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TIME REPARAMETRIZATION OF PIECEWISE PYTHAGOREAN-HODOGRAPH C^1 HERMITE INTERPOLANTS

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ABSTRACT. In this paper, we show two ways of the time reparametrization of piecewise Pythagorean-hodograph C^1 Hermite interpolants. One is the time reparametrization with no shape change, and the other is that with shape change. We show that the first reparametrization does not depend on the boundary data and that it is uniquely determined by the size of parameter domain, up to the general cases. We empirically show that the second parametrization can cause the change of the shape of interpolant.

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1. Introduction

In the computation of offsets and carnal surfaces, one of the main difficulties is the irrationality of the normal vector field: Even though the spline curve is rational, the normal vector might be irrational in general. In 1990, Farouki and Sakkalis introduced the PH (Pythagorean Hodograph) curve as a solution of this problem [1]. PH curves are a special class of polynomial curves with polynomial speed function, guaranteeing polynomial arc length and rational offsets. By this property of PH curves, they are widely used in a number of applications such as CNC machining, interpolation of discrete data, and the control of digital motion along curved paths [2, 3, 4, 5, 6].

Since the first introduction of PH curves, there has been a lot of rigorous researches for both the planar [7, 8, 9, 10] and spatial cases [11, 12, 13, 14] in several directions. There also have been substantial progresses not only on the formal representation of PH curves [1, 15, 16] but also on the application of them to diverse interpolation problems [3, 9, 10, 17, 18, 19, 20, 21, 22].

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Generally, solving various interpolation problems related to practical applications, we obtain several interpolants produced by their own specific methods for the applications. If we have to compare all the possible interpolants of an interpolation problem, first of all, the interplants must be defined on the same parameter domain for fair comparison. Especially if the interpolants consist of consecutive curves with junction points of several types [18, 19, 20], i.e., they are the *Undetermined Junction Point (UJP) interpolants*; the interpolants obtained by the UJP method [20], needless to say, it is obvious. However, when we mainly concentrate our attention on the shapes of interpolants, we sometimes forget it, and moreover ignore the important fact that the different parameter domain for an interpolant means the different motion on it.

In this paper, focussing on the piecewise-connected PH interpolants presented in [20], we handle this problem: how to make the different parameter domains of PH interpolants identified. As answers to the problem, we propose two methods: One is the time parameter reparametrization with no shape change, and the other is that with shape change. (Note that throughout this paper, we will call the parameter of curve *time*.) In addition, we compare all the time-reparameterized interpolants with the original ones, from the viewpoints of the shapes of them and the motions on them.

The last of this paper is organized as follows: In Section 2, we introduce some definitions and two fundamental theorems, which are necessary for further discussions. In Section 3 and 4, we present two kinds of time reparametrizations for PH interpolants obtained by the UJP method. In Section 5, we generalize the results in Section 3 and 4. In Section 6, we conclude our results.

2. Preliminaries

Definition 2.1. Let $\alpha(t) = (x(t), y(t))$ be a planar polynomial curve. Then the *complex representation* of α is defined by $x(t) + iy(t)$.

Note that $x(t) + iy(t)$ is a polynomial with complex coefficients. This means that we can regard a planar polynomial curve $\alpha(t)$ as a complex polynomial, in algebraic and integrodifferential computations. This identification provides us many advantages in handling planar polynomial curves. Throughout this paper, we will use this representation for planar curves.

Definition 2.2. Let $\alpha(t) = x(t) + iy(t)$ be a planar polynomial curve. Then $\alpha(t)$ is a *PH (Pythagorean Hodograph)* curve if and only if there exist a polynomial $\sigma(t)$ which satisfies $x'(t)^2 + y'(t)^2 = \sigma(t)^2$, in the complex representation, $\|\alpha'(t)\|^2 = \|x'(t) + iy'(t)\|^2 = \sigma(t)^2$.

Algebraically, PH polynomial curves can be completely characterized as follows.

Theorem 2.1 ([17]). *Let $\alpha(t) = x(t) + iy(t)$ be a planar polynomial curve. Then $\alpha(t)$ is a PH curve if and only if the roots of the hodograph $\alpha'(t) = x'(t) + i\,y'(t)$*

consist of real roots, complex roots of even multiplicity and pairs of conjugate complex roots.

