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COHERENT AND CONVEX HEDGING ON ORLICZ HEARTS IN INCOMPLETE MARKETS†

JU HONG KIM

Abstract. Every contingent claim is unable to be replicated in the incomplete markets. Shortfall risk is considered with some risk exposure. We show how the dynamic optimization problem with the capital constraint can be reduced to the problem to find an optimal modified claim $\tilde{\psi}H$ where $\tilde{\psi}$ is a randomized test in the static problem. Convex and coherent risk measures defined in the Orlicz hearts spaces, M^{Φ}, are used as risk measure. It can be shown that we have the same results as in [21, 22] even though convex and coherent risk measures defined in the Orlicz hearts spaces, M^{Φ}, are used. In this paper, we use Fenchel duality Theorem in the literature to deduce necessary and sufficient optimality conditions for the static optimization problem using convex duality methods.

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1. Introduction

It is not possible to replicate every contingent claim in incomplete markets, in which the equivalent martingale measures are not unique. There is a dynamic self-financing hedging strategy with arbitrage-free hedging price to super-replicate a contingent claim in complete or incomplete markets. The super-hedging price is the minimal initial capital that an agent or an investor has to invest to find a strategy which dominates the claim payoff with certainty [17]. The super-hedging price of a contingent claim is given by the supremum of the expected values over all equivalent martingale measures.

With the super-hedging price, an agent or an investor could eliminate the shortfall risk completely by choosing a suitable hedging strategy. The prices derived by super-replication are too high and not acceptable in practice. An agent or an investor, who sell a contingent claim, want to get rid of the associated

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shortfall risk by means of a dynamic hedging strategy. The shortfall risk is the difference between the payoff of the contingent claim and the value of the agent's or the investor's hedging strategy at maturity.

With the initial capital less than the super-hedging price, i.e., under the capital constraint an agent or an investor is unable to eliminate all exposed risk associated to the contingent claim completely and so wants to find optimal strategies which minimize the shortfall risk. They are seeking optimal partial hedging strategies with the initial capital less than the super-hedging price by taking some risks.

Usually the dynamic optimization problem which minimizes the shortfall risk can be split into a static optimization problem. The first one is to find an optimal modified claim $\tilde{\psi}H$ where $\tilde{\psi}$ is the solution of the static problem and then to find a superhedging strategy for the modified claim $\tilde{\psi}H$.

Föllmer and Leukert [13] constructed a quantile hedging strategy which maximizes the probability of a successful hedge under the objective measure P under the capital constraint. In the quantile hedging approach, the size of the shortfall is not taken into account but only the probability of its occurrence. Föllmer and Leukert [14] also introduced optimal hedging strategies which minimize the shortfall risk under the capital constraint by using the expected loss functions as risk measures. In [14], the risk measure ρ is the form of $\rho(X) = E^P[\ell(X^+)]$, where X is a random variable on $(\Omega, \mathcal{F})$, P is a fixed probability measure on Ω, and $\ell : \mathbb{R} \to \mathbb{R}$ is a strictly convex function. Nakano [19] uses coherent risk measures as risk measures in the $L^1(\Omega, \mathcal{F}, \mathbb{P})$ random variable spaces instead of the loss function. Notice that the L^1 space is between $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $L^0(\Omega, \mathcal{F}, \mathbb{P})$. Arai [1] obtained robust representation results of shortfall risk measures on Orlicz hearts under the continuous time setting. The Orlicz hearts setting allows us to treat various loss functions and various claims in a unified framework. Coherent risk measure is introduced by Artzner et al. [2, 3] as risk measures, and is extended to general probability spaces by Delbaen [9].

Rudloff [21, 22] uses Fenchel duality to show the existence of the static solution with coherent and convex risk measures defined on L^1. He finds a necessary and sufficient condition for an optimal randomized test ψ, and its randomized test has the typical 0-1-structure. Coherent and convex risk measures using as risk measures are defined L^∞ or L^1 [15], and is represented as a dual form which is the supremum of expectation of acceptable position with respect to different probability spaces.

Cheridito and Li [8] studied risk measures on bigger sets than L^∞, Orlicz hearts. They proved that coherent and convex risk measures on an Orlicz hearts, which is real-valued on a set with non-empty algebraic interior, is real-valued on the whole space and has a robust representation as maximal penalized expectation with respect to different probability spaces. This includes coherent and convex risk measures on L^p-spaces for $1 \leq p < \infty$ and covers a wide range of interesting examples. Also Orlicz hearts have nice dual properties.

In this paper, we use coherent and convex risk measures defined Orlicz hearts in the static optimization problem and get the same results as in Rudloff [21, 22]. We provide modified proofs suitable for Orlicz spaces.

This paper is constructed as follows. Mathematical settings are given in Section 2. The definition and several properties of Orlicz hearts are given in Section 3. The primal and dual problem of the static optimization problem is derived and strong duality holds in Section 4.

2. Mathematical settings

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space. An $\mathbb{R}^n$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ is called a sigmamartingale if there exists an $\mathbb{R}$-valued martingale M and an M-integrable predictable $\mathbb{R}^+$-valued process ξ such that $S_t = \int_0^t \xi_u dM_u$, $t \in [0, T]$. It is assumed that the riskless interest rate is zero for simplicity.

Let $\mathcal{P}_\sigma$ denote the set of probability measures Q equivalent to P such that S is a sigmamartingale with respect to Q. Assume that $\mathcal{P}_\sigma \neq \emptyset$ to exclude arbitrage opportunities. $\mathcal{P}_\sigma \neq \emptyset$ holds if and only if S satisfies the condition of 'no free lunch with vanishing risk'. If the probability space Ω is finite, then the condition 'no free lunch with vanishing risk' can be replaced by the more strong condition 'no arbitrage', and the set $\mathcal{P}_\sigma$ by the set of equivalent martingale measures $\mathcal{M}$ [10, 11].

Definition 2.1. A *self-financing* strategy with initial capital $x \geq 0$ is defined as a predictable process (x, ξ_t) such that the value process (value of the current holdings)

$$X_t = x + \int_0^t \xi_u dS_u, \quad t \in [0, T]$$

is P-a.s. well-defined.

