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Asymptotic Behavior of the Weighted Cross-Variation of a Fractional Brownian Sheet

Yoon Tae Kim[1,a]

[a]Department of Statistics, Hallym University

Abstract

By using the techniques of a Malliavin calculus, we study the asymptotic behavior of the weighted cross-variation of a fractional Brownian sheet with a Hurst parameter $H = (H_1, H_2)$ such that $0 < H_1 < 1/2$ and $0 < H_2 < 1/2$.

Keywords: Malliavin calculus, fractional Brownian sheet, cross-variation, multiple stochastic integral.

1. Introduction

Tudor and Viens (2003) obtain the limit of the following sequence in order to derive Itô formula for fractional Brownian sheet(fBs) $(B_z^H, z \in [0,1])$, with Hurst parameter $1/2 < H_1, H_2 < 1$:

$$Q_n(f) = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right)\left(B^H_{\frac{k+1}{n},\frac{l}{n}} - B^H_{\frac{k}{n},\frac{l}{n}}\right)\left(B^H_{\frac{k}{n},\frac{l+1}{n}} - B^H_{\frac{k}{n},\frac{l}{n}}\right). \tag{1.1}$$

However, the limit of the sequence $\{Q_n(f)\}$ does not exist in the case when $0 < H_1, H_2 < 1/2$.

In this paper, we consider the normalized sequence of $\{Q_n(f)\}$ to study the asymptotic behavior of the sequence $\{Q_n(f)\}$ in the case when $0 < H_1, H_2 < 1/2$. More precisely, we state our main result in the following theorem:

Theorem 1. *Let $B^H = (B^H_{s,t}, (s,t) \in [0,1]^2)$ be fBs with Hurst parameter $H = (H_1, H_2)$. Suppose that $f \in C^4(R)$ and $\sup_{s,t\in[0,1]} \mathbb{E}[|f^{(i)}(B^H_{s,t})|^p] < \infty$ for any $p \in (0,\infty)$ and $i = 0, \ldots, 4$. If $0 < H_1 < 1/2$ and $0 < H_2 < 1/2$, then we have that as $n \to \infty$,*

$$\begin{aligned} Q_n^*(f) &= \frac{n^{2(H_1+H_2)}}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right)\left[\left(B^H_{\frac{k+1}{n},\frac{l}{n}} - B^H_{\frac{k}{n},\frac{l}{m}}\right)\left(B^H_{\frac{k}{n},\frac{l+1}{n}} - B^H_{\frac{k}{n},\frac{l}{n}}\right) - \frac{1}{4n^{2H_1+2H_2}}\right] \\ &\xrightarrow{L^2} \frac{1}{4}\int_0^1\int_0^1 f''\left(B^H_{s,t}\right) s^{2H_1} t^{2H_2}\, ds dt. \end{aligned} \tag{1.2}$$

Here the notation $\xrightarrow{L^2}$ denote the convergence in L^2.

Recently in several works, the asymptotic behavior on the weighted power variations of a fractional Brownian motion has been studied by using Malliavin calculus (see Nourdin, 2008; Nourdin

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[1] Professor, Department of Statistics, Hallym University, Chuncheon, Gangwon-do 200-702, Korea.
E-mail: ytkim@hallym.ac.kr

and Nualart, 2008; Nourdin *et al.*, 2010). For the two-parameter processes, a central limit theorem has been obtained in Réveillac, A. (2009a) for the weighted quadratic variations of a standard Brownian sheet. Furthermore, Réveillac (2009b) proved a central limit theorem for the finite-dimensional laws of the weighted quadratic variations of fBs. In Park *et al.* (2011), the authors consider the central limit theorem and Berry-Essen bounds for the non-weighted cross-variation of this type with respect to a standard Brownian sheet. In addition, Kim (2011) proves the central limit theorem for the non-weighted cross-variation of fBs.

2. Preliminaries

This section briefly reviews some basic facts about Malliavin calculus for Gaussian processes. For a more detailed reference, see Nualart (2006). Suppose that $\mathbb{H}$ is a real separable Hilbert space with a scalar product denoted by $< \cdot, \cdot >_{\mathbb{H}}$. Let $B = (B(h), h \in \mathbb{H})$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $E(B(h)B(g)) =< h, g >_{\mathbb{H}}$. In particular, if B is fBs B^H with Hurst parameter $H = (H_1, H_2)$, then the scalar product is given by

$$\langle \mathbf{1}_{[0,a]}, \mathbf{1}_{[0,b]} \rangle_{\mathbb{H}} = \frac{1}{4} \prod_{i=1}^{2} \left(a_i^{2H_i} + b_i^{2H_i} - |a_i - b_i|^{2H_i} \right), \quad \text{for } a, b \in [0,1]^2. \tag{2.1}$$