Remark 2.1. A polynomial curve $\alpha(t) = x(t) + i\,y(t)$ is said to be *regular* if $\|\alpha'(t)\| = \|x'(t)+i\,y'(t)\| \neq 0$ for all t. Thus, $\alpha(t)$ is regular if $\|\alpha'(t)\|$ has no real root. If $\|\alpha'(t)\|$ has only non-real complex roots of even multiplicity, of which any two are not conjugate mutually, then $\alpha(t)$ is said to be *strongly regular*. For example, a strongly regular PH cubic can be written as $\int k(t-c)^2 dt$, where c is a non-real complex number and k is a constant complex number.

In this paper, we will handle several PH interpolants of various types; single PH interpolants, two-piece PH interpolants, three-piece interpolants, etc.. We assume that all the pieces of interpolants are strongly regular.

Piecewise PH curves have been used, as the need arises, in some researches [18, 19]. Recently, another method to solve efficiently Hermite interpolation problems with them, the UJP method, is introduced in [20], with some methodological variations. By the method, although we can not solve C^1 Hermite interpolation problems with single PH cubics, for a C^1 Hermite data, we can construct always the interpolants which consist of two consecutive PH *cubics*:

Definition 2.3. Let $\alpha_1 : [0,\,1] \to \mathbb{R}^2$ and $\alpha_2 : [0,\,1] \to \mathbb{R}^2$ be two continuous plane curves. A point Q is called *the C^1 junction point* of $\alpha_1(t)$ and $\alpha_2(t)$ if $\alpha_1(1) = Q = \alpha_2(0)$ and $\alpha_1'(1) = \alpha_2'(0)$.

Theorem 2.2 ([20])**.** *For a given C^1 Hermite data $H_C^1 = \{P_0, P_1, V_0, V_1\}$, there generically exist four interpolants, each of which consists of two PH cubics with a C^1 junction point.*

The following example shows how the UJP interpolants stated in Theorem 2.2 can be obtained practically for a given C^1 Hermite data.

Example 2.3. Consider a C^1 Hermite data $H_C^1 = \{0,\, 5,\, 1+3i,\, 1-3i\}$. Let $\alpha_1(t)$ and $\alpha_2(t)$ be PH cubics. Then, by Remark 2.1, we have $\alpha_1(t) = \frac{1}{3}k_1(t-c_1)^3 + d_1$ and $\alpha_2(t) = \frac{1}{3}k_2(t-c_2)^3 + d_2$. Assume that $\alpha_1(0) = 0$, $\alpha_2(1) = 5$,$\alpha_1'(0) = 1+3i$ and $\alpha_2'(1) = 1-3i$, with an undetermined C^1 junction point Q , so that $Q = \alpha_1(1) = \alpha_2(0)$ and $\alpha_1'(1) = \alpha_2'(0)$. Then we obtain

$$\begin{aligned} -\frac{1}{3}k_1c_1^3 + d_1 &= 0, \quad \frac{1}{3}k_2(1-c_2)^3 + d_2 = 5,\\ \frac{1}{3}k_1(1-c_1)^3 + d_1 &= -\frac{1}{3}k_2c_2^3 + d_2 = Q,\\ k_1c_1^2 &= 1+3i, \quad k_2(1-c_2)^2 = 1-3i,\\ k_1(1-c_1)^2 &= k_2c_2^2. \end{aligned}$$

Next, solving this system of equations, as stated in the previous theorem, we can find four two-piece PH interpolants with a C^1 junction point, satisfying H_C^1, as follows;

$$\alpha_1^1(t) = (t + 1.782t^2 - 0.282t^3) + (3t - 0.995t^2 - 0.337t^3)\,i,$$

$$
\begin{aligned}
\alpha_2^1(t) =&(2.5 + 3.1718t - 0.936t^2 - 0.282t^3) + (1.668 - 2.005t^2 + 0.337t^3)\, i;\\
\alpha_1^2(t) =&(t + 3.141t^2 - 0.441t^3) + (3t - 1.155t^2 - 1.095t^3)\, i,\\
\alpha_2^2(t) =&(3.700 + 5.959t - 9.020t^2 + 4.360t^3) + (0.750 - 2.595t + 5.940t^2 - 4.095t^3)\, i;\\
\alpha_1^3(t) =&(t - 4.060t^2 + 4.360t^3) + (3t - 6.345t^2 + 4.095t^3)\, i,\\
\alpha_2^3(t) =&(1.30 + 5.960t - 1.818t^2 - 0.441t^3) + (0.750 + 2.595t - 4.440t^2 + 1.095t^3)\, i;\\
\alpha_1^4(t) =&(t - 5.863t^2 + 7.363t^3) + (3t - 6.505t^2 + 3.337t^3)\, i,\\
\alpha_2^4(t) =&(2.50 + 11.363t - 16.226t^2 + 7.363t^3) + (-0.168 + 3.505t^2 - 3.337t^3)\, i,
\end{aligned}
$$