Definition 2.2. A self-financing strategy (x, ξ_t) is called *admissible* if there exists some constant $c > 0$ such that

$$\forall t \in [0, T] \quad x + \int_0^t \xi_u dS_u \geq -c \quad P - a.s..$$

Consider a contingent claim whose payoff is given by a $\mathcal{F}_T$-measurable, non-negative random variable H. The superhedging price U_0 is given by the dual characterization

$$U_0 = \sup_{Q\in\mathcal{P}_\sigma} E^Q[H] < +\infty.$$

With the superhedging price U_0, a contingent claim H can be perfectly hedged but the superhedging strategy is too expensive to be used in practice. When the agent or the investor is unwilling to invest the superhedge price in a hedging strategy, they seek the best strategy the agent or an investor can achieve with a smaller amount $x_0 < U_0$. They look for an admissible strategy $(\tilde{x}, \xi_t)$ with initial endowment $\tilde{x} \le x_0$ that minimizes the coherent or convex shortfall risk

$$\rho(-(H - X_T)^+). \tag{2.1}$$

To construct the optimal hedging strategy, firstly we solve the static problem of minimizing

$$\rho(-(H - Y)^+)$$

among all $\mathcal{F}_T$-measurable random variables $Y \ge 0$ satisfying the constraints

$$\sup_{Q \in \mathcal{P}_\sigma} E^Q[Y] \le \tilde{x}.$$

If Y^* is a solution in the static problem then so is $\tilde{Y} := H \wedge Y^*$, since $0 \le \tilde{Y} \le H$, $E^Q[\tilde{Y}] \le \tilde{x}$ and $(H - \tilde{Y})^+ = (H - H \wedge Y^*)^+ = (H - Y^*)^+$. So we may assume that $0 \le Y^* \le H$, or equivalently, that $Y^* = H\psi^*$ for $\psi^* \in R_0$,
where R_0 is defined as

$$R_0 \quad := \quad \Big\{ \psi \in R \;\Big|\; \psi : \Omega \to [0,1], \psi \text{ is } \mathcal{F}_T - measurable, \sup_{Q \in \mathcal{M}} E^Q[\psi H] \le \tilde{x} \Big\}.$$

The dynamic optimal problem (2.1) with $\tilde{x} \le x_0$ can be expressed as the static problem

$$\min_{\psi \in R_0} \rho((\psi - 1)H). \tag{2.2}$$

The first one is to find an optimal modified claim $\tilde{\psi}H$ where $\tilde{\psi}$ is the solution of the static problem

$$\min_{\psi \in R_0} \rho((\psi - 1)H) = \rho((\tilde{\psi} - 1)H). \tag{2.3}$$

The second one is to find a superhedging strategy for the modified claim $\tilde{\psi}H$.

Theorem 2.1 ([19]). *Let $\tilde{\psi}$ be a solution to the minimization problem (2.3) and let $(\tilde{x}_0, \tilde{\xi})$ be the admissible strategy determined by the optional decomposition of the claim $\tilde{\psi}H$. Then the strategy $(\tilde{x}_0, \tilde{\xi})$ is the optimal solution of the problem (2.1) with the constraint $\tilde{x} \le x_0$.*

3. Robust representation of risk measure ρ on Orlicz hearts

We consider the shortfall risk assuming that the risk measure ρ is $(-\infty, \infty]$-valued coherent or convex risk measures on maximal subspaces of Orlicz classes. L^0 denotes the space of all $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{F})$.

Definition 3.1. Let $\mathcal{X}$ be a linear subspace of L^0 that contains all constants. A monetary risk measure $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying for all X, $Y \in \mathcal{X}$

(1) $\rho(0) \in \mathbb{R}$ (Finiteness),
(2) $\rho(X) \ge \rho(Y)$ if $X \le Y$ (Monotonicity) ,
(3) $\rho(X + m) = \rho(X) - m$ for all $m \in \mathbb{R}$ (Cash invariance).

A monetary risk measure is called convex if it satisfies

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y) \text{ for all } \lambda \in [0,1] \text{ (Convexity).}$$

A convex monetary risk measure is called coherent if it satisfies

$$\rho(\lambda X) = \lambda\rho(X) \text{ for all } \lambda \ge 0 \text{ (Positive homogeneity).}$$

When $\rho(X)$ is positive, the number $\rho(X)$ can be thought of as the minimum extra cash the agent has to add to the risky position X, and invest in the reference instrument, to be allowed to proceed with his/her plans. When $\rho(X)$ is negative, the amount of cash $-\rho(X)$ can be withdrawn from the position or it can be received as restitution, as in the case of organized markets for financial futures [2].

Under the assumption of the positive homogeneity, the convexity is equivalent to

$$\rho(X+Y) \le \rho(X) + \rho(Y) \text{ (Subadditivity),}$$

which means the downside risk of a position is reduced if the payoff profile is increased.

Definition 3.2. Let $\mathcal{X}$ be a linear subspace of L^0 that contains all constants. The acceptance set of a monetary risk measure $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is given by

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \le 0\}.$$

Let $\mathcal{X}$ be a Banach lattices and $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$. We denote

$$\text{dom } f := \{x \in \mathcal{X} : f(x) \in \mathbb{R}\}.$$

A subset U of $\mathcal{X}$ is an algebraic neighborhood of $x \in \mathcal{X}$ if for every $y \in \mathcal{X}$, there exists an $\epsilon > 0$ such that

$$x + ty \in U \text{ for all } 0 \le t \le \epsilon.$$

The algebraic interior of a subset A of $\mathcal{X}$, denoted by core(A), consists of all $x \in A$ that have an algebraic neighborhood in A.

A convex function f from a topological vector space $\mathcal{X}$ to $\mathbb{R} \cup \{\pm\infty\}$ is said to be proper if $f(x) < \infty$ for at least one $x \in \mathcal{X}$ and $f(x) > -\infty$ for all $x \in \mathcal{X}$. We call it subdifferentiable at $x \in \mathcal{X}$ if $x \in \mathbb{R}$ and there exists an element x^* in the topological dual $\mathcal{X}^*$ of $\mathcal{X}$ such that $x^*(y) \le f(x+y) - f(x)$ for all $y \in \mathcal{X}$. For every proper convex function f, the conjugate

$$f^*(x^*) := \sup_{x \in \mathcal{X}} \{x^*(x) - f(x)\}$$

is a $\sigma(\mathcal{X}^*, \mathcal{X})$-lower semicontinuous, convex function from $\mathcal{X}^*$ to $\mathbb{R} \cup \{+\infty\}$. If f is subdifferentiable at $x \in \mathcal{X}$, then

$$f(x) = \max_{x^* \in \mathcal{X}^*} \{x^*(x) - f^*(x^*)\}.$$

We call a function $\Phi : [0,\infty) \to [0,\infty)$ a Young function if it is left-continuous, convex, $\lim_{x \downarrow 0} \Phi(x) = \Phi(0) = 0$, and $\lim_{x \to \infty} \Phi(x) = \infty$. It follows from these properties that Φ is increasing and continuous except possibly at a single point, where it jumps to ∞. So the condition of left-continuity is needed at that one point. The conjugate(or polar) function Ψ of Φ is defined as

$$\Psi(y) := \sup_{x\geq 0}\{xy - \Phi(x)\}, \quad y \geq 0.$$

The function Ψ is a Young function and its conjugate function is Φ. The Orlicz hearts corresponding to Φ defined as

$$M^{\Phi} := \left\{X \in L^0 \,:\, E^P[\Phi(c|X|)] < \infty \text{ for all } c > 0\right\}.$$