Kim *et al.* (2008) and Kim *et al.* (2009) have developed the theory of stochastic calculus for fBs B^H. For every $n \geq 1$, let $\mathcal{H}_n$ be the n^{th} Wiener chaos of B^H, that is the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_n(B^H(h)) : h \in \mathbb{H}, ||h||_{\mathbb{H}} = 1\}$, where H_n is the n^{th} Hermite polynomial. We define a linear isometric mapping $I_n : \mathbb{H}^{\odot n} \to \mathcal{H}_n$ by $I_n(h^{\otimes n}) = n! H_n(B^H(h))$, where $\mathbb{H}^{\odot n}$ is the symmetric tensor product. The following duality formula holds

$$\mathbb{E}\left[F I_n(h)\right] = \mathbb{E}\left[\langle D^n F, h \rangle_{\mathbb{H}^{\otimes n}}\right], \tag{2.2}$$

for any element $h \in \mathbb{H}^{\odot n}$ and any random variable $F \in \mathbb{D}^{n,2}$. Here $\mathbb{D}^{n,2}$ is the closure of the set of smooth random variables with respect to the norm

$$||F||_{n,2}^2 = \mathbb{E}\left[F^2\right] + \sum_{k=1}^{n} \mathbb{E}\left[\left\|D^k F\right\|_{\mathbb{H}^{\otimes k}}^2\right],$$

where D^k is the iterative Malliavin derivative.

In this paper we will only use multiple stochastic integrals with respect to a fBs $B^H = (B_z^H, z \in [0,1]^2)$, and in this case the scalar product in $\mathbb{H}$ is defined by (2.1). We will use this notation $\mathbb{H}$ throughout this paper.

If $f \in \mathbb{H}^{\odot p}$, the Malliavin derivative of the multiple stochastic integrals is given by

$$D_z I_n(f_n) = n I_{n-1}\left(f_n(\cdot, z)\right), \quad \text{for } z \in [0,1]^2.$$

Let $\{e_l, l \geq 1\}$ be a complete orthonormal system in $\mathbb{H}$. If $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$, the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, is the element of $\mathbb{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{l_1,\ldots,l_r=1}^{\infty} \langle f, e_{l_1} \otimes \cdots \otimes e_{l_r} \rangle_{\mathbb{H}^{\otimes r}} \langle g, e_{l_1} \otimes \cdots \otimes e_{l_r} \rangle_{\mathbb{H}^{\otimes r}}. \tag{2.3}$$

Notice that the tensor product $f \otimes g$ and the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, are not necessarily symmetric even though f and g are symmetric. We will denote their symmetrizations by $f \tilde{\otimes} g$ and $f \tilde{\otimes}_r g$, respectively. The following formula for the product of the multiple stochastic integrals will be frequently used to prove the main result in this paper:

Proposition 1. *Let $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$ be two symmetric functions. Then*

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{2.4}$$

From Proposition 1, we have

$$\mathbb{E}\left[I_p(f)I_q(g)\right] = \begin{cases} 0, & \text{if } p \neq q, \\ p! \left\langle \tilde{f}, \tilde{g} \right\rangle_{\mathbb{H}^{\otimes p}}, & \text{if } p = q, \end{cases} \tag{2.5}$$

where $\tilde{f}$ denotes the symmetrization of f.

3. The Proof of Theorem

For simplicity, we introduce the following notations:

$$\epsilon_{k,l} = \mathbf{1}_{[\frac{k}{n},\frac{k+1}{n}]\times[0,\frac{l}{n}]}, \qquad \theta_{k,l} = \mathbf{1}_{[0,\frac{k}{n}]\times[\frac{l}{n},\frac{l+1}{n}]} \quad \text{and} \quad \xi_{k,l} = \mathbf{1}_{[0,\frac{k}{n}]\times[0,\frac{l}{n}]}.$$

By using the multiplication formula (2.4) of the multiple stochastic integral, we write

$$\begin{aligned} Q_n^*(f) &= \frac{n^{2(H_1+H_2)}}{n^2} \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) I_2\left(\epsilon_{k,l}\otimes\theta_{k,l}\right) + \frac{n^{2(H_1+H_2)}}{n^2} \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right)\left[\langle \epsilon_{k,l},\theta_{k,l}\rangle_{\mathbb{H}} - \frac{1}{4n^{2H_1+2H_2}}\right] \\ &:= Q_{1,n}^*(f) + Q_{2,n}^*(f). \end{aligned}$$

