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Bayesian Estimators Using Record Statistics of Exponentiated Inverse Weibull Distribution

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Abstract

The inverse Weibull distribution(IWD) is a complementary Weibull distribution and plays an important role in many application areas. In this paper, we develop a Bayesian estimator in the context of record statistics values from the exponentiated inverse Weibull distribution(EIWD). We obtained Bayesian estimators through the squared error loss function (quadratic loss) and LINEX loss function. This is done with respect to the conjugate priors for shape and scale parameters. The results may be of interest especially when only record values are stored.

Keywords: Bayesian estimation, exponentiated inverse Weibull distribution, record statistics.

1. Introduction

The probability density function(pdf) and cumulative distribution function(cdf) of the random variable X having the exponentiated inverse Weibull distribution are given by

$$f(x;\alpha,\beta,\gamma) = \frac{\alpha\gamma}{\beta^\gamma}\exp\left(-\alpha(\beta x)^{-\gamma}\right)x^{-\gamma-1} \tag{1.1}$$

and

$$F(x;\alpha,\beta,\gamma) = \exp\left(-\alpha(\beta x)^{-\gamma}\right), \quad x>0,\ \alpha,\beta,\gamma>0. \tag{1.2}$$

The k^{th} moment of this distribution that was introduced by Ali *et al.* (2007) is

$$E\left(X^k\right) = \frac{\alpha^{\frac{k}{\gamma}}}{\beta^k}\Gamma\left(1-\frac{k}{\gamma}\right), \quad \gamma>k. \tag{1.3}$$

Therefore, the mean and the variance of the exponentiated inverse Weibull distribution can be written as follows.

$$E(X) = \frac{\alpha^{\frac{1}{\gamma}}}{\beta}\Gamma\left(1-\frac{1}{\gamma}\right) \tag{1.4}$$

and

$$\mathrm{Var}(X) = \frac{\alpha^{\frac{2}{\gamma}}}{\beta^2}\left[\Gamma\left(1-\frac{2}{\gamma}\right)-\left\{\Gamma\left(1-\frac{1}{\gamma}\right)\right\}^2\right], \quad \text{for } \gamma>2. \tag{1.5}$$

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Both the mean (1.4) and the variance (1.5) increase as α increases, when $\gamma > 2$. From (1.2), the reliability function of the exponentiated inverse Weibull distribution is given by

$$R(t) = 1 - F(t) = 1 - \exp\left(-\alpha(\beta t)^{-\gamma}\right), \quad t > 0. \tag{1.6}$$

Note that the inverse Weibull distribution is a special case of (1.1) when $\alpha = 1$. The inverse Weibull distribution is the complementary Weibull distribution and plays an important role in many applications including the dynamic components of diesel engines, the times to breakdown of an insulating fluid subject to the action of constant tension and flood data (see Nelson, 1982; Maswadah, 2003). In addition, it has been used quite extensively when the data indicate a monotone hazard function because of the flexibility of the pdf and its corresponding hazard function. Studies for the inverse Weibull distribution have been conducted by many authors. Calabria and Pulcini (1994) studied Bayes 2-sample prediction for the inverse Weibull distribution. Mahmoud *et al.* (2003) considered the order statistics arising from the inverse Weibull distribution and derived the exact expression for the single moments of order statistics. They also obtained variances and covariances based on the moments of order statistics.

Chandler (1952) introduced the study of record values and documented many of the basic properties of records. Record values arise in many real-life situations that involve the weather, sports, economics and life tests. Record model is related to the order statistics model, both of which appear in many statistical applications and are widely used in statistical modeling and inference because it can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of observations. In particular, Balakrishnan *et al.* (1992) established some recurrence relations for the single and double moments of lower record values from Gumble distribution. Soliman *et al.* (2006) obtained Bayes estimators based on record statistics for two unknown parameters of the Weibull distribution. Recently, Sultan (2008) derived the Bayes estimators and obtained the estimators of the reliability and hazard functions for the unkonwn parameters of the inverse Weibull distribution based on lower record values.

The squared error loss function(SELF) is a symmetric loss function assigning equal losses to overestimation and underestimation. Therefore, under the SELF, Bayes estimator is defined by the posterior expectation. However, such a restriction may be impractical because an overestimate is usually more serious than an underestimate in the estimation of reliability and failure rate functions. In this case the use of a symmetrical loss function might be inappropriate. To cover this drawback, we consider two types asymmetric loss functions known as the LINEX loss function(LLF) and the SQUAREX loss function(SLF). The LLF was introduced by Varian (1975) and received significant popularity due to Zellner (1986). It may be expressed as $L(\triangle) \propto \exp(c\triangle) - c\triangle - 1$, $c \neq 0$, where $\triangle = \hat{\theta} - \theta$ and $\hat{\theta}$ is an estimator of θ. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively. When c is positive, the overestimation is more serious than underestimation and the situation is reverse when c is negative. For $c = 1$, the LLF is quite asymmetric (about zero) with overestimation being more costly than underestimation. If c is close to zero, the LINEX loss is approximately the squared error loss and therefore almost symmetric. By Zellner (1986), the Bayes estimator of θ under the LLF was given by $\hat{\theta}_L = -(1/c)\log[E_\pi(e^{-c\theta})]$, provided that the expectation exists and is finite.

Secondly, the SLF proposed by Thompson and Basu (1996), is a generalization of LLF. It has the following form. $L(\triangle) \propto \exp(c\triangle) + d\triangle^2 - c\triangle - 1$, $d > 0$, c and $\triangle$ are as before. Hence, if $d = 0$, the SLF is identical with the LLF; if $c = 0$, it reduces to the SELF. Under the SLF, the Bayes estimator of θ is $\hat{\theta}_{SL} = \hat{\theta}_L + (1/c)\log[1 + (2d/c)(\hat{\theta}_s - \hat{\theta}_{SL})]$.