The Orlicz space for Φ is defined as

$$L^{\Phi} := \left\{X \in L^0 \,:\, E^P[\Phi(c|X|)] < \infty \text{ for some } c > 0\right\}.$$

The Luxemburg norm and the Orlicz norm are respectively defined as

$$\|X\|_{\Phi} := \inf\{c > 0 : E[(|X/c|)] \leq 1\}$$

and

$$\|X\|^*_{\Psi} := \sup\{E^P[XY] : \|Y\|_{\Psi} \leq 1\}.$$

The above two norms are equivalent on L^{Φ}.

If Φ jumps to ∞, then M^{Φ} is equal to the trivial space $\{0\}$. In this paper, it assumed that Φ is real-valued. So Φ is continuous function.

Theorem 3.1. *Suppose that Φ is finite. Then M^{Φ} is the $\|\cdot\|_{\Phi}$-closure of L^{∞} and the Banach dual of M^{Φ} with the Luxemburg norm is L^{Ψ} with the Orlicz norm, i.e. $(M^{\Phi})^* = L^{\Psi}$.*

The following examples show the several Orlicz spaces and the relations among M^{Φ}, L^{Ψ}, L^{∞}, L^p and L^1 depending on the choice of the Young function Φ.

Example 3.2 ([8])**.** If $\Phi(x) = x$, then we have

$$\Psi(y) = \begin{cases} 0, & \text{for } y \leq 1 \\ \infty, & \text{for } y > 1, \end{cases}$$

and

$$M^{\Phi} = L^{\Phi} = L^1, \quad \|\cdot\|_{\Phi} = \|\cdot\|_1, \quad L^{\Psi} = L^{\infty}, \quad \|\cdot\|^*_{\Phi} = \|\cdot\|_{\infty}.$$

If $\Phi(x) = x^p$ for $p \in (1, \infty)$, then $\Psi(y) = p^{1-q}q^{-1}y^q$, and we have

$$M^{\Phi} = L^{\Phi} = L^p, \quad \|\cdot\|_{\Phi} = \|\cdot\|_p, \quad L^{\Psi} = L^q, \quad \|\cdot\|^*_{\Phi} = \|\cdot\|_q.$$

If $\Phi(x) = exp(\lambda x) - 1$ for $\lambda > 0$, then we have

$$\Psi(y) = \begin{cases} 0, & \text{for } y \leq \lambda \\ (y/\lambda)\ln(y/\lambda) - y/\lambda + 1, & \text{for } y > \lambda, \end{cases}$$

and we have

$$L^{\infty} \subset M^{\Phi} \subset L^p \subset L^{\Psi} \subset L^1 \quad \text{for all } p \in (1, \infty).$$

We identify a probability measure Q on $(\Omega, \mathcal{F})$ that is absolutely continuous with respect to P with the Radon-Nikodym derivative $\varphi = dQ/dP \in L^1$. The set

$$\mathcal{D} := \{\varphi \in L^1 \,:\, \varphi \geq 0, E^P[\varphi] = 1\}$$

represents all probability measures on $(\Omega, \mathcal{F})$ that is absolutely continuous with respect to P. Let $\mathcal{D}^\Psi$ be denoted by the intersection

$$\mathcal{D}^\Psi = \mathcal{D} \cap L^\Psi.$$

A mapping $\gamma \,:\, \mathcal{D}^\Psi \to (-\infty, \infty]$ is called a penalty function on $\mathcal{D}^\Psi$ if it is bounded from below and $\gamma \not\equiv \infty$.

Definition 3.3. It is called that γ satisfies the growth condition (G) if there exist constants $a \in \mathbb{R}$ and $b > 0$ such that

$$\gamma(Q) \geq a + b\|Q\|_\Phi^* \quad \text{ for all } Q \in \mathcal{D}^\Psi.$$

For a penalty function γ on $\mathcal{D}^\Psi$, define ρ_γ as

$$\rho_\gamma(X) := \sup_{Q \in \mathcal{D}^\Psi} \{E^Q[-X] - \gamma(Q)\} \quad X \in M^\Phi,$$

which is called a robust representation of ρ_γ.

ρ_γ defines a lower semicontinuous convex risk measure on M^Φ with values in $(-\infty, \infty)$. The bilinear form $< Y, X >$ between L^Ψ and M^Φ is defined as $< Y, X > := E[XY]$ for $Y \in L^\Psi$ and $X \in M^\Phi$.

Theorem 3.3 ([8])**.** *Let $\rho \,:\, M^\Phi \to (-\infty, \infty]$ be a convex monetary risk measure with core(dom ρ) $\neq \emptyset$. Then ρ is real-valued, $\rho^\#$ is a penalty function on $\mathcal{D}^\Psi$ satisfying the growth condition (G), and*

$$\rho(X) := \max_{Q \in \mathcal{D}^\Psi} \{E^Q[-X] - \rho^\#(Q)\} \quad X \in M^\Phi. \tag{3.4}$$

Moreover, if $\rho = \rho_\gamma$ for a penalty function γ on $\mathcal{D}^\Psi$, then $\rho^\#$ is the greatest convex, $\sigma(L^\Psi, M^\Phi)$-lower semicontinuous minorant of γ, where $\rho^\#$ is given by

$$\rho^\#(Q) := \sup_{X \in M^\Phi} \{E^Q[-X] - \rho(X)\} \quad Q \in \mathcal{D}^\Psi.$$

Notice that $\rho^\#(Q) = f^*(Q)$ for all $Q \in \mathcal{D}^\Psi$, where f is defined as $f(X) = \rho(-X)$. By the definition of convex duality,

$$f^*(Y) = \sup_{X \in M^\Phi} \{Y(X) - f(X)\} = \sup_{X \in M^\Phi} \{E[YX] - \rho(-X)\} \text{ for } Y \in \mathcal{D}^\Psi. \tag{3.5}$$