We compute the limit of the square of the expectation of $Q_{1,n}^*(f)$. Using the formula for the multiplication (2.4) of the multiple stochastic integral, we write

$$\begin{aligned} \mathbb{E}\left[\left|Q_{1,n}^*(f)\right|^2\right] &= \frac{n^{4(H_1+H_2)}}{n^4} \sum_{k,l=0}^{n-1}\sum_{k',l'=0}^{n-1} \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\times I_2\left(\epsilon_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\theta_{\frac{k}{n},\frac{l}{n}}\right) I_2\left(\epsilon_{\frac{k'}{n},\frac{l'}{n}}\tilde{\otimes}\theta_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \\ &= \frac{n^{4(H_1+H_2)}}{n^4} \sum_{r=0}^{2}\sum_{k,l=0}^{n-1}\sum_{k',l'=0}^{n-1} r!\binom{2}{r}^2 \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\times I_{4-2r}\left(\left(\epsilon_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\theta_{\frac{k}{n},\frac{l}{n}}\right)\tilde{\otimes}_r\left(\epsilon_{\frac{k'}{n},\frac{l'}{n}}\tilde{\otimes}\theta_{\frac{k'}{n},\frac{l'}{n}}\right)\right)\right] \\ &:= \frac{n^{4(H_1+H_2)}}{n^4}\left(\ell_0^n + \ell_1^n + \ell_2^n\right). \end{aligned}$$

By using the binomial formula for the derivative and the duality formula (2.2), we obtain

$$\begin{aligned} \ell_0^n &= \sum_{k,l,k',l'=0}^{n-1} \mathbf{E}\left[\left\langle D^4\left(f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right), \left(\epsilon_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\theta_{\frac{k'}{n},\frac{l'}{n}}\right)\tilde{\otimes}\left(\theta_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\epsilon_{\frac{k'}{n},\frac{l'}{n}}\right)\right\rangle_{\mathbb{H}^{\otimes 4}}\right] \\ &= \sum_{p=0}^{4}\binom{4}{p}\sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f^{(p)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(4-p)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right]\times\left\langle \xi^{\otimes p}_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\xi^{\otimes(4-p)}_{\frac{k'}{n},\frac{l'}{n}}, \left(\epsilon_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\theta_{\frac{k'}{n},\frac{l'}{n}}\right)\tilde{\otimes}\left(\theta_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\epsilon_{\frac{k'}{n},\frac{l'}{n}}\right)\right\rangle_{\mathbb{H}^{\otimes 4}} \\ &:= \sum_{p=0}^{4}\ell_{0p}^n. \end{aligned}$$

Define, for $e = 1, 2$,

$$a^{H_e}(i,j) = (i+1)^{2H_e} - i^{2H_e} + |j-i|^{2H_e} - |j-i-1|^{2H_e}$$
$$b^{H_e}(i,j) = |i-j+1|^{2H_e} + |i-j-1|^{2H_e} - 2|i-j|^{2H_e}$$
$$c^{H_e}(i,j) = i^{2H_e} + j^{2H_e} - |i-j|^{2H_e}.$$

Then we have the following estimates:

$$\sup_{i,j=0,\dots,n-1} \left|a^{H_e}(i,j)\right| \le C, \qquad \sum_{i,j=0}^{n-1} \left|a^{H_e}(i,j)\right| \le C n^{2H_e+1} \quad \text{and} \quad \left|c^{H_e}(i,j)\right| \le 2 i^{H_e} j^{H_e}. \tag{3.1}$$

The sum ℓ_{00}^n can be written as

$$\frac{n^{2(H_1+H_2)}}{n^2}\ell_{00}^n = \frac{n^{2(H_1+H_2)}}{n^2}\frac{n^{-8(H_1+H_2)}}{64}\sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right)f^{(4)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right]$$
$$\times (k')^{2H_1} a^{H_1}(k,k') c^{H_1}(k',k) a^{H_1}(k',k') \times (l')^{2H_2} c^{H_2}(l',l) a^{H_2}(l,l') a^{H_2}(l',l').$$

By using (3.1) and the assumption on f, we estimate

$$\frac{n^{4(H_1+H_2)}}{n^4}|\ell_{00}^n| \le C\frac{n^{4(H_1+H_2)}}{n^4}\frac{n^{-8(H_1+H_2)}}{64}\sum_{k,k'=0}^{n-1} k^{H_1}(k')^{3H_1}\left|a^{H_1}(k,k')\right| \times \sum_{l,l'=0}^{n-1} l^{H_2}(l')^{3H_2}\left|a^{H_2}(l,l')\right|$$
$$\le C\frac{n^{4(H_1+H_2)}}{n^4}\frac{n^{-8(H_1+H_2)}}{64} n^{4H_1}\sum_{k,k'=0}^{n-1}\left|a^{H_1}(k,k')\right| \times n^{4H_2}\sum_{l,l'=0}^{n-1}\left|a^{H_2}(l,l')\right|$$
$$\le C\frac{n^{4(H_1+H_2)}}{n^4}\frac{n^{-8(H_1+H_2)}}{64} n^{6(H_1+H_2)+2} \le C\frac{n^{2(H_1+H_2)}}{n^2}.$$