The exponentiated inverse Weibull distribution is the most attractive generalization of inverse

Weibull distribution and provides a better fit for real life data compared to an inverse Weibull distribution. In this paper, we develop a Bayesian estimator in the context of record statistics values from the EIWD. We obtained Bayesian estimators using the squared error loss function (quadratic loss) and LINEX loss function. This is done with respect to the conjugate priors for the shape and scale parameters.

The outline of the remaining sections is as follows. In Section 2, we develop the exact form of the single moment and the maximum likelihood estimators(MLEs) of lower record values from the EIWD. Section 3 details Bayesian estimation in the context of record statistics values from the EIWD under three types loss functions. In Section 4, we also analyze application examples to illustrate the application of different derived estimators. Finally, in the estimated risks, the Bayes estimators are compared with MLEs through Monte Carlo simulations.

2. Maximum Likelihood Estimation

In this section, we consider the MLEs of the unknown parameters and reliability function $R(t)$ in an exponentiated inverse Weibull distribution based on lower record values. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed(iid) random variables with cdf $F(x)$ and pdf $f(x)$. Setting $Y_n = \min(X_1, X_2, \ldots, X_n)$, $n \geq 1$, we say that X_j is a lower record and denoted by $X_{L(j)}$ if $Y_j < Y_{j-1}$, $j > 1$. The indices at which the lower record values occur are given by the record times $\{L(n),\ n \geq 1\}$, where $L(n) = \min\{j | j > L(n-1),\ X_j < X_{L(n-1)}\}$, $n > 1$, with $L(1) = 1$. The corresponding likelihood function of the first n lower record values, $x_{L(1)}, \ldots, x_{L(n)}$ is

$$L = f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{L(i)})}{F(x_{L(i)})}. \tag{2.1}$$

Suppose we observe n lower record values $x_{L(1)}, \ldots, x_{L(n)}$ from the exponentiated inverse Weibull distribution with pdf (1.1). It follows, from (1.1), (1.2), and (2.1), that

$$L(\alpha, \beta, \gamma) = \left(\frac{\alpha\gamma}{\beta^{\gamma}}\right)^{n} \exp\left(-\frac{\alpha}{(\beta x_{L(n)})^{\gamma}}\right) \prod_{i=1}^{n} x_{L(i)}^{-\gamma-1}. \tag{2.2}$$

As a property of lower record values, its k^{th} moment can be obtained by

$$E\left(X_{L(n)}^{k}\right) = \frac{\alpha^{\frac{k}{\gamma}}}{\beta^{k}} \frac{\Gamma(n - k/\gamma)}{\Gamma(n)}, \quad \gamma > k. \tag{2.3}$$

Now, we derive the MLEs of the parameters of the exponentiated inverse Weibull distribution when record values are given as data. From (2.2), the natural logarithm of the likelihood function is given by

$$\log L(\alpha, \beta, \gamma) = n \log \alpha - n\gamma \log \beta + n \log \gamma - \frac{\alpha}{(\beta x_{L(n)})^{\gamma}} - (\gamma + 1) \sum_{i=1}^{n} \log x_{L(i)}. \tag{2.4}$$

From the log-likelihood function (2.4), we obtain the likelihood equations for α, β, and γ as

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \left(\frac{1}{\beta x_{L(n)}}\right)^{\gamma} = 0, \tag{2.5}$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{\gamma n}{\beta} - \frac{\alpha\gamma}{\beta} \left(\frac{1}{\beta x_{L(n)}}\right)^{\gamma} = 0, \tag{2.6}$$

and

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} - n\log\beta + \frac{\alpha \log(\beta x_{L(n)})}{(\beta x_{L(n)})^{\gamma}} - \sum_{i=1}^{n} \log x_{L(i)} = 0. \tag{2.7}$$

By solving the above equations, we can find the following MLEs of the unknown parameters α, β, and γ.

$$\hat{\alpha} = n\left(\hat{\beta} x_{L(n)}\right)^{\hat{\gamma}}, \tag{2.8}$$

$$\hat{\beta} = \left(\frac{\hat{\alpha}}{n}\right)^{\frac{1}{\hat{\gamma}}} x_{L(n)}^{-1}, \tag{2.9}$$

and

$$\hat{\gamma} = \frac{n}{n\log\hat{\beta} + \sum_{i=1}^{n} \log x_{L(i)} - \hat{\alpha}\left(\hat{\beta} x_{L(n)}\right)^{-\hat{\gamma}} \log\left(\hat{\beta} x_{L(n)}\right)}. \tag{2.10}$$

The MLE $\hat{\gamma}$ in (2.10), in conjunction with the MLE $\hat{\beta}$ in (2.9), reduces to

$$\hat{\gamma} = \frac{n}{\sum_{i=1}^{n} \log x_{L(i)} - n\log x_{L(n)}}. \tag{2.11}$$

By the invariance property of the MLE, we can obtain the MLE of reliability function $R(t)$ to be

$$\hat{R}(t) = 1 - \exp\left(-\frac{\hat{\alpha}}{\left(\hat{\beta} t\right)^{\hat{\gamma}}}\right). \tag{2.12}$$

3. Bayesian Estimation

In this section, we estimate α, β, γ, and $R(t)$, through consideration of symmetric loss function and two types of asymmetric loss functions and discuss method to obtain hyperparameters.

3.1. Unknown parameter α

Under the assumption that parameters β and γ are known, a natural conjugate prior for the parameter α is a gamma prior as follows.

$$\pi(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-\alpha b}, \quad \alpha > 0,\ a, b > 0. \tag{3.1}$$

It follows, from (3.1), that the posterior distribution of α is given by

$$\pi(\alpha|\mathbf{x}) = \frac{\left(b + (\beta x_{L(n)})^{-\gamma}\right)^{n+a}}{\Gamma(n+a)} \alpha^{n+a-1} e^{-\alpha(b+(\beta x_{L(n)})^{-\gamma})}, \tag{3.2}$$

which is a Gamma $(n + a,\ b + (\beta x_{L(n)})^{-\gamma})$.