For $X \in \mathcal{A}_\rho$, we have $\rho(-X) \geq 0$ since $0 = \rho(X - X) \leq \rho(X) + \rho(-X)$ implies $-\rho(-X) \leq \rho(X) \leq 0$. Thus the equation (3.5) becomes

$$\rho^*(-Q) = \rho^\#(Q) = \sup_{X \in \mathcal{A}_\rho} E^Q[X] \quad \text{ for } Q \in \mathcal{D}^\Psi. \tag{3.6}$$

Theorem 3.4 ([8])**.** *Let $\rho : M^\Phi \to \mathbb{R} \cup \{+\infty\}$ be a coherent risk measure with acceptance set*

$$\mathcal{A}_\rho := \{X \in M^\Phi \,:\, \rho(X) \le 0\}.$$

If core(dom ρ) $\neq \emptyset$, then ρ is real-valued and can be represented as

$$\rho(X) = \max_{Q \in \mathcal{Q}} E^Q[-X], \quad X \in M^\Phi, \tag{3.7}$$

for the $\|\cdot\|_\phi^*$-bounded, convex set

$$\mathcal{Q} := \{Q \in \mathcal{D}^\Psi \,:\, E^Q[X] \ge 0 \text{ for all } X \in \mathcal{A}_\rho\}.$$

Moreover, if $\mathcal{R}$ is a subset of $\mathcal{D}^\Psi$ such that $\rho = \rho_\mathcal{R}$, then $\mathcal{Q}$ is the $\sigma(L^\Psi, M^\Phi)$-closed, convex hull of $\mathcal{R}$.

We designate by V and V^* two vector spaces placed in duality by a bilinear pairing denoted by $<,>$.

Theorem 3.5 ([20])**.** *Suppose X and Y are normed spaces. To each $A : X \to Y$ corresponds a unique $A^* : Y^* \to X^*$ that satisfies*

$$< Ax, y^* > = < x, A^* y^* >$$

for all $x \in X$ and all $y^* \in Y^*$.

Suppose that ρ is a coherent risk measure on M^Φ. Then $E^Q[-X] \le \rho(X)$ holds for all $X \in M^\Phi$ and $Q \in \mathcal{D}^\Psi$. If $X \in \mathcal{A}_\rho$, then $E^Q[X] \ge 0$ since $\rho(X) \le 0$. Conversely, suppose that $E^Q[X] \ge 0$ holds for all $X \in \mathcal{A}_\rho$. Since $\rho(X + \rho(X)) = 0$, $X + \rho(X)$ belongs to $\mathcal{A}_\rho$ and so $E^Q[X + \rho(X)] \ge 0$ by assumption, i.e. $E^Q[-X] \le \rho(X)$ holds for all $X \in \mathcal{A}_\rho$. Thus we obtain the following equivalent relations

$$\begin{aligned} \mathcal{Q} &:= \{Q \in \mathcal{D}^\Psi \,:\, E^Q[-X] \le \rho(X) \quad \forall X \in \mathcal{A}_\rho\} \\ &= \{\varphi \in \mathcal{D}^\Psi \,:\, E[\varphi X] \ge 0 \quad \forall X \in \mathcal{A}_\rho\}. \end{aligned} \tag{3.8}$$

Theorem 3.6 ([7])**.** *(Fenchel duality) Let X be a Banach space and let Y be a barrelled locally convex topological vector space. Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be a lower semicontinuous convex functions and let $A : X \to Y$ be a closed densely defined linear map. Suppose that f and g satisfy the condition*

$$0 \in core(dom\, g - A\, dom\, f).$$

If p is defined as

$$p = \inf_{x \in X} \{f(x) + g(Ax)\}, \tag{3.9}$$

then the dual expression of p is given by

$$d = \sup_{\phi \in Y^*} \{-f^*(A^*\phi) - g^*(-\phi)\}. \tag{3.10}$$

Moreover, $p = d$ and the supremum in the dual problem (3.10) is attained whenever finite.

Let V be a real vector space. We denote the indicator functional of a convex set $\mathcal{A} \subset V$ with

$$\chi_{\mathcal{A}}(\phi) := \begin{cases} 0, & \phi \in \mathcal{A} \\ +\infty, & \phi \notin \mathcal{A}, \end{cases}$$

and the convex conjugate of the indicator functional $\chi_{\mathcal{A}}(\cdot)$ with $\chi^*_{\mathcal{A}} : V^* \to \mathbb{R} \cup \{+\infty\}$,

$$\chi^*_{\mathcal{A}}(u^*) = \sup_{u \in V}\{< u, u^* > -\chi_{\mathcal{A}}(u)\} = \sup_{u \in \mathcal{A}} < u, u^* >, \quad u^* \in V^*.$$

The indicator functional $\chi_{\mathcal{A}}$ is convex lower semicontinuous function if $\mathcal{A}$ is convex set, termed the support function of $\mathcal{A}$, and $\chi^{**}_{\mathcal{A}} = \chi_{\overline{co}\mathcal{A}}$, where $\overline{co}\mathcal{A}$ denotes the closure of the convex combination of $\mathcal{A}$.

4. The Primal and Dual Problem in the Static Problem

Assume that the contingent claim H belongs to M^Φ. Let the static problem (2.2) be the primal problem with value

$$\begin{aligned} p \quad &:= \quad \min_{\psi \in R_0} \rho((1-\psi)H) & (4.11)\\ &= \quad \min_{\psi \in L^\infty} \{\rho((1-\psi)H) + \chi_{R_0}(\psi)\}, & (4.12) \end{aligned}$$

where $\chi_{R_0}(\psi)$ is the indicator function.