This estimate gives, from $H_1 + H_2 < 1$, that

$$\lim_{n\to\infty}\frac{n^{4(H_1+H_2)}}{n^4}|\ell_{00}^n| = 0.$$

However, we estimate $|\ell_{01}^n| \le C(A_1^n + A_2^n + A_3^n + A_4^n)$, where

$$A_1^n = \sum_{k,l,k',l'=0}^{n-1}\left|\left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\right|,$$

$$A_2^n = \sum_{k,l,k',l'=0}^{n-1}\left|\left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\right|,$$

$$A_3^n = \sum_{k,l,k',l'=0}^{n-1}\left|\left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\right|,$$

$$A_4^n = \sum_{k,l,k',l'=0}^{n-1}\left|\left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k}{n},\frac{l}{n}}\right\rangle_{\mathbb{H}}\right|.$$

Then, by using (3.1), we obtain

$$
\begin{aligned}
A_1^n &= \frac{n^{-8(H_1+H_2)}}{32} \sum_{k,k'=0}^{n-1} (k')^{2H_1} \left| a^{H_1}(k,k) a^{H_1}(k',k') c^{H_1}(k',k) \right| \times \sum_{l,l'=0}^{n-1} l^{2H_2} (l')^{2H_2} \left| a^{H_2}(l',l') a^{H_2}(l,l') \right| \\
&\le C n^{-8(H_1+H_2)} \times n^{4(H_1+H_2)} \sum_{k=0}^{n-1} \left| a^{H_1}(k,k) \right| \sum_{k'=0}^{n-1} \left| a^{H_1}(k',k') \right| \sum_{l,l'=0}^{n-1} \left| a^{H_2}(l,l') \right| \\
&\le C n^{-4(H_1+H_2)} \left(n + n^{2H_2} \right)^2 n^{1+2H_2} \le C n^{-4H_1-2H_2+3}.
\end{aligned}
$$

Similarly, we can estimate

$$
\begin{aligned}
A_2^n &\le C n^{-4(H_1+H_2)} \sum_{k,k'=0}^{n-1} \left| a^{H_1}(k',k)) a^{H_1}(k',k') \right| \sum_{l,l'=0}^{n-1} \left| a^{H_2}(l',l) a^{H_2}(l,l') \right| \\
&\le C n^{-4(H_1+H_2)} \sum_{k,k'=0}^{n-1} \left| a^{H_1}(k',k)) \right| \sum_{l,l'=0}^{n-1} \left| a^{H_2}(l,l') \right| \\
&\le C n^{-4(H_1+H_2)} n^{2+2(H_1+H_2)} \le C n^{2-2(H_1+H_2)}.
\end{aligned}
$$

By a similar estimate as for A_1^n, the sum A_3^n can be estimated as

$$
\begin{aligned}
A_3^n &\le C n^{-4(H_1+H_2)} \sum_{k,k'=0}^{n-1} \left| a^{H_1}(k,k') \right| \sum_{l=0}^{n-1} \left| a^{H_2}(l,l) \right| \sum_{l'=0}^{n-1} \left| a^{H_2}(l',l') \right| \\
&\le C n^{-4(H_1+H_2)} n^{1+2H_1} \left(n + n^{2H_2} \right)^2 \le C n^{3-2H_1-4H_2}.
\end{aligned}
$$

Using the same arguments as for A_2^n yields $A_4^n \le C n^{2-2(H_1+H_2)}$. Hence we have

$$
\lim_{n\to\infty} \frac{n^{4(H_1+H_2)}}{n^4} \left| \ell_{01}^n \right| = 0.
$$

As for ℓ_{02}^n, we write

$$
\begin{aligned}
\ell_{02}^n &= \binom{4}{2} \sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}} \right) f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}} \right) \right] \times \left\langle \xi^{\otimes 2}_{\frac{k}{n},\frac{l}{n}} \tilde{\otimes} \xi^{\otimes(2)}_{\frac{k'}{n},\frac{l'}{n}}, \left(\epsilon_{\frac{k}{n},\frac{l}{n}} \tilde{\otimes} \theta_{\frac{k'}{n},\frac{l'}{n}} \right) \tilde{\otimes} \left(\theta_{\frac{k}{n},\frac{l}{n}} \tilde{\otimes} \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right) \right\rangle_{\mathbb{H}^{\otimes 4}} \\
&= \sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}} \right) f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}} \right) \right] \sum_{i=1}^{6} B_i^n(k,l,k',l') \\
&:= \sum_{p=1}^{6} \ell_{02p}^n,
\end{aligned}
$$

where $B_i^n(k,l,k',l')$ for $k = 1,\dots,6$ are given by

$$\begin{aligned}
B_1^n(k,l,k',l') &= \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}},\\
B_2^n(k,l,k',l') &= \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}},\\
B_3^n(k,l,k',l') &= \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}},\\
B_4^n(k,l,k',l') &= \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}},\\
B_5^n(k,l,k',l') &= \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}},\\
B_6^n(k,l,k',l') &= \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}}.
\end{aligned}$$

