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THE RELATIONSHIP BETWEEN NONCOMMUTATIVE AND LORENTZ-VIOLATING PARAMETERS IN QUANTUM

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ABSTRACT. When it comes to Lorentz symmetry violation, there are generally two approaches to studying noncommutative field theory: 1) conventional fields are equivalent to noncommutative fields; however, symmetry groups are larger. 2) The symmetry group is the same as conventional standard model's symmetry group; but fields here are written based on the Seiberg–Witten map. Here by adopting the first approach, we aim to connect Lorentz violation coefficients with noncommutative parameters and compare the results with the second approach's results. Through the experimental values obtained for the Lorentz-violating parameters, we obtain a limit of noncommutative symmetry.

1. INTRODUCTION

All the physical interactions remain invariant under Lorentz transformations and charge-parity-time (CPT) inversion. These two transformations correlate through the CPT theorem, which mooted by Bell, Lüders, and Pauli in 1954 and claims that theories that have Lorentz symmetry absolutely have CPT symmetry as well, and theories in which CPT symmetry is broken contain Lorentz violation; however, the reverse is not always true [1]. Lorentz symmetry is divided into two types: observer Lorentz transformation and particle Lorentz transformation [2]. Lorentz symmetry occurs when these two transformations remain invariant. The CPT symmetry is a combination of charge conjugation (C), parity (P) and time reversal (T). Laws of physics remain unchanged under the CPT operator. Many theories and experiments have been carried out to investigate the correctness of these symmetries. Nevertheless, in recent years, some considerable theoretical evidences of Lorentz and CPT symmetry violation have been observed in the Planck scale domain [3].

Based on these evidences, the quantization of gravity is impossible without breaking Lorentz symmetry. All the theories that are proposed in this regard like the string theory and noncommutative field theory have Lorentz symmetry violation. The standard-model extension (SME) is the best framework to describe theories containing Lorentz and CPT symmetry violation. In this model, all the interaction terms added to the standard model have observer Lorentz symmetry but break particle Lorentz symmetry [2, 4].

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The most general form of the Lagrangian normalized in the quantum-electrodynamic version of SME (QEDE) is written as:

$$\mathcal{L}_{QEDE} = \mathcal{L}_{Fermion} + \mathcal{L}_{Photon} \tag{1.1}$$

where the interaction of Fermions:

$$\mathcal{L}_{Fermion} = \frac{1}{2} i\bar{\psi}\Gamma^{\nu}\overleftrightarrow{D}_{\nu}\psi - M\bar{\psi}\psi \tag{1.2}$$

in which Γ and M are extended as:

$$\Gamma^{\nu} = \gamma^{\nu} + c^{\mu\nu}\gamma_{\mu} + d^{\mu\nu}\gamma_5\gamma_{\mu} \tag{1.3}$$

$$M = m + a_{\mu}\gamma^{\mu} + b_{\mu}\gamma_5\gamma^{\mu} + \frac{1}{2}H^{\mu\nu}\sigma_{\mu\nu} \tag{1.4}$$

Also, the photon term is given by:

$$\mathcal{L}_{Photon} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}k_{F_{\alpha\beta\mu\nu}}F^{\alpha\beta}F^{\mu\nu} + \frac{1}{2}\left(k_{AF}\right)^{\alpha}\varepsilon_{\alpha\beta\mu\nu}A^{\beta}F^{\mu\nu} \tag{1.5}$$

where a_{μ}, b_{μ}, $c_{\mu\nu}$, $d_{\mu\nu}$, $H_{\mu\nu}$, k_{AF}, k_F are constant and real. These coefficients are considered as background tensor fields and responsible for the spontaneous breaking of Lorentz symmetry. Moreover, they are sensitive to special tests and, in fact, control Lorentz and CPT violation terms [4, 28].

Tests that are sensitive to Lorentz and CPT symmetries include particle accelerators, low-energy atomic experiments, and astrophysics. Lorentz violation coefficients are limited through these tests [5] such as comparison of atomic clocks on the earth and in the space [6, 7], investigation of hydrogen and antihydrogen atomic spectra [8], cosmology calculations [9], observation of neutrino oscillations [10], and so forth.