Lemma 4.1. *There exists a $\psi \in R_0$ satisfying the static problem (2.2).*

Proof. We reproduce the proof in [22]. The set of $R = \{\psi : \Omega \to [0,1], \mathcal{F}_T-$ measurable$\}$ is weak* compact as a weak* closed subset of the weak* compact unit ball in L^∞. Since the map $\psi \to \sup_{Q^* \in \mathcal{P}_\sigma} E^{Q^*}[\psi H]$ is lower semicontinuous in the weak* topology, the set R_0 is weak* closed, and hence weak* compact. Since $\psi \to \sup_{Q^* \in \mathcal{P}_\sigma}\{E^{Q^*}[\psi H] - \rho^{\#}(Q^*)$ is lower semicontinuous in the weak* topology, there exists a $\tilde{\psi} \in R_0$ solving (2.2). □

If a convex risk measure is expressed as (3.4), then the expression of the primal problem can be written as

$$p = \min_{\psi \in R_0}\Big\{\sup_{Q \in \mathcal{D}^\Psi}\Big\{E^Q[(1-\psi)H] - \rho^{\#}(Q)\Big\}\Big\}. \qquad (4.13)$$

If a coherent risk measure is expressed as (3.7), then the expression of the primal problem can be written as

$$p = \min_{\psi \in R_0}\Big\{\sup_{Q \in \mathcal{Q}} E^Q[(1-\psi)H]\Big\}. \qquad (4.14)$$

Theorem 4.2. *Let $\rho : M^\Phi \to (-\infty, \infty]$ be a coherent or convex monetary risk measure with core(dom ρ) $\neq \emptyset$. Then the dual problem of the primal problem (4.13) or (4.14) is given by*

$$d \quad = \quad \sup_{Q \in \mathcal{D}^\Psi} \inf_{\psi \in R_0} \{E^Q[(1-\psi)H] - \rho^{\#}(Q)\}, \ \textit{or} \qquad (4.15)$$

$$d \quad = \quad \sup_{Q\in\mathcal{Q}} \inf_{\psi\in R_0} \{E^Q[(1-\psi)H]\}, \tag{4.16}$$

respectively. Also the strong duality holds, i.e. $p = d$ in each case of convex or coherent risk measures. If $\tilde{\psi}$ is the solution of (4.13) or (4.14), and $\tilde{\varphi}_Q = d\tilde{Q}/dP$ is the solution of (4.15) or (4.16), then $(\tilde{\varphi}_Q, \tilde{\psi})$ is a saddle point of the function $E^Q[(1-\psi)H] - \rho^{\#}(Q)$ or $E^Q[(1-\psi)H]$, respectively. Consequently we get

$$\min_{\psi\in R_0} \max_{Q\in\mathcal{D}^\Psi} \{E^Q[(1-\psi)H] - \rho^{\#}(Q)\} = \max_{Q\in\mathcal{D}^\Psi} \min_{\psi\in R_0} \{E^Q[(1-\psi)H] - \rho^{\#}(Q)\}, \tag{4.17}$$

$$\min_{\psi\in R_0} \max_{Q\in\mathcal{Q}} E^Q[(1-\psi)H] = \max_{Q\in\mathcal{Q}} \min_{\psi\in R_0} E^Q[(1-\psi)H]. \tag{4.18}$$

Proof. We follow the proof in [21, 22] with keeping in mind we are considering the risk measures ρ defined on the Orlicz hearts, M^Φ. Consider the primal problem (4.12)

$$p = \min_{\psi\in L^\infty} \{\rho((\psi-1)H) + \chi_{R_0}(\psi)\}.$$

To apply Fenchel duality Theorem (3.6), define A, f and g as

$$\begin{aligned} &A : L^\infty \to M^\Phi \text{ by } A\psi := H\psi, \\ &f : L^\infty \to \mathbb{R}\cup\{+\infty\} \text{ by } f(\psi) := \chi_{R_0}(\psi), \\ &g : M^\Phi \to \mathbb{R}\cup\{+\infty\} \text{ by } g(X) := \rho(X-H), \end{aligned}$$

respectively.

Notice that L^Φ is a Banach space with the Luxemberg (or gauge) norm

$$\|X\|_\Phi := \inf\Big\{\lambda > 0 \,:\, E\Big[\Phi\Big(\Big|\frac{X}{\lambda}\Big|\Big)\Big] \le 1\Big\},$$

and that $L^\infty \subset L^\Phi \subset L^1(P)$ since Φ is finite, regular on its proper domain and convex. M^Φ, which is a linear subspace of L^Φ, is always closed and hence it is a Banach space with the Luxemberg norm. M^Φ is the $\|\cdot\|_\Phi$-closure of L^∞ in L^Φ.

Clearly A is linear. Suppose that ψ_n converges to ψ in $\|\cdot\|_\infty$. Then from the inequality

$$E[|H(\psi_n-\psi)|] \le \|\psi_n-\psi\|_\infty E[H], \quad \sup_{Q\in\mathcal{P}_\sigma} E^Q[H] < \infty,$$

we have $H(\psi_n-\psi)$ converges to 0 P-a.s.. Since Φ is continuous, for $\lambda > 0$

$$\Phi\Big(\Big|\frac{H(\psi_n-\psi)}{\lambda}\Big|\Big) \to \Phi(0) = 0 \quad P-a.s. \text{ as } n\to\infty.$$

Hence $\|H(\psi_n-\psi)\|_\Phi \to 0$ P-a.s. as $n\to\infty$. Thus $A : L^\infty \to M^\Phi$ is a bounded linear operator, and by Theorem (3.5) we have

$$\begin{aligned} &\forall\phi\in L^\infty\ \forall Y^*\in(M^\Phi)^* \quad < A\phi, Y^* > = < \phi, A^*Y^* >, i.e. \\ &\forall\phi\in L^\infty\ \forall Y^*\in(M^\Phi)^* \quad \int_\Omega H\phi Y^* dP = \int_\Omega \phi A^*Y^* dP. \end{aligned} \tag{4.19}$$

If there exists $\Omega_1 \subset \Omega$ such that $P(\Omega_1) > 0$ and $A^*Y^* < HY$ on Ω_1, then $\int_\Omega HY^*dP > \int_\Omega A^*Y^*dP$ for the function ϕ defined as $\phi(\omega) = 1$ on Ω_1 and 0 otherwise. This is a contradiction to (4.19). So there is no such Ω_1. Similarly we can show that there exists no $\Omega_2 \subset \Omega$ such that $P(\Omega_2) > 0$ and $A^*Y^* > HY^*$ on Ω_2. Therefore $A^*Y^* = HY^* = AY^*\ \forall Y^* \in (M^\Phi)^* = L^\Psi$, i.e. $A = A^*$. The operator A is self-adjoined.