By a similar estimate as for ℓ_{01}^n, we have

$$\begin{aligned}
\ell_{021}^n &\le n^{-4(H_1+H_2)} \sum_{k,k'=0}^{n-1} \left| a^{H_1}(k',k)) \right| \sum_{l,l'=0}^{n-1} \left| a^{H_2}(l,l') \right| \\
&\le C n^{2-2(H_1+H_2)}.
\end{aligned}$$

Using the same arguments as for ℓ_{021}^n, we can easily show that

$$\lim_{n\to\infty} \frac{n^{4(H_1+H_2)}}{n^4} \left| \ell_{02p}^n \right| = 0, \quad \text{for } p = 1,\dots,5.$$

As for ℓ_{026}^n, we write

$$\begin{aligned}
\frac{n^{4(H_1+H_2)}}{n^4} \ell_{026}^n = {} & \frac{n^{4(H_1+H_2)}}{n^4} \times \frac{n^{-8(H_1+H_2)}}{16} \sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \\
& \times k^{2H_1}\left((k+1)^{2H_1} - k^{2H_1} - 1\right)(k')^{2H_1}\left((k'+1)^{2H_1} - (k')^{2H_1} - 1\right) \\
& \times l^{2H_2}\left((l+1)^{2H_2} - l^{2H_2} - 1\right)(l')^{2H_2}\left((l'+1)^{2H_2} - (l')^{2H_2} - 1\right). \qquad (3.2)
\end{aligned}$$

Hence it follows from (3.2) that

$$\begin{aligned}
& \frac{n^{4(H_1+H_2)}}{n^4} \left(\ell_{026}^n - \sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \times \frac{1}{16} \left(\frac{k}{n}\right)^{2H_1} \left(\frac{k'}{n}\right)^{2H_1} \left(\frac{l}{n}\right)^{2H_2} \left(\frac{l'}{n}\right)^{2H_2} n^{-4(H_1+H_2)} \right) \\
& = \frac{1}{16} \frac{n^{4(H_1+H_2)}}{n^4} \sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \\
& \quad \times \left(\frac{k}{n}\right)^{2H_1} \left(\frac{k'}{n}\right)^{2H_1} \left\{ \left[\left(\frac{k+1}{n}\right)^{2H_1} - \left(\frac{k}{n}\right)^{2H_1} \right] \left[\left(\frac{k'+1}{n}\right)^{2H_1} - \left(\frac{k'}{n}\right)^{2H_1} \right] \right. \\
& \quad \left. - \left(\frac{1}{n}\right)^{2H_1} \left[\left(\frac{k'+1}{n}\right)^{2H_1} - \left(\frac{k'}{n}\right)^{2H_1} \right] - \left(\frac{1}{n}\right)^{2H_1} \left[\left(\frac{k+1}{n}\right)^{2H_1} - \left(\frac{k}{n}\right)^{2H_1} \right] \right\}
\end{aligned}$$

$$\times\left(\frac{l}{n}\right)^{2H_2}\left(\frac{l'}{n}\right)^{2H_2}\left\{\left[\left(\frac{l+1}{n}\right)^{2H_2}-\left(\frac{l}{n}\right)^{2H_2}\right]\left[\left(\frac{l'+1}{n}\right)^{2H_2}-\left(\frac{l'}{n}\right)^{2H_2}\right]\right.$$
$$\left.-\left(\frac{1}{n}\right)^{2H_2}\left[\left(\frac{l'+1}{n}\right)^{2H_2}-\left(\frac{l'}{n}\right)^{2H_2}\right]-\left(\frac{1}{n}\right)^{2H_2}\left[\left(\frac{l+1}{n}\right)^{2H_2}-\left(\frac{l}{n}\right)^{2H_2}\right]\right\}. \tag{3.3}$$

For $e = 1, 2$, we estimate

$$\sum_{k,k'=0}^{n-1}\left|k^{2H_e}\left((k+1)^{2H_e}-k^{2H_e}-1\right)(k')^{2H_e}\left((k'+1)^{2H_e}-(k')^{2H_e}-1\right)-1\right|$$
$$=\sum_{k,k'=0}^{n-1}k^{2H_e}(k')^{2H_e}\left((k+1)^{2H_e}-k^{2H_e}\right)\left((k'+1)^{2H_e}-(k')^{2H_e}\right)$$
$$+\sum_{k,k'=0}^{n-1}k^{2H_e}(k')^{2H_e}\left((k+1)^{2H_e}-k^{2H_e}\right)+\sum_{k,k'=0}^{n-1}k^{2H_e}(k')^{2H_e}\left((k'+1)^{2H_e}-(k')^{2H_e}\right)$$
$$\leq n^{8H_e}+2n^{6H_e+1}. \tag{3.4}$$