where α_1^j and α_2^j are two consecutive PH cubics which determine one PH interpolant for each $j = 1, 2, 3, 4$. (See Figure 1.)

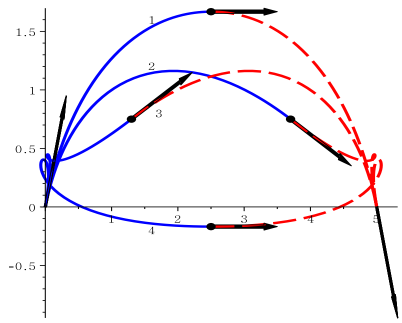

FIGURE 1. Four C^1 Hermite PH cubic interpolats obtained by the UJP method for $H_C^1 = \{0,\, 5,\, 1+3i,\, 1-3i\}$.

Note that, even though various piecewise PH interpolants are generated by several techniques [20], in this paper, we will state our results, giving priority to two-piece PH cubic interpolants with a C^1 junction point and their simple variations. In the last part of this paper, we will see that our methods can work for general piecewise PH interpolants.

3. Time reparametrization with no shape change

From now on, we present our main results; time reparametrizations. We first consider the time reparametrization with no shape change. Before starting our main, we need some definitions;

Definition 3.1. Let $\alpha(\tilde{t})$ be a planar curve with $\tilde{t} \in [0, 1]$, satisfying the boundary conditions; $\alpha(0) = p_0$, $\alpha(1) = p_1$, $\alpha'(0) = v_0$, and $\alpha'(1) = v_1$. Then a *time reparametrization* of $\alpha(\tilde{t})$ with respect to an interval $I_\rho = [0, \rho]$ is defined by a monotone increasing *polynomial* $\tilde{t} = \phi(t)$ where $t \in I_\rho$ with $\phi(0) = 0$, $\phi(\rho) = 1$, $\frac{d}{dt}\alpha(\phi(t))|_{t=0} = \phi'(0) \cdot \alpha'(\phi(0)) = v_0$ and $\frac{d}{dt}\alpha(\phi(t))|_{t=\rho} = \phi'(\rho) \cdot \alpha'(\phi(\rho)) = v_1$.

Now, let $\alpha(t)$ be a C^1 curve defined on $[0, \frac{1}{2}]$. Then, by the following lemma, we can find the special time reparametrization of $\alpha(t)$ uniquely determined by the size of time domain, which does not depend on the curve $\alpha(t)$.

Lemma 3.1. *Let $\alpha(\tilde{t})$ be a regular curve with $\alpha(0) = p_0$, $\alpha(1) = p_1$, $\alpha'(0) = v_0$, and $\alpha'(1) = v_1$. Assume that the time reparametrization $\tilde{t} = \phi(t)$ of $\alpha(\tilde{t})$ has degree 3. Then, when $n \geq 2$, $\tilde{t} = \phi(t)$ is uniquely determined with respect to the interval $I_{\frac{1}{n}} = [0, \frac{1}{n}]$.*

Proof. Let $\phi(t)$ be a monotone increasing polynomial defined on the interval $[0, \frac{1}{n}]$, with

$$\phi(0) = 0,\ \phi(\frac{1}{n}) = 1,\ \phi'(0) = 1,\ \phi'(\frac{1}{n}) = 1.$$

Then, since $\phi(0) = 0,\ \phi(\frac{1}{n}) = 1$, we can find a polynomial $g(t)$ such that

$$\phi(t) = tg(t),\ \text{ where } g(\frac{1}{n}) = n. \tag{1}$$

In addition, since $\phi'(0) = 1$ and $\phi'(\frac{1}{n}) = 1$, $g(t)$ satisfies the following;