Let us show that $0 \in$ core(dom g - A dom f). Since dom $f = M^\Phi =$ dom g and $0 \in M^\Phi$, we have $M^\Phi \subset U :=$ dom g - A dom $f = M^\Phi - HM^\Phi$. For all $X \in M^\Phi$

$$0 + tX \in M^\Phi \subset U \quad \text{ for all } 0 \le t \le \epsilon, \text{ for some } \epsilon > 0.$$

Hence $0 \in U$. Since R_0 is closed and convex set, the indicator function $f : L^\infty \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous. Since $\rho : M^\Phi \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous convex monetary risk measure, the function g is lower semicontinuous convex function. By Theorem (3.6), the dual problem d of the primal problem (4.14) can be written by

$$d = \sup_{\phi \in (M^\Phi)^*} \{-f^*(A^*\phi) - g^*(-\phi)\}. \tag{4.20}$$

Notice that $(M^\Phi)^* = L^\Psi$ and $A^* : (M^\Phi)^* \to (L^\infty)^*$. When Φ is finite-valued, M^Φ is a norm closed band of L^Φ and its bidual is itself, i.e. $M^\Phi = (M^\Phi)^{**} = (L^\Psi)^*$.

Let's derive the conjugate functions f^* of f and g^* of g. From the convex duality, we have

$$\begin{aligned} f^*(A^*\phi) &= \sup_{X \in L^\infty} \{< A^*\phi, X > -f(X)\} = \sup_{X \in L^\infty} \{< H\phi, X > -\chi_{R_0}(X)\} \\ &= \sup_{\psi \in R_0} \{< H\phi, \psi >\} = \sup_{\psi \in R_0} E[\psi H\phi], \quad \phi \in (M^\Phi)^*. \end{aligned}$$

Also we have for $Y^* \in L^\Psi$

$$\begin{aligned} g^*(Y^*) &= \sup_{X \in M^\Phi} \{< Y^*, X > -g(X)\} = \sup_{X \in M^\Phi} \{< Y^*, X > -\rho(X - H)\} \\ &= \sup_{X' \in M^\Phi} \{< Y^*, X' + H > -\rho(X')\} = \rho^*(Y^*)+ < Y^*, H > . \end{aligned}$$

If ρ is a convex risk measure, then from the equation (3.6)

$$\rho^*(Y^*) = \sup_{X \in \mathcal{A}_\rho} E[-Y^*X] = \rho^\#(-Y^*) \text{ for } Y^* \in \mathcal{D}^\Psi.$$

Secondly, if ρ is a coherent risk measure, then its conjugate function dually represented as

$$\rho(X) \quad = \quad \sup_{Q \in \mathcal{Q}} E^Q[-X] = \sup_{\varphi_Q \in \mathcal{Q}} < X, -\varphi_Q >= \chi^*_{-\mathcal{Q}}(Y^*).$$

Since $\mathcal{Q}$ is the $\sigma(L^\Psi, M^\Phi)$-closed, convex hull of $\mathcal{R} \subset \mathcal{D}^\Psi$ if $\rho = \rho_\mathcal{R}$, we have

$$\rho^*(Y^*) = \chi^{**}_{-\mathcal{Q}}(Y^*) = \chi_{-\overline{co}\mathcal{Q}}(Y^*) = \chi_{-\mathcal{Q}}(Y^*).$$

Hence the dual problem (4.20) becomes

$$\begin{aligned} d &= \sup_{Y^* \in L^\Psi} \{- \sup_{\psi \in R_0} E[\psi H Y^*] + < Y^*, H > -\rho^*(-Y^*)\} \\ &= \sup_{Y^* \in L^\Psi} \{ \inf_{\psi \in R_0} E[(1-\psi) H Y^*] - \rho^*(-Y^*)\}. \end{aligned} \tag{4.21}$$

If ρ is a convex risk measure, then d in (4.21) becomes

$$d = \sup_{Q \in \mathcal{D}^\Psi} \{ \inf_{\psi \in R_0} E^Q[(1-\psi)H] - \rho^{\#}(Q)\}.$$

If ρ is a coherent risk measure, then d in (4.21) becomes

$$d = \sup_{Q \in \mathcal{Q}} \{ \inf_{\psi \in R_0} E^Q[(1-\psi)H]\}.$$

Since $H \in M^\Phi$ and $\sup_{Q \in \mathcal{M}} E^Q[\psi H] \leq \tilde{x}$, d is finite and the strong duality hold $p = d$. Consider the case of the convex risk measure ρ. It can be shown similarly in case of the coherent risk measure. Let $\tilde{\psi}$ be the solution of the primal problem (4.13) and $\tilde{\varphi}_Q$ the solution of the dual problem (4.15).

Since

$$\begin{aligned} p &= \sup_{Q \in \mathcal{D}^\Psi} \{E^Q[(1-\tilde{\psi})H] - \rho^{\#}(Q)\} \geq E[\tilde{\varphi}_Q(1-\tilde{\psi})H] - \rho^{\#}(\tilde{\varphi}_Q) \quad \text{and} \\ d &= \inf_{\psi \in R_0} \{E[\tilde{\varphi}_Q(1-\psi)H] - \rho^{\#}(Q)\} \leq E[\tilde{\varphi}_Q(1-\tilde{\psi})H] - \rho^{\#}(\tilde{\varphi}_Q), \end{aligned}$$

we have

$$[\tilde{\varphi}_Q(1-\tilde{\psi})H] - \rho^{\#}(\tilde{\varphi}_Q) \leq p = d \leq E[\tilde{\varphi}_Q(1-\tilde{\psi})H] - \rho^{\#}(\tilde{\varphi}_Q).$$

Hence (4.17) holds. $(\tilde{\varphi}_Q, \tilde{\psi})$ is a saddle point of the function $E[\varphi_Q(1-\psi)H] - \rho^{\#}(\varphi_Q)$. □

5. The Existence of Minimum of a Primal Problem

Definition 5.1. Let f be a proper function defined on a topological vector space V and let $x_0 \in \text{dom} f$. A vector $x^* \in V^*$ is said to be a subgradient of a convex function f at a point x_0 if

$$f(x_0) \geq f(x) + < x^*, x_0 - x >, \quad \forall x \in V. \tag{5.22}$$

The set of all subgradients of f at x_0 is called the subdifferential of f at x_0 and is denoted by $\partial f(x_0)$, which is a closed convex subset of the dual V^*.

The condition (5.22) has a simple geometric meaning when f is finite at x, i.e. it says that the graph of the affine function $h(z) = f(x) + < x^*, z - x >$ is a non-vertical supporting hyperplane to the convex set epi$f$ at the point $(x, f(x))$.