Using the estimate (3.4) and the equation (3.3), we have

$$\frac{n^{4(H_1+H_2)}}{n^4}\left|\ell_{026}^n-\sum_{k,l,k',l'=0}^{n-1}\mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right)f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right]\times\frac{1}{16}\left(\frac{k}{n}\right)^{2H_1}\left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l}{n}\right)^{2H_2}\left(\frac{l'}{n}\right)^{2H_2}n^{-4(H_1+H_2)}\right|$$
$$\leq C\frac{n^{4(H_1+H_2)}}{n^4}\left(1+n^{-2H_1+1}+n^{-2H_2+1}+n^{-2(H_1+H_2)+2}\right). \tag{3.5}$$

Since $2H_1+4H_2 < 3$ and $4H_1+2H_2 < 3$, the right-hand side of (3.5) tends to zero as n goes to infinity. As for ℓ_{03}^n, by a similar estimate as for ℓ_{01}^n, we can easily show that

$$\lim_{n\to\infty}\frac{n^{4(H_1+H_2)}}{n^4}\left|\ell_{03}^n\right| = 0.$$

For ℓ_{04}^n, we note that $|\langle\xi_{k/n,l/n}^{\otimes 4},(\epsilon_{k/n,l/n}\tilde{\otimes}\theta_{k'/n,l'/n})\tilde{\otimes}(\theta_{k/n,l/n}\tilde{\otimes}\epsilon_{k'/n,l'/n})\rangle_{\mathbb{H}^{\otimes 4}}| \leq Cn^{2-2(H_1+H_2)}$. From this estimate, it follows that

$$\lim_{n\to\infty}\frac{n^{4(H_1+H_2)}}{n^4}\left|\ell_{04}^n\right| = 0.$$

In order to estimate ℓ_1^n and ℓ_2^n, we compute the contraction $(\epsilon_{k/n,l/n}\tilde{\otimes}\theta_{k/n,l/n})\tilde{\otimes}_r(\epsilon_{k'/n,l'/n}\tilde{\otimes}\theta_{k'/n,l'/n})$ for $r = 1, 2$. When $r = 1$, for $a, b \in [0,1]^2$,

$$\left(\epsilon_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\theta_{\frac{k}{n},\frac{l}{n}}\right)\tilde{\otimes}_1\left(\epsilon_{\frac{k'}{n},\frac{l'}{n}}\tilde{\otimes}\theta_{\frac{k'}{n},\frac{l'}{n}}\right)(a,b)$$
$$=\frac{1}{4}\left(\left\langle\theta_{\frac{k}{n},\frac{l}{n}},\theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\epsilon_{\frac{k}{n},\frac{l}{n}}\otimes\epsilon_{\frac{k'}{n},\frac{l'}{n}}+\left\langle\theta_{\frac{k}{n},\frac{l}{n}},\epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\epsilon_{\frac{k}{n},\frac{l}{n}}\otimes\theta_{\frac{k'}{n},\frac{l'}{n}}\right.$$
$$\left.+\left\langle\epsilon_{\frac{k}{n},\frac{l}{n}},\theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\theta_{\frac{k}{n},\frac{l}{n}}\otimes\epsilon_{\frac{k'}{n},\frac{l'}{n}}+\left\langle\epsilon_{\frac{k}{n},\frac{l}{n}},\epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\theta_{\frac{k}{n},\frac{l}{n}}\otimes\theta_{\frac{k'}{n},\frac{l'}{n}}\right)(a,b). \tag{3.6}$$

In addition, when $r = 2$,

$$\left(\epsilon_{\frac{k}{n},\frac{l}{n}}\tilde{\otimes}\theta_{\frac{k}{n},\frac{l}{n}}\right)\otimes_2\left(\epsilon_{\frac{k'}{n},\frac{l'}{n}}\tilde{\otimes}\theta_{\frac{k'}{n},\frac{l'}{n}}\right)=\frac{1}{2}\left(\left\langle\theta_{\frac{k}{n},\frac{l}{n}},\theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\left\langle\epsilon_{\frac{k}{n},\frac{l}{n}},\epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}+\left\langle\theta_{\frac{k}{n},\frac{l}{n}},\epsilon_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\left\langle\epsilon_{\frac{k}{n},\frac{l}{n}},\theta_{\frac{k'}{n},\frac{l'}{n}}\right\rangle_{\mathbb{H}}\right). \tag{3.7}$$

From (3.6), the term ℓ_1^n can be written as $\ell_1^n = \sum_{p=1}^4 \ell_{1p}^n$, where

$$\ell_{11}^n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right) I_2\left(\epsilon_{\frac{k}{n},\frac{l}{n}} \otimes \epsilon_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \left\langle \theta_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}},$$

$$\ell_{12}^n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right) I_2\left(\epsilon_{\frac{k}{n},\frac{l}{n}} \otimes \theta_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \left\langle \theta_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}},$$

$$\ell_{13}^n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right) I_2\left(\theta_{\frac{k}{n},\frac{l}{n}} \otimes \epsilon_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \left\langle \epsilon_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}},$$

$$\ell_{14}^n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right) I_2\left(\theta_{\frac{k}{n},\frac{l}{n}} \otimes \theta_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \left\langle \epsilon_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}}.$$