$$g(0) = 1, \tag{2}$$

$$g'(\frac{1}{n}) = -n^2 + n. \tag{3}$$

Here, we assume that $g(t)$ is a quadratic polynomial. Then, by (1), (2) and (3), we obtain

$$\phi(t) = 2(n^2 - n^3)t^3 - 3(n - n^2)t^2 + t. \tag{4}$$

Next, we check the monotone increasing property of $\phi(t)$. Note that

$$\phi'(t) = 6(n^2 - n^3)(t - \frac{1}{2n})^2 - \frac{3(1-n)}{2}.$$

Since $\phi'(0) = \phi'(\frac{1}{n}) = 1$ and $n^2 - n^3 < 0$ for $n \geq 2$, we consequently have $\phi'(t) > 0$ on the interval $I_{\frac{1}{n}}$. This complete the proof. □

Note that, as shown in Eq. (4), the time reparametrization depends only on the number n. That's, the time reparametrization is not influenced by any other boundary data. This means that the time reparametrization does not change the shape of the original curve but the speed of motion on the curve.

Next, we apply Lemma 3.1 to two-piece PH cubic interpolants generated by the UJP method.

Theorem 3.2. *Let $H_C^1 = \{P_0, P_1, V_0, V_1\}$ be a C^1 Hermite data and let $\alpha_*(t)$ be a two-piece PH cubic interpolant satisfying H_C^1, which consists of two consecutive PH cubics $\alpha_1(t)$ and $\alpha_2(t)$ with the standard time domain $[0, 1]$;*

$$\alpha_*(t) = \begin{cases} \alpha_1(t), & t \in [0,1]; \\ \alpha_2(t-1), & t \in [1,2]. \end{cases} \tag{5}$$

Then, the time-reparameterized curve $\beta^*(t)$ given by

$$\beta_*(t) = \begin{cases} \alpha_1 \circ \phi_1(t), & t \in [0, \frac{1}{2}]; \\ \alpha_2 \circ \phi_2(t), & t \in [\frac{1}{2}, 1], \end{cases}$$

where $\phi_1(t) = -8t^3 + 6t^2 + t$ and $\phi_2(t) = \phi_1(t - \frac{1}{2})$, is also a two-piece PH interpolant satisfying H_C^1.

Proof. By Theorem 2.2, there exists four possible two-piece PH cubic interpolants satisfying H_C^1. Let $\alpha_*(t)$ be one of them.

By Lemma 3.1, we can find the time reparametrizations $\phi_1(t)$ and $\phi_2(t)$ of $\alpha_1(t)$ and $\alpha_2(t)$ with respect to the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, as follows; $\phi_1(t) = -8t^3 + 6t^2 + t$ and $\phi_2(t) = \phi_1(t - \frac{1}{2})$. Note that $\alpha_1 \circ \phi_1(t)$ and $\alpha_2 \circ \phi_2(t)$ are polynomial curves with the time domains $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. Moreover, they are consecutively connected by the junction point $\alpha_1 \circ \phi_1(\frac{1}{2}) = \alpha_2 \circ \phi_2(\frac{1}{2})$ with $\frac{d}{dt}\alpha_1 \circ \phi_1|_{t=\frac{1}{2}} = \frac{d}{dt}\alpha_2 \circ \phi_2|_{t=\frac{1}{2}}$, Additionally, they satisfy the boundary conditions; $\alpha_1 \circ \phi_1(0) = P_0$, $\frac{d}{dt}\alpha \circ \phi_1(t)|_{t=0} = V_0$, $\alpha_2 \circ \phi_2(1) = P_1$ and $\frac{d}{dt}\alpha \circ \phi_1(t)|_{t=1} = V_1$. This completes the proof. □

Note that $\alpha_*(t)$ and $\beta_*(t)$ have the same image in $\mathbb{R}^2$. Thus, $\alpha_*(t)$ and $\beta_*(t)$ represent different motions along the same image curve. We can empirically confirm this in the following example.