Here first, we study the noncommutative field theory (NFT) as a theory containing Lorentz symmetry violation and introduce the proposed approaches to expressing the NFT. Then, we find the relationship between Lorentz violation coefficients and noncommutative (NC) parameters through the first approach and compare its results with the results of the second approach. Finally, we obtain some limits comparable to others' works for noncommutative symmetry.

2. NONCOMMUTATIVE FIELD THEORY

In recent years, many efforts have been made to study the NFT and its phenomenological results [11]. The idea of noncommutative space-time seriously began from the problem of quantizing an open string in the presence of background field [11].

In comparison to the NFT's counterpart in the commutative space, there are new interactions here that are of great significance in particle and cosmological physics and can open a window to new physics. In the framework of noncommutative space-time, the coordinates are defined as operators and their commutation equation is written as:

$$\left[\hat{x}_{\mu}, \hat{x}_{\nu}\right] = i\theta_{\mu\nu} \tag{2.1}$$

where the NC parameter ($\theta_{\mu\nu}$) is constant, antisymmetric, and real. As (2.1) shows, $\theta_{\mu\nu}$ has square-length dimension. Therefore, it will correlate with the energy scale (s) through $\theta^{\mu\nu} \propto 1/\Lambda_{NC}^2$. The constant NC parameter causes a preferable direction in the space-time and eliminates the isotropy of space-time. This matter motivates us to interconnect noncommutative theories with SME. In other words, the NFT can be a subcategory of SME. The NC parameter serves as background fields in the SME and leads to the breaking of particle Lorentz symmetry [12].

The noncommutative space is constructed by substituting noncommutative functions for commutative functions and converting the normal product into star product in the commutative field theory. Considering (2.1), the star production is expressed as:

$$f * g(x) = \exp\left[\frac{1}{2} i\theta^{\mu\nu} \partial_{x^\mu} \partial_{y^\nu}\right] f(x) g(y) \big| x = y \tag{2.2}$$

which is known as Weyl–Moyal correspondence. The NFT is not easily constructed using this correspondence. The electric charge quantization is one the problems. The group $U(1)$ in noncommutative QED can only describe particles with charge 0 and ± 1 [13]. To solve this problem, two different approaches have been proposed to construct the NFT with gauge symmetry: 1) NFT without using Seiberg–Witten map [14] and NFT by using Seiberg–Witten map [15].

In the first approach, we consider NC fields to be conventional and enlarge the theory's symmetry group. For instance, the standard-model group is considered $U(3)\times U(2)\times U(1)$. Finally, it is reduced to the conventional-standard-model group by two spontaneous symmetry breakings; consequently, two extra Higgs particles are obtained. It suggests that there are more particles in this theory than the standard model [14].

In the second approach, the symmetry group is considered the same as the standard-model symmetry group, $SU(3)\times SU(2)\times U(1)$; however, the fields are a function of NC parameter and determined by Seiberg–Witten map [15]. According to this map, the fields are extended to every arbitrary order of NC parameter [16]. Furthermore, the number of particles here is the same as that of the corresponding theory in the commutative space.

Physicist have made many efforts to study the NFT and its phenomenological aspects through these two approaches and achieved different and interesting results. As an example in the noncommutative standard model, considering Seiberg–Witten map, there is not any vertex factor for the interaction between left-handed neutrino and photon whereas in the second approach for such an interaction, the vertex factor is the same for both right-handed and left-handed neutrinos [17]. In the noncommutative approach, there are only Majorana neutrinos because of Seiberg–Witten map whereas there are Dirac neutrinos in addition to Majorana neutrinos in the opposite approach [18].

Now, we want to consider another difference between these two approaches in Lorentz symmetry violation.