Theorem 5.1. *Let f be a convex function. Then $0 \in \partial f(x^*)$ if and only if f attains its minimum at x^*.*

In the primal problem (4.12), the convex function $h : \psi \to \rho((\psi - 1)H) + \chi_{R_0}(\psi)$ attains its minimum at $\tilde{\psi}$ if and only if

$$0 \in \partial\{\rho((\psi - 1)H) + \chi_{R_0}(\psi)\}$$

if and only if there exist a φ such that

$$-\varphi \in \partial\{\rho((\psi - 1)H)\} \text{ and } \varphi \in \partial\{\chi_{R_0}(\psi)\} \tag{5.23}$$

by the Theorem (5.1).

Proposition 5.2. *If $\rho : M^\Phi \to \mathbb{R} \cup \{\infty\}$ is a coherent risk measure with core (dom ρ) $\neq \emptyset$ in (5.23), then we have*

$$\max_{\psi \in R_0} E[\varphi_Q \psi H] = E[\tilde{\varphi}_Q \tilde{\psi} H],$$
$$\tilde{\varphi}_Q = \arg\min_{Q \in \mathcal{Q}} E[\varphi_Q(\tilde{\psi} - 1)H].$$

Proof. First we calculate $\partial\chi_{R_0}(\tilde{\psi})$. The φ belongs to $\partial\chi_{R_0}(\tilde{\psi})$ if and only if $\chi_{R_0}(\tilde{\psi}) \geq \chi_{R_0}(\psi) + < \varphi, \tilde{\psi} - \psi > \forall\psi \in M^\Phi$ if and only if $\tilde{\psi} \in R_0$ and $< \varphi, \psi - \tilde{\psi} > \leq 0 \ \forall\psi \in R_0$. Thus we have

$$E[\varphi\psi] \leq E[\varphi\tilde{\psi}] \quad \forall\psi \in R_0 \Longleftrightarrow \max_{\psi \in R_0} E[\varphi\psi] = E[\varphi\tilde{\psi}].$$

Secondly, define function f as $f(\psi) = g(A\psi) = \rho(A\psi - H)$ to calculate $\partial\{\rho((\psi - 1)H)\}$. $\varphi \in \partial g(A\tilde{\psi})$ if and only if $g(A\tilde{\psi}) \geq g(A\psi) + < \varphi, A\tilde{\psi} - A\psi >= g(A\psi) + < A^*\varphi, \tilde{\psi} - \psi >$ if and only if $f(\tilde{\psi}) \geq f(\psi) + < A^*\varphi, \tilde{\psi} - \psi >$ if and only if $A^*\varphi \in \partial f(\tilde{\psi})$.

Since shifting by a constant does not change the subdifferential of a convex function, we have $\partial\{\rho((\psi - 1)H)\} = A^*\partial\rho(A\psi - H) = A^*\partial\rho(A\psi)$. By the definition of the subdifferential,

$$-\varphi \in \partial\rho(\tilde{X}) \Longleftrightarrow \rho(X) \geq \rho(\tilde{X}) + < -\varphi, X - \tilde{X} > \quad \forall X \in M^\Phi. \tag{5.24}$$

Putting $X = \lambda\tilde{X}$ for $\lambda > 0$ and the positive homogeneity of ρ gives

$$(\lambda - 1) < -\varphi, \tilde{X} > \leq (\lambda - 1)\rho(\tilde{X}).$$

Since $(\lambda - 1)$ is either positive or negative, we have

$$\rho(\tilde{X}) = < -\varphi, \tilde{X} > .$$

So the inequality (5.24) reduces to

$$\rho(X) \geq < -\varphi, X > = -E[\varphi X] \quad \forall X \in M^\Phi.$$

From the relation (3.8), we can see $\varphi \in \mathcal{D}^\Psi$ is a Radon-Nikodym derivative of a measure $\tilde{Q} \in \mathcal{Q}$, which is denoted by $\tilde{\varphi}_Q = d\tilde{Q}/dP$. Conversely, if $\rho(\tilde{X}) = < -\varphi, \tilde{X} >$, then by the expression of ρ (3.7)

$$\rho(X) \geq -E[\tilde{\varphi}_Q X] = \rho(\tilde{X}) + < -\tilde{\varphi}_Q, X - \tilde{X} > .$$

This implies that $-\tilde{\varphi}_Q \in \partial\rho(\tilde{X})$. Thus we have

$$\partial\rho(\tilde{X}) = \{-\tilde{\varphi}_Q \in \mathcal{D}^\Psi \,|\, Q \in \mathcal{Q} \text{ and } \rho(\tilde{X}) = -E[\tilde{\varphi}_Q \tilde{X}]\}.$$

Therefore, we have the relation

$$-E[\tilde{\varphi}_Q \tilde{X}] = \rho(\tilde{X}) = \sup_{Q\in\mathcal{Q}} E^Q[-\tilde{X}] = -\inf_{Q\in\mathcal{Q}} E^Q[\tilde{X}].$$

This becomes

$$E[\tilde{\varphi}_Q \tilde{X}] = \inf_{Q\in\mathcal{Q}} E^Q[\tilde{X}], \text{ i.e. } \tilde{\varphi}_Q = \arg\min_{Q\in\mathcal{Q}} E[\varphi_Q \tilde{X}].$$

Hence the subdifferential of ρ is

$$\partial\rho(\tilde{X}) = \{-\tilde{\varphi}_Q \in \mathcal{D}^\Psi \,|\, \tilde{\varphi}_Q \in \arg\min_{Q\in\mathcal{Q}} E[\varphi_Q \tilde{X}]\}.$$

We conclude that

$$\partial\{\rho((\tilde{\psi}-1)H)\} = \{-\varphi \in \mathcal{D}^\Psi \,|\, \varphi = H\tilde{\varphi}_Q, \tilde{\varphi}_Q \in \arg\min_{Q\in\mathcal{Q}} E[\varphi_Q(\tilde{\psi}-1)H)]\}$$

since $A^* = H$. □

For each $Q \in \mathcal{Q}$ define $p(Q)$ as

$$p(Q) := \max_{\psi\in R_0} E^Q[\psi H]. \tag{5.25}$$

In the following Theorem (5.3), it is shown that Fenchel duality $d(Q)$ of $p(Q)$ is given by

$$d(Q) := \inf_{\lambda\in\Lambda_+} \Big\{ \int_\Omega \Big[H\varphi_Q - H\int_{\mathcal{P}_\sigma} z_{Q^*} d\lambda\Big] dP + \tilde{x}\lambda(\mathcal{P}_\sigma)\Big\}. \tag{5.26}$$