Example 3.3. Let $\alpha_*(t)$ be one of the two-piece PH cubic interpolants introduced in Example 2.3. Then, as shown in the example, there are four possible pairs of two consecutive PH cubics; $\alpha_1^j(t)$ and $\alpha_2^j(t)$ $(j = 1, 2, 3, 4)$, such that

$$\alpha_*(t) = \begin{cases} \alpha_1^j(t), & t \in [0, 1]; \\ \alpha_2^j(t-1), & t \in [1, 2]. \end{cases}$$

Then, by Theorem 3.2, we can obtain four possible time reparametrization of $\alpha_*(t)$ with respect to the interval $[0, 1]$, as follows:

$$\beta_*(t) = \begin{cases} \alpha_1^j \circ \phi_1(t), & t \in [0, \frac{1}{2}]; \\ \alpha_2^j \circ \phi_2(t), & t \in [\frac{1}{2}, 1], \end{cases}$$

where $\phi_1(t) = -8t^3 + 6t^2 + t$, $\phi_2(t) = \phi_1(t - \frac{1}{2})$ and $j = 1, 2, 3, 4$.

$\alpha_*(t)$ and $\beta_*(t)$ have the same image curve in $\mathbb{R}^2$. Thus the practically meaningful difference between them is to be observed only by comparing the motions created by them on the image curve. Figure 2 shows the comparison of the motions created by $\alpha_*(t)$ and $\beta_*(t)$ when $j = 1$, on the same image curve annotated by 1 in Figure 1. In the figure, we clearly observe that, at the same moment t, $\alpha_*(t)$ and $\beta_*(t)$ assign different positions on the image curve. This means that $\alpha_*(t)$ and $\beta_*(t)$ represent two motions with different speed on the same trajectory.

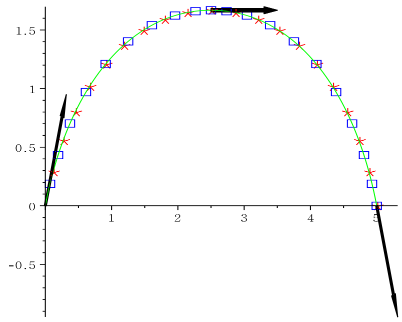

FIGURE 2. The comparison of the motions along the interpolants $\alpha_*(t)$ and $\beta_*(t)$, which are denoted respectively by small boxes and stars, on the same image curve when $j = 1$.

4. Time reparametrization with shape change

In this section, we introduce, using $\phi(t) = nt$, the second type parameter reparametrization, i.e., the time reparametrization with shape change. We first consider the following: Let $H_C^1 = \{P_0, P_1, V_0, V_1\}$ be a C^1 Hermite data and let $\alpha(t)$ be an interpolant for H_C^1 with the general parameter domain. Here, if we reparametrize the time variable of $\alpha(t)$ by $\phi(t) = nt$, even though we can transform the time domain of $\alpha(t)$ into the target time domain $I_{\frac{1}{n}} = [0, \frac{1}{n}]$, the time-reparametrized curve $\alpha \circ \phi(t)$ can not satisfy H_C^1 but the new C^1 Hermite data ${H_C^1}^* = \{P_0, P_1, nV_0, nV_1\}$. This means that, if we use $\phi(t)$ for time reparametrization, different from the previous one, we need to modify something more. The following theorem shows what we need.

Theorem 4.1. *Let $H_C^1 = \{P_0, P_1, V_0, V_1\}$ be a C^1 Hermite data and let $\alpha^*(t)$ be a two-piece PH cubic interpolant satisfying ${H_C^1}^* = \{P_0, P_1, \frac{1}{2}V_0, \frac{1}{2}V_1\}$, which consists of two consecutive PH cubics $\alpha_1(t)$ and $\alpha_2(t)$ with the standard time domain $[0, 1]$;*

$$\alpha_*(t) = \begin{cases} \alpha_1(t), & t \in [0, 1]; \\ \alpha_2(t-1), & t \in [1, 2]. \end{cases} \tag{6}$$

Then, the following time-reparameterized curve $\beta_*(t)$ is a two-piece PH interpolant for H_C^1 with the general time domain $[0, 1]$;

$$\beta_*(t) = \begin{cases} \alpha_1 \circ \phi_1(t), & t \in [0, \frac{1}{2}]; \\ \alpha_2 \circ \phi_2(t), & t \in [\frac{1}{2}, 1], \end{cases}$$

where $\phi_1(t) = 2t$ and $\phi_2(t) = 2t - 1$.