From (3.2), the Bayes estimators of α and $R(t)$ based on the SELF can be derived, respectively, as

$$\hat{\alpha}_s = \int_0^{\infty} \alpha\pi(\alpha|\mathbf{x}) d\alpha = \frac{n+a}{b + (\beta x_{L(n)})^{-\gamma}} \tag{3.3}$$

and

$$\hat{R}_s(t) = \int_0^\infty R(t)\pi(\alpha|\mathbf{x})d\alpha = 1 - \left(1 + \frac{(\beta t)^{-\gamma}}{b + (\beta x_{L(n)})^{-\gamma}}\right)^{-(n+a)}. \tag{3.4}$$

Likewise, the Bayes estimators of α and $R(t)$ based on the LLF can be derived, respectively, as

$$\hat{\alpha}_L = -\frac{1}{c}\log\int_0^\infty e^{-c\alpha}\pi(\alpha|\mathbf{x})d\alpha = \frac{n+a}{c}\log\left(1 + \frac{c}{b + (\beta x_{L(n)})^{-\gamma}}\right) \tag{3.5}$$

and

$$\begin{aligned}\hat{R}_L(t) &= -\frac{1}{c}\log\int_0^\infty e^{-cR(t)}\pi(\alpha|\mathbf{x})d\alpha \\ &= -\frac{1}{c}\log\left[e^{-c}\sum_{m=0}^{\infty}\frac{c^m}{m!}\left(1 + \frac{m(\beta t)^{-\gamma}}{b + (\beta x_{L(n)})^{-\gamma}}\right)^{-(n+a)}\right] \\ &= 1 - \frac{1}{c}\log\left[\sum_{m=0}^{\infty}\frac{c^m}{m!}\left(1 + \frac{m(\beta t)^{-\gamma}}{b + (\beta x_{L(n)})^{-\gamma}}\right)^{-(n+a)}\right].\end{aligned} \tag{3.6}$$

Also, the Bayes estimators of α and $R(t)$ based on the SLF are

$$\hat{\alpha}_{SL} = \hat{\alpha}_L + \frac{1}{c}\log\left[1 + \frac{2d}{c}(\hat{\alpha}_s - \hat{\alpha}_{SL})\right] \tag{3.7}$$

and

$$\hat{R}_{SL}(t) = \hat{R}_L(t) + \frac{1}{c}\log\left[1 + \frac{2d}{c}\left(\hat{R}_s(t) - \hat{R}_{SL}(t)\right)\right]. \tag{3.8}$$

Since the Bayes estimators (3.7) and (3.8) are not explicit form, we can solve through the use of a numerical method such as Newton-Raphson.

To assess the performance of the MLEs and the Bayes estimators, we simulate the estimated risks of all derived estimators through a Monte Carlo simulation method when the parameters β and γ are known. The following procedure is required to obtain estimated risks. After setting $E(\alpha)$ and $\mathrm{Var}(\alpha)$ from the prior density (3.1), we obtain the hyperparameters a and b of the gamma prior (3.1) by solving them. Note that $E(\alpha)$ is the actual value for α. We generate the lower record values from the exponentiated inverse Weibull distribution with $\alpha = E(\alpha)$. By using these values, we can finally obtain the Bayes estimators. The estimated risks for each estimator are calculated as the average of their squared deviations for 10,000 repetitions. It is expressed as

$$\frac{1}{n}\sum_{i=1}^{n}\left(\theta_t - \hat{\theta}\right)^2.$$

Here θ_t and $\hat{\theta}$ is the actual value and the estimate of θ, respectively. The estimated risks of α and $R(t)$ are given in Table 1. The Bayes estimators based on the SELF, the LLF, and the SLF are denoted by BS, BL, and BSL, respectively.

3.2. Unknown parameters (α, β)

In the case of the two parameters problem, we need to specify a general joint prior for α and β that may lead to computational complexities. To avoid this problem, we consider Soland's method. Soland (1969) considered a family of joint prior distribution that places continuous distribution on the scale parameter and discrete distributions on the shape parameter.

Suppose that β is restricted to the values $\beta_1, \beta_2, \ldots, \beta_J$ with prior probabilities $\eta_1, \eta_2, \ldots, \eta_J$, that is,

$$\pi(\beta_j) = P[\beta = \beta_j] = \eta_j, \quad j = 1, 2, \ldots, J. \tag{3.9}$$

Further, suppose that the conditional α upon $\beta = \beta_j$ has a natural conjugate prior as gamma(a_j, b_j) with pdf

$$\pi(\alpha|\beta = \beta_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \alpha^{a_j - 1} e^{-\alpha b_j}, \quad \alpha > 0,\ a_j, b_j > 0. \tag{3.10}$$

Combining (2.2) and (3.10), we get the conditional posterior of $\alpha|\beta = \beta_j$ as

$$\pi(\alpha|\beta = \beta_j,\ \mathbf{x}) = \frac{\left(b_j + (\beta_j x_{L(n)})^{-\gamma}\right)^{n+a_j}}{\Gamma(n + a_j)} \alpha^{n+a_j-1} e^{-\alpha\left(b_j + (\beta_j x_{L(n)})^{-\gamma}\right)} \tag{3.11}$$

which is a Gamma$(n + a_j,\ b_j + (\beta_j x_{L(n)})^{-\gamma})$.

In view of the discrete version of Bayes theorem, we obtain the marginal posterior of β as

$$\begin{aligned}\pi_M(\beta_j|\mathbf{x}) &\propto \int_0^\infty L(\alpha, \beta)\pi(\alpha|\beta = \beta_j)\pi(\beta_j) d\alpha \\ &= \frac{\eta_j b_j^{a_j} \Gamma(n + a_j)\gamma^n \prod_{i=1}^n x_{L(i)}^{-\gamma-1}}{\Gamma(a_j)\beta_j^{\gamma n} \left(b_j + (\beta_j x_{L(n)})^{-\gamma}\right)^{n+a_j}}.\end{aligned} \tag{3.12}$$