Theorem 5.3 ([21, 22])**.** *Strong duality holds, i.e.*

$$d(Q) = p(Q) \quad \forall Q \in \mathcal{Q}.$$

Moreover, for each $Q \in \mathcal{Q}$ there exists a solution $\tilde{\lambda}_Q$ to (5.26). The optimal randomized test $\tilde{\psi}_Q$ of (5.25) has the following structure.

$$\tilde{\psi}_Q(\omega) := \begin{cases} 1, & H\varphi_Q > H\int_{\mathcal{P}_\sigma} z_{Q^*} d\tilde{\lambda}_Q(Q^*) \\ 0, & H\varphi_Q < H\int_{\mathcal{P}_\sigma} z_{Q^*} d\tilde{\lambda}_Q(Q^*), \end{cases} \quad P-a.s.$$

with

$$E^{Q^*}[\tilde{\psi}_Q H] = \tilde{x} \quad \tilde{\lambda}_Q - a.s.$$

Proof. Th proof can be done in the same fashion as in [22] except the proof of the continuity of the linear operator $B : (M^\Phi, \|\cdot\|_\Phi) \to (\mathcal{L}, \tau(\mathcal{L}, \Lambda))$. Let $\mathcal{S}$ be a σ-algebra generated by all subsets of $\mathcal{P}_\sigma$. Let $\mathcal{L}$ be the linear space of all bounded and measurable real functions on $(\mathcal{P}_\sigma, \mathcal{S})$. The order is given on the space $\mathcal{L}$ like

$$l_1 \le l_2,\ l_1, l_2 \in \mathcal{L} \Longleftrightarrow l_2 - l_1 \in \mathcal{L} := \{l \in \mathcal{L} \,|\, l(Q) \ge 0\ \forall Q \in \mathcal{P}_\sigma\}.$$

Let Λ be the space of all σ-additive signed measures on $(\mathcal{P}_\sigma, \mathcal{S})$ with bounded variation. For $l \in \mathcal{L}$ and $Q \in \Lambda$ define the bilinear form $< l, Q >= \int_{\mathcal{P}_\sigma} l\, dQ$. If we consider the space $\mathcal{L}$ with the Mackey topology $\tau(\mathcal{L}, \Lambda)$, then the topological dual of $(\mathcal{L}, \tau(\mathcal{L}, \Lambda))$ is Λ and $\mathcal{L}$ is a barrelled space.

Define a linear operator $B : (M^{\Phi}, \|\cdot\|_{\Phi}) \to (\mathcal{L}, \tau(\mathcal{L}, \Lambda))$ by

$$(B\psi)(Q) = -\int H\psi\, dQ \quad \text{for } Q \in \mathcal{P}_{\sigma}.$$

Then it can be shown that B is continuous. Let $\psi_n \to \psi$ in $(M^{\Phi}, \|\cdot\|_{\Phi})$, i.e. $\|\psi_n - \psi\|_{\Phi} \to 0$ as $n \to \infty$. Since $|E^Q[(\psi_n - \psi)]| \le \|Q\|^*_{\Phi}\|(\psi_n - \psi)\|_{\Phi}$, ψ_n converges to ψ $\mathcal{P}_{\sigma}$-a.s. Hence $\|\psi_n - \psi\|_{\infty} \to 0$ as $n \to \infty$ by Theorem.

$$\|B(\psi_n - \psi)(Q)\|_{\Phi} = |E^Q[H(\psi_n - \psi)]| \le \sup_{Q \in \mathcal{P}_{\sigma}} E^Q[H]\|\psi_n - \psi\|_{\infty}.$$

Also $B\psi_n$ converges in the weaker topology $\tau(\mathcal{L}, \Lambda)$.

So $B\psi_n \to B\psi$ in $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$, where $\|l\|_{\mathcal{L}} := \sup_{Q \in \mathcal{P}_{\sigma}} |l(Q)|$. See [22] for the rest of the proof. □

From the equation (4.15), we have

$$\begin{aligned}
&\max_{Q \in \mathcal{D}^{\Psi}} \min_{\psi \in R_0} \{E^Q[(1-\psi)H] - \rho^{\#}(Q)\} = \max_{Q \in \mathcal{D}^{\Psi}} \{E^Q[H] - p(Q) - \rho^{\#}(Q)\} \\
&= \max_{Q \in \mathcal{D}^{\Psi}} \{E^Q[H] - d(Q) - \rho^{\#}(Q)\} \\
&= \max_{Q \in \mathcal{D}^{\Psi}, \lambda \in \Lambda_+} \Big\{E^P\Big[H\varphi_Q \wedge H \int_{\mathcal{P}_{\sigma}} \varphi_{Q^*} d\lambda\Big] - \tilde{x}\lambda(\mathcal{P}_{\sigma}) - \rho(Q)^{\#}\Big\}. \qquad (5.27)
\end{aligned}$$

There exists $\tilde{Q}$ maximizing the equation (4.15) with respect to $Q \in \mathcal{D}^{\Psi}$. From the Theorem (5.3), there exists a solution $\tilde{\lambda} = \tilde{\lambda}_{\tilde{Q}}$ to (5.26). Thus there exists a solution $(\tilde{Q}, \tilde{\lambda})$ of the equation (5.27).

Theorem 5.4 ([22]). *Let $(\tilde{Q}, \tilde{\lambda})$ be the solution pair of (5.27). Then the solution of the static optimization problem (2.2) is*

$$\psi(\omega) := \begin{cases} 1, & H\varphi_Q > H\int_{\mathcal{P}_{\sigma}} \varphi_{Q^*} d\tilde{\lambda}_Q(Q^*) \\ 0, & H\varphi_Q < H\int_{\mathcal{P}_{\sigma}} \varphi_{Q^*} d\tilde{\lambda}_Q(Q^*), \end{cases} \quad P-a.s.$$

with

$$E^{Q^*}[\tilde{\psi}H] = \tilde{x} \quad \tilde{\lambda} - a.s.$$

$(\tilde{\psi}, \tilde{\varphi}_Q)$ is the saddle point of Theorem (4.2). $(\tilde{x}, \tilde{\varphi})$ solves the dynamic convex hedging problem (2.1), where $\tilde{\varphi}$ is the superhedging strategy of the modified claim $\tilde{\psi}H$.

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Ju Hong Kim received his BS from Seoul National University and Ph.D at the State University of New York at Stony Brook. Since 1998 he has been at Sungshin Women's University. His current research interests focus on risk measure and related topics in mathematical finance and actuarial sciences.

Department of Mathematics, Sungshin Women's University, Seoul 136 - 742, Korea.
e-mail: jhkkim@sungshin.ac.kr