Proof. Note that

$$\beta_*(0) = \alpha_1(0) = P_0,\ \beta_*(1) = \alpha_2(1) = P_1,$$

$$\beta_*{}'(0) = \phi_1'(0) \cdot \alpha_1'(0) = 2 \cdot \frac{1}{2} \cdot V_0 = V_0,$$
$$\beta_*{}'(1) = \phi_2'(1) \cdot \alpha_2'(1) = 2 \cdot \frac{1}{2} \cdot V_1 = V_1,$$
$$\beta_*(\frac{1}{2}) = \alpha_1(1) = \alpha_2(0),\ \alpha_1'(1) = \alpha_2'(0).$$

Thus $\beta_*(t)$ is a two-piece PH cubic interpolant for H_C^1 with the general time domain $[0, 1]$. □

Remark 4.1. Note that, at the first step in the second time reparametrization, we modify the given Hermite data. So, applying the UJP method to this modified data, we obtain new interpolants, which are different from the ones satisfying the original data. This means that, even though we can modify the new interpolants, by the time reparametrization, to satisfy the original data, their shapes must be different from the original ones. (See Figure 4 in the following example.)

Example 4.2. Consider again the C^1 Hermite data, $H_C^1 = \{0,\ 5,\ 1+3i,\ 1-3i\}$, given in Example 2.3 and let ${H_C^1}^*$ be the new C^1 Hermite data obtained by reducing the terminal speeds of H_C^1 by half, i.e., ${H_C^1}^* = \{0,\ 5,\ \frac{1+3i}{2},\ \frac{1-3i}{2}\}$. Here, applying the UJP method to ${H_C^1}^*$, we can obtain four new C^1 Hermite interpolants satisfying ${H_C^1}^*$, which consist of two consecutive PH cubic curves $\alpha_1^j(t)$ and $\alpha_2^j(t)$ for $j = 1, 2, 3, 4$, as follows:

$$\alpha_1^1(t) = (t + 6.913t^2 + 2.174t^3) + (3t + 0.424t^2 - 4.566t^3)\, i,$$
$$\alpha_2^1(t) = (2.500 + 9.543t - 10.174t^2 + 2.174t^3) + (1.035 - 6.424t^2 + 4.566t^3)\, i;$$
$$\alpha_1^2(t) = (t + 9.772t^2 + 3.583t^3) + (3t + 0.206t^2 - 9.411t^3)\, i,$$
$$\alpha_2^2(t) = (3.391 + 13.459t - 36.529t^2 + 32.092t^3) + (0.375 - 3.853t + 16.911t^2 - 21.411t^3)\, i;$$
$$\alpha_1^3(t) = (t - 11.610t^2 + 32.092t^3) + (3t - 15.206t^2 + 21.411t^3)\, i,$$
$$\alpha_2^3(t) = (1.609 + 13.459t - 15.147t^2 + 3.583t^3) + (0.375 + 3.853t - 13.911t^2 + 9.411t^3)\, i;$$
$$\alpha_1^4(t) = (t - 15.075t^2 + 46.151t^3) + (3t - 15.424t^2 + 16.566t^3)\, i,$$
$$\alpha_2^4(t) = (2.500 + 20.538t - 54.151t^2 + 46.151t^3) + (-0.285 + 9.424t^2 - 16.566t^3)\, i.$$

(See Figure 3.)

Next, we reparametrize the new interpolants by the time reparametrization functions $\phi_1(t) = 2t$ and $\phi_2(t) = 2t - 1$, so that the four new curves $\beta_*^j(t)$ ($j = 1, 2, 3, 4$) given by

$$\beta_*^j(t) = \begin{cases} \alpha_1^j \circ \phi_1(t), & t \in [0, \frac{1}{2}]; \\ \alpha_2^j \circ \phi_2(t), & t \in [\frac{1}{2}, 1] \end{cases}$$

satisfy H_C^1. Thus they are new two-piece PH interpolants, with the general time domain $[0, 1]$, satisfying H_C^1. Especially, we should pay attention to the fact that, since we change the terminal speeds of H_C^1 in the first step, the new interpolants have shapes different from the old ones obtained only by the UJP method. (See Figure 4.)