Hence, we get

$$\pi_M(\beta_j|\mathbf{x}) = G(\beta) \frac{\eta_j b_j^{a_j} \Gamma(n + a_j)\gamma^n \prod_{i=1}^n x_{L(i)}^{-\gamma-1}}{\Gamma(a_j)\beta_j^{\gamma n} \left(b_j + (\beta_j x_{L(n)})^{-\gamma}\right)^{n+a_j}}, \tag{3.13}$$

where $G(\beta)$ is the normalizing constant given by

$$G^{-1}(\beta) = \sum_{j=1}^J \frac{\eta_j b_j^{a_j} \Gamma(n + a_j)\gamma^n \prod_{i=1}^n x_{L(i)}^{-\gamma-1}}{\Gamma(a_j)\beta_j^{\gamma n} \left(b_j + (\beta_j x_{L(n)})^{-\gamma}\right)^{n+a_j}}. \tag{3.14}$$

Multiplying (3.11) by (3.13), we also obtain the marginal posterior of α as

$$\pi_{M_1}(\alpha|\mathbf{x}) = \sum_{j=1}^J \pi(\alpha|\beta = \beta_j, \mathbf{x})\pi_M(\beta_j|\mathbf{x}). \tag{3.15}$$

From (3.13) and (3.15), the Bayes estimators of α, β, and $R(t)$ based on the SELF are derived, respectively, as

$$\begin{aligned}
\hat{\alpha}_s &= \int_0^\infty \alpha\pi_{M_1}(\alpha|\mathbf{x})d\alpha = \sum_{j=1}^J \pi_M(\beta_j|\mathbf{x}) \int_0^\infty \alpha\pi(\alpha|\beta = \beta_j, \mathbf{x})d\alpha \\
&= \sum_{j=1}^J \pi_M(\beta_j|\mathbf{x}) \frac{\left(b_j + (\beta_j x_{L(n)})^{-\gamma}\right)^{n+a_j}}{\Gamma(n + a_j)} \int_0^\infty e^{-\alpha(b_j+(\beta_j x_{L(n)})^{-\gamma})}\alpha^{n+a_j}d\alpha \\
&= \sum_{j=1}^J \pi_M(\beta_j|\mathbf{x}) \frac{n + a_j}{b_j + (\beta_j x_{L(n)})^{-\gamma}}, \qquad (3.16)
\end{aligned}$$

$$\hat{\beta}_s = \sum_{j=1}^J \beta_j\pi_M(\beta_j|\mathbf{x}), \qquad (3.17)$$

and

$$\begin{aligned}
\hat{R}_s(t) &= \int_0^\infty \pi_{M_1}(\alpha|\mathbf{x})R(t)d\alpha = \int_0^\infty \pi_{M_1}(\alpha|\mathbf{x})\left(1 - e^{-\alpha(\beta_j t)^{-\gamma}}\right)d\alpha \\
&= \int_0^\infty e^{-\alpha(b_j+(\beta_j x_{L(n)})^{-\gamma})}\alpha^{n+a_j-1}\left(1 - e^{-\alpha(\beta_j t)^{-\gamma}}\right)d\alpha \times \sum_{j=1}^J \frac{\left(b_j + (\beta_j x_{L(n)})^{-\gamma}\right)^{n+a_j}}{\Gamma(n + a_j)} \\
&= 1 - \sum_{j=1}^J \pi_M(\beta_j|\mathbf{x})\left(1 + \frac{(\beta_j t)^{-\gamma}}{b_j + (\beta_j x_{L(n)})^{-\gamma}}\right)^{-(n+a_j)}. \qquad (3.18)
\end{aligned}$$

The Bayes estimator of a function $g(\alpha, \beta)$ based on the LLF is given by

$$\hat{g}(\alpha, \beta) = -\frac{1}{c}\log\left[E\left(e^{-cg(\alpha,\beta)}\right)\right] \qquad (3.19)$$

which can be written as

$$\hat{g}(\alpha, \beta) = -\frac{1}{c}\log\left[\sum_{j=1}^J \pi_M(\beta_j|\mathbf{x}) \int e^{-cg(\alpha,\beta)}\pi(\alpha|\beta = \beta_j, \mathbf{x})d\alpha\right]. \qquad (3.20)$$

By using (3.20), the Bayes estimators of α, β, and $R(t)$ based on the LLF are derived, respectively, as

$$\begin{aligned}
\hat{\alpha}_L &= -\frac{1}{c}\log\int_0^\infty \pi_{M_1}(\alpha|\mathbf{x})e^{-c\alpha}d\alpha \\
&= -\frac{1}{c}\log\left[\sum_{j=1}^J \pi_M(\beta_j|\mathbf{x})\left(\frac{b_j + (\beta_j x_{L(n)})^{-\gamma}}{b_j + (\beta_j x_{L(n)})^{-\gamma} + c}\right)^{n+a_j}\right] \\
&= -\frac{1}{c}\log\left[\sum_{j=1}^J \pi_M(\beta_j|\mathbf{x})\left(1 + \frac{c}{b_j + (\beta_j x_{L(n)})^{-\gamma}}\right)^{-(n+a_j)}\right], \qquad (3.21)
\end{aligned}$$

$$\hat{\beta}_L = -\frac{1}{c}\log\left[\sum_{j=1}^J \pi_M(\beta_j|\mathbf{x})e^{-c\beta_j}\right], \qquad (3.22)$$

and

$$\begin{aligned}\hat{R}_L(t) &= -\frac{1}{c}\log\int_0^\infty \pi_{M_1}(\alpha|\mathbf{x})e^{-cR(t)}d\alpha\\ &= -\frac{1}{c}\log\left[e^{-c}\sum_{j=1}^{J}\sum_{m=0}^{\infty}\pi_M(\beta_j|\mathbf{x})\frac{c^m}{m!}\left(1+\frac{m(\beta_j t)^{-\gamma}}{b_j+(\beta_j x_{L(n)})^{-\gamma}}\right)^{-(n+a_j)}\right]\\ &= 1-\frac{1}{c}\log\left[\sum_{j=1}^{J}\sum_{m=0}^{\infty}\pi_M(\beta_j|\mathbf{x})\frac{c^m}{m!}\left(1+\frac{m(\beta_j t)^{-\gamma}}{b_j+(\beta_j x_{L(n)})^{-\gamma}}\right)^{-(n+a_j)}\right] \end{aligned} \tag{3.23}$$

because of

$$\exp\left(ce^{-\alpha(\beta t)^{-\gamma}}\right) = \sum_{m=0}^{\infty}\frac{c^m}{m!}\exp\left(-\frac{\alpha m}{(\beta t)^{\gamma}}\right). \tag{3.24}$$