Using the duality formula (2.2) and the assumption on f, we estimate $|\ell_{11}^n| \le C \sum_{p=1}^3 |\ell_{11p}|$, where

$$\ell_{111}^n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \theta_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}}$$

$$\ell_{112}^n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} \left[\left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \theta_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} + \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \theta_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \right]$$

$$\ell_{113}^n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \left\langle \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}} \left\langle \theta_{\frac{k}{n},\frac{l}{n}}, \theta_{\frac{k'}{n},\frac{l'}{n}} \right\rangle_{\mathbb{H}}.$$

For ℓ_{111}^n, we first note that for $e = 1, 2$,

$$\sum_{k,k'=0}^{n-1} \left| b^{H_e}(k,k') \right| \le n \sum_{p=-\infty}^{\infty} \left| |p+1|^{H_e} + |p-1|^{2H_e} - 2|p|^{2H_e} \right| \le Cn. \tag{3.8}$$

From (3.8), we estimate

$$\begin{aligned} \left|\ell_{111}^n\right| &\le Cn^{-4H_1-2H_2} \sum_{k,k'=0}^{n-1} \left| a^{H_1}(k',k) \right| \sum_{l,l'=0}^{n-1} \left| b^{H_2}(l,l') \right| \\ &\le Cn^{-4H_1-2H_2} n^{1+2H_1} n \le Cn^{2-2(H_1+H_2)}. \end{aligned}$$

This estimate implies

$$\lim_{n\to\infty} \frac{n^{4(H_1+H_2)}}{n^4} \left|\ell_{111}^n\right| = 0.$$

By a similar estimate as for ℓ_{111}^n, we can easily show that

$$\lim_{n\to\infty} \frac{n^{4(H_1+H_2)}}{n^4} \left|\ell_{11p}^n\right| = 0, \quad \text{for } p = 2, 3.$$

We can easily show that

$$\lim_{n\to\infty} \frac{n^{4(H_1+H_2)}}{n^4} \left|\ell_{1p}^n\right| = 0, \quad \text{for } p = 2,3,4.$$

As for ℓ_2^n, we estimate, from (3.7),

$$\begin{aligned}\left|\ell_2^n\right| &\le Cn^{-2(H_1+H_2)} \sum_{k,k'=0}^{n-1} \left|b^{H_1}(k,k')\right| \sum_{l,l'=0}^{n-1} \left|b^{H_2}(l,l')\right| + n^{-4(H_1+H_2)} \sum_{k,k'=0}^{n-1} \left|a^{H_1}(k,k')\right| \sum_{l,l'=0}^{n-1} \left|a^{H_2}(l,l')\right| \\ &\le Cn^{2-2(H_1+H_2)},\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \frac{n^{4(H_1+H_2)}}{n^4} \left|\ell_2^n\right| = 0.$$

Combining the above results for ℓ_0, ℓ_1^n and ℓ_2^n, we obtain that as $n \to \infty$,

$$\mathbb{E}\left[\left|Q_{1,n}^*(f)\right|^2\right] \to \frac{1}{16} \int_{[0,1]^4} \mathbb{E}\left[f^{(2)}\left(B_{s,t}^H\right) f^{(2)}\left(B_{u,v}^H\right)\right] (su)^{2H_1} (tv)^{2H_2} dsdtdudv. \tag{3.9}$$

Now we investigate the asymptotic behavior on the sequence of the following expectation:

$$\alpha_n = \mathbb{E}\left[Q_{1,n}^*(f) \frac{1}{4n^2} \sum_{k',l'=0}^{n-1} f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l'}{n}\right)^{2H_2}\right].$$

Then the sequence $\{\alpha(n)\}$ can be written as

$$\begin{aligned}\alpha(n) &= \frac{n^{2(H_1+H_2)}}{4n^4} \sum_{k,l,k',l'=0}^{n-1} \left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l'}{n}\right)^{2H_2} \mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \times \left\langle \xi^{\otimes 2}_{\frac{k}{n},\frac{l}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \otimes \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \\ &+ 2\frac{n^{2(H_1+H_2)}}{4n^4} \sum_{k,l,k',l'=0}^{n-1} \left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l'}{n}\right)^{2H_2} \mathbb{E}\left[f^{(1)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(3)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \times \left\langle \xi_{\frac{k}{n},\frac{l}{n}} \otimes \xi_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \otimes \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \\ &+ \frac{n^{2(H_1+H_2)}}{4n^4} \sum_{k,l,k',l'=0}^{n-1} \left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l'}{n}\right)^{2H_2} \mathbb{E}\left[f\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(5)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right] \times \left\langle \xi^{\otimes 2}_{\frac{k'}{n},\frac{l'}{n}}, \epsilon_{\frac{k}{n},\frac{l}{n}} \otimes \theta_{\frac{k}{n},\frac{l}{n}} \right\rangle_{\mathbb{H}} \\ &:= \alpha_1(n) + \alpha_2(n) + \alpha_3(n).\end{aligned}$$

