüser, Boston, Basel, Berlin, 1999.
7. A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funcktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

George A. Anastassiou is a Professor of Mathematics at the University of Memphis and he received his PhD from University of Rochester in 1984. His research interests lie in the broad areas of Approximation Theory, Inequalities and related fields. He has published numerous of articles and several books. He is associate editor in many mathematical journals and editor in chief in three journals most notably in the Journal of Computational Analysis and Applications. He has been awarded best young (1990) and senior (1999, 2008) researcher awards from the University of Memphis and best Greek Analyst from Academy of Athens, Greece (2001). Also he received a Honorary Doctoral Degree from University of Oradea, Romania (2007).

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.
`e-mail: ganastss@memphis.edu`