On the basis of the Bayes estimators based on the SELF and the LLF, we can find the following Bayes estimators based on the SLF.

$$\hat{\alpha}_{SL} = \hat{\alpha}_L + \frac{1}{c}\log\left[1+\frac{2d}{c}\left(\hat{\alpha}_s-\hat{\alpha}_{SL}\right)\right], \tag{3.25}$$

$$\hat{\beta}_{SL} = \hat{\beta}_L + \frac{1}{c}\log\left[1+\frac{2d}{c}\left(\hat{\beta}_s-\hat{\beta}_{SL}\right)\right], \tag{3.26}$$

and

$$\hat{R}_{SL}(t) = \hat{R}_L(t) + \frac{1}{c}\log\left[1+\frac{2d}{c}\left(\hat{R}_s(t)-\hat{R}_{SL}(t)\right)\right]. \tag{3.27}$$

As mentioned in the Section 3.1, $\hat{\alpha}_{SL}, \hat{\beta}_{SL}$, and $\hat{R}_{SL}(t)$ are obtained by applying the numerical method.

3.3. Unknown parameters (α, β, γ)

For the same reason, we expand the method employed by Soland (1969). Suppose that β and γ are restricted to a finite number of values $\beta_1, \beta_2, \ldots, \beta_J$ and $\gamma_1, \gamma_2, \ldots, \gamma_K$ with prior probabilities $\eta_1, \eta_2, \ldots, \eta_J$ and $\zeta_1, \zeta_2, \ldots, \zeta_K$, respectively. That is,

$$\pi(\beta_j) = P[\beta = \beta_j] = \eta_j, \quad j = 1, 2, \ldots, J \tag{3.28}$$

and

$$\pi(\gamma_k) = P[\gamma = \gamma_k] = \zeta_k, \quad k = 1,\ 2, \ldots, K. \tag{3.29}$$

Now, assume that the conditional α upon $\beta = \beta_j$ and $\gamma = \gamma_k$, $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, K$ has a gamma (a_{jk}, b_{jk}) prior with pdf

$$\pi(\alpha|\beta = \beta_j, \gamma = \gamma_k) = \frac{b_{jk}^{a_{jk}}}{\Gamma(a_{jk})}\alpha^{a_{jk}-1}e^{-\alpha b_{jk}}, \quad \alpha > 0,\ a_{jk}, b_{jk} > 0. \tag{3.30}$$

Then, the conditional posterior of $\alpha|\beta = \beta_j,\ \gamma = \gamma_k$ and the marginal joint posterior of (β, γ) can be obtained by

$$\pi\left(\alpha|\beta = \beta_j,\ \gamma = \gamma_k,\ \mathbf{x}\right) = \frac{\left(b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}\right)^{n+a_{jk}}}{\Gamma(n + a_{jk})}\alpha^{n+a_{jk}-1}e^{-\alpha(b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k})} \tag{3.31}$$

and

$$\pi_M(\beta_j, \gamma_k|\mathbf{x}) = G(\beta, \gamma)\frac{\eta_j\zeta_k b_{jk}^{a_{jk}}\Gamma(n + a_{jk})\gamma_k^n \prod_{i=1}^n x_{L(i)}^{-\gamma_k-1}}{\Gamma(a_{jk})\beta_j^{\gamma_k n}\left(b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}\right)^{n+a_{jk}}}, \tag{3.32}$$

where $G(\beta, \gamma)$ is the normalizing constant given by

$$G^{-1}(\beta, \gamma) = \sum_{j=1}^{J}\sum_{k=1}^{K}\frac{\eta_j\zeta_k b_{jk}^{a_{jk}}\Gamma(n + a_{jk})\gamma_k^n \prod_{i=1}^n x_{L(i)}^{-\gamma_k-1}}{\Gamma(a_{jk})\beta_j^{\gamma_k n}\left(b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}\right)^{n+a_{jk}}}. \tag{3.33}$$

Note that $\alpha|\beta = \beta_j,\ \gamma = \gamma_k$ has a Gamma$(n + a_{jk},\ b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k})$.

Using (3.31) and (3.32), we can obtain the marginal posterior of α as

$$\pi_{M_2}(\alpha|\mathbf{x}) = \sum_{j=1}^{J}\sum_{k=1}^{K}\pi\left(\alpha|\beta = \beta_j, \gamma_k, \mathbf{x}\right)\pi_M\left(\beta_j, \gamma_k|\mathbf{x}\right). \tag{3.34}$$

From (3.32) and (3.34), the Bayes estimators of α, β, γ, and $R(t)$ based on the SELF are derived, respectively, as

$$\begin{aligned}\hat{\alpha}_s &= \int_0^\infty \alpha\pi_{M_2}(\alpha|\mathbf{x})d\alpha = \sum_{j=1}^{J}\sum_{k=1}^{K}\pi_M(\beta_j, \gamma_k|\mathbf{x})\int_0^\infty \alpha\pi(\alpha|\beta = \beta_j, \gamma = \gamma_k, \mathbf{x})d\alpha \\ &= \sum_{j=1}^{J}\sum_{k=1}^{K}\pi_M(\beta_j, \gamma_k|\mathbf{x})\frac{n + a_{jk}}{b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}},\end{aligned} \tag{3.35}$$

$$\hat{\beta}_s = \sum_{j=1}^{J}\sum_{k=1}^{K}\beta_j\pi_M(\beta_j, \gamma_k|\mathbf{x}), \tag{3.36}$$

$$\hat{\gamma}_s = \sum_{j=1}^{J}\sum_{k=1}^{K}\gamma_k\pi_M(\beta_j, \gamma_k|\mathbf{x}), \tag{3.37}$$