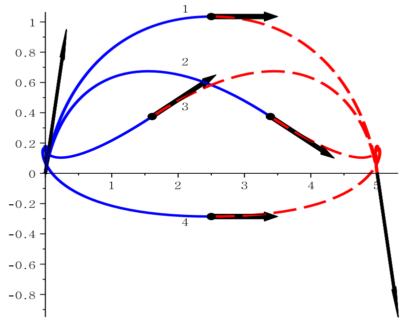

FIGURE 3. Four C^1 Hermite PH interpolats obtained by the second time reparametrization for the same C^1 Hermite data H_C^1 in Example 2.3.

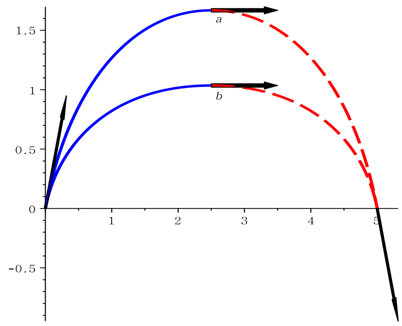

FIGURE 4. The change of the shape of interpolnant by the second time reparametrization: (a) shows the interpolant obtained by the original UJP method, which is annotated by 1 in Figure 1. (b) shows the interpolant obtained by the second time reparametrization, which is annotated by 1 in Figure 3.

5. Generalization of the time reparametrizations

Now, in this section, we consider the time reparametrization for several piece PH C^1 Hermite interpolants, i.e., the generalized version of our previous results. Let $H_C^1 = \{P_0, P_1, V_0, V_1\}$ be a C^1 Hermite data and let $\alpha_*(t)$ be a C^1 Hermite interpolant satisfying H_C^1, which consists of n consecutive PH curves $\alpha_i(t)$ where

$1 \le i \le n$;

$$\alpha_*(t) = \begin{cases} \alpha_1(t), & t \in [0,1]; \\ \vdots & \vdots \\ \alpha_i(t-i+1), & t \in [i-1,i]; \\ \vdots & \vdots \\ \alpha_n(t-n+1), & t \in [n-1,n], \end{cases} \tag{7}$$

with $n-1$ consecutive C^1 junction points $\alpha_1(1), \cdots, \alpha_i(1), \cdots, \alpha_{n-1}(1)$. Then we have the following theorems:

Theorem 5.1. *Let $H_C^1 = \{P_0, P_1, V_0, V_1\}$ be a given C^1 Hermite data and let $\alpha_*(t)$ be an n-piece PH C^1 Hermite interpolant satisfying H_C^1, which is given by Eq. (7). Then the time-reparameterized interpolant $\beta_*(t)$ given by*

$$\beta_*(t) = \begin{cases} \alpha_1 \circ \phi_1(t), & t \in [0, \frac{1}{n}]; \\ \vdots & \vdots \\ \alpha_i \circ \phi_i(t), & t \in [\frac{i-1}{n}, \frac{i}{n}]; \\ \vdots & \vdots \\ \alpha_n \circ \phi_n(t), & t \in [\frac{n-1}{n}, 1], \end{cases}$$

where $\phi_1(t) = 2(n^2-n^3)t^3 - 3(n-n^2)t^2 + t$ and $\phi_i(t) = \phi_1(t - \frac{i-1}{n})$ for $2 \le i \le n$, is also an n-piece PH C^1 Hermite interpolant satisfying H_C^1.

Proof. Note that, since

$$\begin{aligned} &\alpha_*(0) = \alpha_1(0) = P_0, \;\; \alpha_*(n) = \alpha_n(1) = P_1; \\ &\alpha_*{}'(0) = \alpha_1'(0) = V_0, \; \alpha^*{}'(n) = \alpha_n'(1) = V_1; \\ &\alpha^*(i) = \alpha_i(1) = \alpha_{i+1}(0), \;\; \alpha^*{}'(i) = \alpha_i'(1) = \alpha_{i+1}'(0) \;\text{ for }\; 1 \le i \le n-1, \end{aligned}$$