By using the same arguments as for B_6^n term, we obtain

$$\begin{aligned}\alpha_1(n) = \frac{n^{2(H_1+H_2)}}{16n^4} \sum_{k,l,k',l'=0}^{n-1} \mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right) f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right]\left(\frac{k}{n}\right)^{2H_1}\left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l}{n}\right)^{2H_2}\left(\frac{l'}{n}\right)^{2H_2} \\ \times\left[\left(\frac{k+1}{n}\right)^{2H_1} - \left(\frac{k}{n}\right)^{2H_1} - \left(\frac{1}{n}\right)^{2H_1}\right]\left[\left(\frac{l+1}{n}\right)^{2H_2} - \left(\frac{l}{n}\right)^{2H_2} - \left(\frac{1}{n}\right)^{2H_2}\right].\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\alpha_1(n) - \frac{1}{16n^4}\sum_{k,l,k',l'=0}^{n-1}\mathbb{E}\left[f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right)f^{(2)}\left(B^H_{\frac{k'}{n},\frac{l'}{n}}\right)\right]\left(\frac{k}{n}\right)^{2H_1}\left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l}{n}\right)^{2H_2}\left(\frac{l'}{n}\right)^{2H_2}\right| \\
&\le \frac{Cn^{2(H_1+H_2)}}{16n^4}\sum_{k,l,k',l'=0}^{n-1}\left(\frac{k}{n}\right)^{2H_1}\left(\frac{k'}{n}\right)^{2H_1}\left(\frac{l}{n}\right)^{2H_2}\left(\frac{l'}{n}\right)^{2H_2}\left|\left[\left(\frac{k+1}{n}\right)^{2H_1}-\left(\frac{k}{n}\right)^{2H_1}\right]\left[\left(\frac{l+1}{n}\right)^{2H_2}-\left(\frac{l}{n}\right)^{2H_2}\right]\right. \\
&\quad \left. -\left(\frac{1}{n}\right)^{2H_1}\left[\left(\frac{l+1}{n}\right)^{2H_2}-\left(\frac{l}{n}\right)^{2H_2}\right]-\left(\frac{1}{n}\right)^{2H_2}\left[\left(\frac{k+1}{n}\right)^{2H_1}-\left(\frac{k}{n}\right)^{2H_1}\right]\right| \\
&\le C\frac{n^{2(H_1+H_2)}}{n^4}\left(n^2+n^{-2H_1+3}+n^{-2H_2+3}\right)\to 0 \quad \text{as } n\to\infty. \qquad (3.10)
\end{aligned}$$

We can easily show that $\lim_{n\to\infty}\alpha_k(n) = 0$ for $k = 2, 3$. Therefore, from (3.9) and (3.10), it follows that

$$\lim_{n\to\infty}\mathbb{E}\left[\left|Q^*_{1,n}(f) - \frac{1}{4}\sum_{k,l=0}^{n-1}f^{(2)}\left(B^H_{\frac{k}{n},\frac{l}{n}}\right)\left(\frac{k}{n}\right)^{2H_1}\left(\frac{l}{n}\right)^{2H_2}\frac{1}{n^2}\right|^2\right] = 0. \qquad (3.11)$$

Hence the Riemann sum arguments yield, from (3.11), as $n\to\infty$

$$Q^*_{1,n}(f) \to \frac{1}{4}\int_0^1\int_0^1 f^{(2)}\left(B^H_{s,t}\right)s^{2H_1}t^{2H_2}dsdt \quad \text{in } L^2. \qquad (3.12)$$

However, as n tends to infinity, the sequence $\{Q^*_{2,n}(f)\}$ converges to zero by the following estimate:

$$\begin{aligned}
\left|Q^*_{2,n}(f)\right| &\le C\frac{n^{2(H_1+H_2)}}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\left((k+1)^{2H_1}-k^{2H_1}\right)\left((l+1)^{2H_2}-l^{2H_1}\right) \\
&\quad + \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\left((k+1)^{2H_1}-k^{2H_1}\right) + \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\left((l+1)^{2H_2}-l^{2H_1}\right) \\
&\le C\frac{1}{n^2}\left(n^{2(H_1+H_2)}+n^{1+2H_1}+n^{1+2H_2}\right).
\end{aligned}$$

Hence we complete the proof of our main theorem by (3.12). □

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