and

$$\begin{aligned}\hat{R}_s(t) &= \int_0^\infty \pi_{M_2}(\alpha|\mathbf{x})R(t)d\alpha = \int_0^\infty \pi_{M_2}(\alpha|\mathbf{x})\left(1 - e^{-\alpha(\beta_j t)^{-\gamma_k}}\right)d\alpha \\ &= \sum_{j=1}^{J}\sum_{k=1}^{K}\pi_M(\beta_j, \gamma_k|\mathbf{x})\int_0^\infty \pi(\alpha|\beta = \beta_j, \gamma = \gamma_k, \mathbf{x})\left(1 - e^{-\alpha(\beta_j t)^{-\gamma_k}}\right)d\alpha \\ &= 1 - \sum_{j=1}^{J}\sum_{k=1}^{K}\pi_M(\beta_j, \gamma_k|\mathbf{x})\left(1 + \frac{(\beta_j t)^{-\gamma_k}}{b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})}.\end{aligned} \tag{3.38}$$

Similarly, the Bayes estimators of α, β, γ, and $R(t)$ based on the LLF are obtained by

$$\hat{\alpha}_L = -\frac{1}{c}\log\int_0^\infty \pi_{M_2}(\alpha|\mathbf{x})e^{-c\alpha}d\alpha$$

$$= -\frac{1}{c}\log\left[\sum_{j=1}^{J}\sum_{k=1}^{K}\pi_M(\beta_j,\gamma_k|\mathbf{x})\left(1+\frac{c}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})}\right], \tag{3.39}$$

$$\hat{\beta}_L = -\frac{1}{c}\log\left[\sum_{j=1}^{J}\sum_{k=1}^{K}\pi_M(\beta_j,\gamma_k|\mathbf{x})e^{-c\beta_j}\right], \tag{3.40}$$

$$\hat{\gamma}_L = -\frac{1}{c}\log\left[\sum_{j=1}^{J}\sum_{k=1}^{K}\pi_M(\beta_j,\gamma_k|\mathbf{x})e^{-c\gamma_k}\right], \tag{3.41}$$

and

$$\hat{R}_L(t) = -\frac{1}{c}\log\int_0^\infty \pi_{M_2}(\alpha|\mathbf{x})e^{-cR(t)}d\alpha$$

$$= -\frac{1}{c}\log\left[e^{-c}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{m=0}^{\infty}\pi_M(\beta_j,\gamma_k|\mathbf{x})\frac{c^m}{m!}\left(1+\frac{m(\beta_j t)^{-\gamma_k}}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})}\right]$$

$$= 1-\frac{1}{c}\log\left[\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{m=0}^{\infty}\pi_M(\beta_j,\gamma_k|\mathbf{x})\frac{c^m}{m!}\left(1+\frac{m(\beta_j t)^{-\gamma_k}}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})}\right]. \tag{3.42}$$

Finally, the Bayes estimators of α, β, γ, and $R(t)$ based on the SLF are given by

$$\hat{\alpha}_{SL} = \hat{\alpha}_L + \frac{1}{c}\log\left[1+\frac{2d}{c}\left(\hat{\alpha}_s - \hat{\alpha}_{SL}\right)\right], \tag{3.43}$$

$$\hat{\beta}_{SL} = \hat{\beta}_L + \frac{1}{c}\log\left[1+\frac{2d}{c}\left(\hat{\beta}_s - \hat{\beta}_{SL}\right)\right], \tag{3.44}$$

$$\hat{\gamma}_{SL} = \hat{\gamma}_L + \frac{1}{c}\log\left[1+\frac{2d}{c}\left(\hat{\gamma}_s - \hat{\gamma}_{SL}\right)\right], \tag{3.45}$$

and

$$\hat{R}_{SL}(t) = \hat{R}_L(t) + \frac{1}{c}\log\left[1+\frac{2d}{c}\left(\hat{R}_s(t) - \hat{R}_{SL}(t)\right)\right]. \tag{3.46}$$

In order to apply the methods discussed in this section, we should first extract the values of (β_j, η_j), (γ_k, ζ_k) and the hyperparameters (a_{jk}, b_{jk}) in the conjugate prior (3.30). For each choice of (a_{jk}, b_{jk}), it is difficult to find the prior of α conditioned on each value of β_j and γ_k. An alternative method to obtain the values (a_{jk}, b_{jk}) can be based on the expected value of the reliability function $R(t)$ conditional on $\beta = \beta_j$ and $\gamma = \gamma_k$, which is given using (3.30) by

$$E\left[R(t)|\beta=\beta_j,\ \gamma=\gamma_k\right] = \frac{b_{jk}^{a_{jk}}}{\Gamma(a_{jk})}\int_0^\infty\left(1-\exp\left(-\alpha(\beta_j t)^{-\gamma_k}\right)\right)\alpha^{a_{jk}-1}e^{-\alpha b_{jk}}d\alpha$$

$$= 1-\left(1+\frac{(\beta_j t)^{-\gamma_k}}{b_{jk}}\right)^{-a_{jk}}, \quad t>0. \tag{3.47}$$

Table 1: The maximum flood level over a 20 four–year period (1890–1969)

0.654	0.613	0.315	0.449	0.297	0.402	0.379	0.423	0.379	0.324
0.269	0.740	0.418	0.412	0.494	0.416	0.338	0.392	0.484	0.265

If we are able to specify two values $(t_1, R(t_1))$ and $(t_2, R(t_2))$ from prior beliefs about the distribution, the values of a_{jk} and b_{jk} can be obtained numerically from (3.47). Otherwise, a nonparametric procedure can be used to estimate the corresponding two different values of $R(t)$. We use mid-point estimator for $R(t)$ as a nonparametric method.

4. Application

We present two examples to illustrate the methods of inference discussed in the previous sections.

4.1. Real data

Consider the real data given by Dumonceaux and Antle (1973) which represent the maximum flood level (in millions of cubic feet per second) of the Susquehenna River at Harrisburg, Pennsylvania over a 20 four–year period (1890–1969). This data given in Table 1 has been utilized by some authors such as Maswadah (2003) and Sultan (2008). Maswadah (2003) showed that this real data follow an inverse Weibull distribution giving a rough indication of the goodness of fit for the model.