we have

$$\begin{aligned} &\beta_*(0) = \alpha_1(\phi_1(0)) = P_0, \; \beta_*(1) = \alpha_n(\phi_n(1)) = P_1, \\ &\beta_*{}'(0) = \phi_1'(0) \cdot \alpha_1'(0) = 1 \cdot V_0 = V_0, \\ &\beta_*{}'(1) = \phi_n'(1) \cdot \alpha_n'(n) = 1 \cdot V_1 = V_1; \\ &\beta_*(\frac{i}{n}) = \alpha_i(\phi_i(\frac{i}{n})) = \alpha_i(\phi_1(\frac{1}{n})) = \alpha_{i+1}(\phi_1(0)) = \alpha_{i+1}(\phi_{i+1}(\frac{i}{n})), \\ &\beta_*{}'(\frac{i}{n}) = \alpha_i{}'(\phi_i(\frac{i}{n})) \cdot \phi_i{}'(\frac{i}{n}) = \alpha_i{}'(\phi_1(\frac{1}{n})) \cdot \phi_1{}'(\frac{1}{n})) = \alpha_{i+1}{}'(\phi_1(0)) \cdot \phi_1{}'(0) \\ &\qquad\qquad = \alpha_{i+1}{}'(\phi_{i+1}(\frac{i}{n})) \cdot \phi_{i+1}{}'(\frac{i}{n}) \;\text{ for }\; 1 \le i \le n-1. \end{aligned}$$

This means that $\beta_*(t)$ is an n-piece PH interpolant satisfying H_C^1, defined on the general time domain $[0,1]$, with $n-1$ consecutive C^1 junction points. $\square$

Theorem 5.2. *Let $H_C^1 = \{P_0, P_1, V_0, V_1\}$ be a C^1 Hermite data and let $\alpha^*(t)$ be an n-piece C^1 Hermite interpolant satisfying ${H_C^1}^* = \{P_0, P_1, \frac{1}{n}V_0, \frac{1}{n}V_1\}$, which is given by Eq. (7). Then the following time-reparameterized curve $\beta_*(t)$ is an n-piece PH interpolant for H_C^1 with the general time domain $[0,1]$;*

$$\beta_*(t) = \begin{cases} \alpha_1 \circ \phi_1(t), & t \in [0, \frac{1}{n}]; \\ \vdots & \vdots \\ \alpha_i \circ \phi_i(t), & t \in [\frac{i-1}{n}, \frac{i}{n}]; \\ \vdots & \vdots \\ \alpha_n \circ \phi_n(t), & t \in [\frac{n-1}{n}, 1], \end{cases}$$

where $\phi_i(t) = nt - i + 1$ for $1 \le i \le n$.

Proof. Note that, since

$$\alpha^*(0) = \alpha_1(0) = P_0, \;\; \alpha^*(n) = \alpha_n(1) = P_1;$$
$$\alpha^{*\prime}(0) = \alpha_1'(0) = \frac{V_0}{n}, \;\; \alpha^{*\prime}(n) = \alpha_n'(1) = \frac{V_1}{n};$$
$$\alpha^*(i) = \alpha_i(1) = \alpha_{i+1}(0), \;\; \alpha^{*\prime}(i) = \alpha_i'(1) = \alpha_{i+1}'(0) \;\text{ for } 1 \le i \le n-1,$$

we have

$$\beta_*(0) = \alpha_1(\phi_1(0)) = P_0, \;\; \beta_*(1) = \alpha_n(\phi_n(1)) = P_1;$$
$${\beta_*}'(0) = \phi_1'(0) \cdot \alpha_1'(0) = n \cdot \frac{V_0}{n} = V_0;$$
$${\beta_*}'(1) = \phi_n'(1) \cdot \alpha_n'(1) = n \cdot \frac{V_1}{n} = V_1;$$
$$\beta_*(\frac{i}{n}) = \alpha_i(\phi_i(\frac{i}{n})) = \alpha_i(1) = \alpha_{i+1}(0) = \alpha_{i+1}(\phi_{i+1}(\frac{i}{n})),$$
$$\begin{aligned} {\beta_*}'(\frac{i}{n}) &= {\phi_i}'(\frac{i}{n}) \cdot \alpha_i'(\phi_i(\frac{i}{n})) = n \cdot {\alpha_i}'(1) = n \cdot {\alpha_{i+1}}'(0) \\ &= {\phi_{i+1}}'(\frac{i}{n}) \cdot \alpha_{i+1}'(\phi_{i+1}(\frac{i}{n})) \;\text{ for } 1 \le i \le n-1. \end{aligned}$$

Thus $\beta_*(t)$ is an n-piece PH interpolant satisfying H_C^1 with $n-1$ consecutive C^1 junction points on the general time domain $[0,1]$. □

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