During this period, 6 lower records of the maximum flood level are observed, they are

$$0.654, \quad 0.613, \quad 0.315, \quad 0.297, \quad 0.269, \quad 0.265.$$

In this example, we use gamma prior for the parameter α and discrete priors for the parameters β and γ. The values of β_j, γ_k and the hyperparameters of the gamma prior (3.30) are derived by the following steps. First, we estimate two values of the reliability function using the mid-point estimator for $R(t_i = x_{L(i)}) = (n-i+0.5)/n$, $i = 1, 2, \ldots, n$. Here, we assume that the reliability for $t_1 = 0.613$ and $t_2 = 0.269$ are, respectively, $R(t_1) = 0.25$, and $R(t_2) = 0.75$. Next, we obtain the MLE $\hat{\gamma} = 2.93565$ from (2.11) based on the above 6 lower record values when $\beta = 1$. Finally, we assume that $\gamma_k = 2.6(0.1)3.2$ and $\beta_j = 0.8(0.1)1.2$. Therefore, the values of the hyperparameters a_{jk} and b_{jk} at each value of β_j and γ_k are obtained by solving the following equations using Newton-Raphson method.

$$1 - \left(1 + \frac{(\beta_j\, 0.613)^{-\gamma_k}}{b_{jk}}\right)^{-a_{jk}} = 0.25 \tag{4.1}$$

and

$$1 - \left(1 + \frac{(\beta_j\, 0.269)^{-\gamma_k}}{b_{jk}}\right)^{-a_{jk}} = 0.75. \tag{4.2}$$

Table 2 shows the values of the hyperparameters and posterior probabilities obtained for each β_j and γ_k. By using entries of Table 2, the ML estimates, and the Bayes estimates of α, β, γ, and $R(t)$ are calculated. The results are given in Table 3. Note that the positive value of c is considered here because overestimation is more serious than underestimation in this example. We see that the MLEs are nearly equal to the Bayes estimates. To check the goodness of fit for the exponentiated inverse Weibull distribution with $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$, we conduct a simple test. A simple plot of 6 lower records of the maximum flood level against the expected values of the first exponentiated inverse Weibull lower record values indicate a very strong correlation (0.895). Besides, we have nearly the same results for

Table 2: Prior information, hyperparameters and the posterior probabilities

$j\,(\eta_j = 1/5)$	β_j	$k\,(\xi_k = 1/7)$	γ_k	a_{jk}	b_{jk}	π_{jk}
1	0.8	1	2.6	1.087650	21.060400	0.027100
		2	2.7	0.963320	19.674200	0.028722
		3	2.8	0.866560	18.673900	0.029543
		4	2.9	0.789100	17.950100	0.029683
		5	3.0	0.725400	17.419000	0.029269
		6	3.1	0.672210	17.045500	0.028424
		7	3.2	0.626990	16.791200	0.027258
2	0.9	1	2.6	1.087670	15.505300	0.027100
		2	2.7	0.963390	14.316900	0.028721
		3	2.8	0.866620	13.429800	0.029543
		4	2.9	0.789070	12.755200	0.029683
		5	3.0	0.725440	12.235600	0.029269
		6	3.1	0.672230	11.832200	0.028424
		7	3.2	0.627070	11.521700	0.027259
3	1.0	1	2.6	1.087550	11.787600	0.027101
		2	2.7	0.963320	10.770700	0.028722
		3	2.8	0.866610	9.998600	0.029543
		4	2.9	0.789040	9.396400	0.029683
		5	3.0	0.725470	8.920400	0.029269
		6	3.1	0.672240	8.535700	0.028424
		7	3.2	0.627030	8.223100	0.027258
4	1.1	1	2.6	1.087570	9.200600	0.027101
		2	2.7	0.963320	8.327000	0.028722
		3	2.8	0.866710	7.658500	0.029543
		4	2.9	0.789140	7.129100	0.029683
		5	3.0	0.725490	6.702600	0.029270
		6	3.1	0.672170	6.350600	0.028423
		7	3.2	0.627140	6.064100	0.027260
5	1.2	1	2.6	1.087660	7.339000	0.027100
		2	2.7	0.963380	6.584200	0.028722
		3	2.8	0.866710	6.002500	0.029543
		4	2.9	0.789080	5.538300	0.029683
		5	3.0	0.725530	5.163300	0.029270
		6	3.1	0.672220	4.850000	0.028424
		7	3.2	0.627080	4.589200	0.027259

the Bayes estimates of α, β, and γ. Therefore, the assumption that these record values are from the exponentiated inverse Weibull distribution seems quite reasonable. The data given in Table 4 are the expected values of the first exponentiated inverse Weibull lower record values with the MLEs $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$.

4.2. Simulation study

Similarly, we also consider simulated data consisting of 6 record values from an exponentiated inverse Weibull distribution. A set of lower record values is generated from the standard exponentiated inverse Weibull distribution with $\alpha = 0.5$ and $\gamma = 2.5$. The actual generated population values of $R(t = 0.5)$ is 0.94089. The following are the simulated 6 lower record values.

$$1.20300, \quad 1.06597, \quad 0.47784, \quad 0.40798, \quad 0.40367, \quad 0.38334.$$

Using the formulae presented in Section 2 and Section 3, we obtain the Bayes and ML estimates of α, β, γ, and $R(t)$. For α and γ, the Bayes estimates are closer to the actual values. Specially, the closest estimate to the actual value is asymmetric Bayes estimate under SLF with $c = 1.5$ and $d = 3.0$. We

Table 3: Estimates of α, β, γ, and $R(t = 0.5)$ for real data

	MLE	BS	BL (BSL)		
			$c = 0.5$ ($d = 0.5$)	$c = 1.5$ ($d = 0.5$)	$c = 2.5$ ($d = 0.5$)
α	0.12162	0.13884	0.13684 (0.13844)	0.13310 (0.13486)	0.12965 (0.13092)
β	1.00000	1.00000	0.99500 (0.99900)	0.98507 (0.98966)	0.97533 (0.97872)
γ	2.93565	2.90122	2.89134 (2.89925)	2.87179 (2.88081)	2.85291 (2.85953)
$R(t)$	0.60565	0.58662	0.58106 (0.58551)	0.56982 (0.57498)	0.55847 (0.56234)

Table 4: Expected values and real data for the simple plot

i	1	2	3	4	5	6
$E\left(X_{L(i)}\right)$	0.66710	0.43986	0.36494	0.32350	0.29595	0.27579
Real Data	0.654	0.613	0.315	0.297	0.269	0.265

Table 5: Estimates of α, β, γ, and $R(t = 0.5)$ for simulated data

	MLE	BS	BL (BSL)		
			$c = 0.5$ ($d = 3.0$)	$c = 1.0$ ($d = 3.0$)	$c = 1.5$ ($d = 3.0$)
α	0.60122	0.51592	0.49625 (0.51558)	0.47863 (0.51094)	0.46277 (0.50130)
β	1.00000	1.00000	0.99500 (0.99983)	0.99002 (0.99860)	0.98507 (0.99592)
γ	2.39933	2.58071	2.57109 (2.58043)	2.56156 (2.57808)	2.55229 (2.57293)
$R(t)$	0.95807	0.91231	0.91050 (0.91224)	0.90858 (0.91778)	0.90657 (0.91074)

also see that the asymmetric Bayes estimates under SLF draw closer to the actual values as c increases; however, the MLEs are the closest to the actual values for β and $R(t)$. These values are given in Table 5.

In general, it is difficult to judge which one is better estimator through a set of sample. A simulation study is conducted to see the efficiency of the Bayes and ML estimation methods in terms of estimated risks. The estimated risks for each estimator are calculated as the average of their squared deviations for 10,000 repetitions according to method discussed in Chapter 4. Samples of lower record values with size $n = 10$, are generated from the exponentiated inverse Weibull distribution with $\alpha = 0.05, \beta = 0.6$, and $\gamma = 1.2$. For $\beta = 0.6$ and $\gamma = 1.2$, we consider the prior over the interval $(0.1, 1.0)$ and $(0.7, 1.6)$ by the discrete priors with β and γ taking the 10 values, each with probability 0.1. To obtain the Bayes estimates, we first calculate two values of the reliability function $R(t_2 = x_{L(2)})$ and $R(t_9 = x_{L(9)})$ and then can obtain the hyperparameters a_{jk} and b_{jk} using the expected value of the $R(t)$ in (3.47). The posterior probabilities are easily calculated from a_{jk} and b_{jk} at each value of β_j and γ_k. Through these steps, we obtain the Bayes estimates. By 10,000 repeating this procedure, the estimated risks for α, β, γ, and $R(t)$ are obtained. For $\alpha = 0.05, \beta = 2$, and $\gamma = 1.5$, the same simulation method is carry out. The results are presented in Table 6. From the table, we can see that the Bayes estimators are generally better than their corresponding MLEs. For α and $R(t)$, the Bayes estimators relative to asymmetric loss function are more efficient than the Bayes estimators under symmetric loss function such as SELF. In addition, the estimated risks of them decrease as c increases for fixed d.

Table 6: The estimated risks of α, β, γ, and $R(t)$ when record values of size is 10

Actual values: (α, β, γ, $R(t = 0.5)$) = (0.05, 0.6, 1.2, 0.19107)

	MLE	BS	BL (BSL) $c = 1.0$ ($d = 0.05$)	BL (BSL) $c = 3.0$ ($d = 0.05$)	BL (BSL) $c = 5.0$ ($d = 0.05$)
α	0.51779	0.21478	0.15282 (0.15798)	0.09004 (0.09111)	0.05991 (0.06033)
β	0.02719	0.00250	0.00826 (0.00760)	0.027400 (0.02698)	0.05048 (0.05017)
γ	0.20427	0.07541	0.08267 (0.08197)	0.09711 (0.09684)	0.11086 (0.11069)
$R(t)$	0.41108	0.27623	0.26283 (0.26403)	0.23540 (0.23583)	0.20828 (0.20852)

Actual values: (α, β, γ, $R(t = 0.5)$) = (0.05, 2, 1.5, 0.04877)

	MLE	BS	BL (BSL) $c = 1.0$ ($d = 0.05$)	BL (BSL) $c = 3.0$ ($d = 0.05$)	BL (BSL) $c = 5.0$ ($d = 0.05$)
α	0.69794	0.14711	0.12347 (0.12553)	0.09815 (0.09884)	0.08557 (0.08609)
β	0.02173	0.00250	0.01226 (0.01170)	0.02196 (0.02150)	0.03307 (0.03268)
γ	0.31917	0.07057	0.07915 (0.07832)	0.09196 (0.09157)	0.11086 (0.09993)
$R(t)$	0.25340	0.08046	0.07190 (0.07226)	0.06673 (0.06694)	0.06204 (0.062176)

For β and γ, the symmetric Bayes estimators are more efficient than the asymmetric Bayes estimators. Not only that but, their estimated risks rather increase as c increases for fixed d.

5. Concluding Remarks

In this paper, we develop Bayes estimators in the context of record statistics values from the exponentiated inverse Weibull distribution. Given non-informative prior distribution for β, it is not clear whether posterior distribution is proper or not. Therefore, we consider joint conjugate prior distribution used by Soland (1969). Using this prior, we can guarantee the existence of Bayes estimators as well as avoiding computational complexities of a joint prior. We derive the Bayes estimators for unknown parameters and reliability function $R(t)$ using Soland's method. Their corresponding MLEs are also obtained. The MLEs are compared with Bayes estimators based on the symmetric and two types asymmetric loss functions in terms of estimated risks. Our results show that the Bayes estimators superior to the MLEs under the informative (conjugate) prior. Specially, the asymmetric Bayes estimators are generally better than the symmetric Bayes estimators provided using a suitable value of c and d.

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