3. NFT AND LORENTZ SYMMETRY VIOLATION WITHOUT SEIBERG–WITTEN MAP

Here, through the first approach in NFT, we get the relationship between Lorentz violation coefficients and NC parameters. This paper is based on only the quantum-electrodynamic part of the standard model. To do so, we consider NC fields to be conventional and develop our gauge symmetry group to a larger symmetry group $U_*(1)$. This group operates in the Weyl–Moyal space constructed based on star product. Since this group has the same form as $U(1)$ [14], its Lagrangian is considered conventional and the normal product is converted into star product.

$$\mathcal{L}_{NC} = \frac{1}{2} i\bar{\hat{\psi}} * \gamma^{\mu} \overleftrightarrow{\hat{D}}_{\mu} \hat{\psi} - m\bar{\hat{\psi}} * \hat{\psi} - \frac{1}{4q^2} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \tag{3.1}$$

The noncommutative photon and Fermion fields are defined respectively as follows:

$$\hat{F}_{\mu\nu} = \partial_{\mu} \hat{A}_{\nu} - \partial_{\nu} \hat{A}_{\mu} - i\left[\hat{A}_{\mu}, \hat{A}_{\nu}\right]_* \tag{3.2}$$

$$\bar{\hat{\psi}} * \overleftrightarrow{\hat{D}}_{\mu} \hat{\psi} \equiv \bar{\hat{\psi}} * \hat{D}_{\mu} \hat{\psi} - \hat{D}_{\mu} \bar{\hat{\psi}} * \hat{\psi} \tag{3.3}$$

The definition of covariant derivation is also obtained as usual and just by substituting star product for normal product.

$$\hat{D}_{\mu} \hat{\psi} = \partial_{\mu} \hat{\psi} - i\hat{A}_{\mu} * \hat{\psi} \tag{3.4}$$

The NFT is considered as a gauge theory. It means that the gauge symmetry remains unchanged under the following transformations [19]:

$$\begin{aligned} &\hat{F}_{\mu\nu} \to U(x) * \hat{F}_{\mu\nu} * U^{-1}(x) \\ &\hat{A}_{\mu} \to U(x) * \hat{A}_{\mu} * U^{-1}(x) + iU(x) * \partial_{\mu} U^{-1}(x) \\ &\hat{\psi} \to U(x) * \hat{\psi} \end{aligned} \tag{3.5}$$

The unitary transformation is written as:

$$\begin{aligned} &U(x) \equiv e_*^{i\lambda} = 1 + i\lambda + \frac{1}{2} \lambda * \lambda + \cdots \\ &U(x) * U^{-1}(x) = 1 \end{aligned} \tag{3.6}$$

Now, the NC fields are applied to the Lagrangian (3.1). Because of the integration over the whole space and antisymmetric NC parameter (see Appendix A), the correction terms that have Lorentz symmetry violation are obtained as:

$$\begin{aligned} \mathcal{L}_{NC}^{LIV} &= \mathcal{L}_{QED} + \frac{1}{4} iq\theta^{\alpha\beta} F_{\alpha\mu} \bar{\psi} \gamma^{\mu} \overleftrightarrow{D}_{\beta} \psi + \frac{1}{4} iq\theta^{\alpha\beta} \left(\partial_{\mu} A_{\alpha}\right) \bar{\psi} \gamma^{\mu} \overleftrightarrow{D}_{\beta} \psi \\ &- \frac{1}{2} q\theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} F^{\mu\nu} + \frac{1}{8} q\theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{3.7}$$

where $\mathcal{L}_{QED}$ denotes the similar Lagrangian in the commutative field theory. The correction terms of the photon part in both proposed approaches are renormalizable on the single-ring surface in the noncommutative space [19]. Equation (3.7) shows that the noncommutative effects vanish in the neutral particles.

Now, selecting a background electromagnetic field, we extend our investigations to more physical cases. To calculate the noncommutative effective Lagrangian, we apply the transformations $A_{\mu} \to A_{\mu}^{ext} + A_{\mu}$ (where A_{μ}^{ext} is the external field) and

$F_{\mu\nu} \to f_{\mu\nu} + F_{\mu\nu}$ (where $f_{\mu\nu}$ is the background constant field corresponding to A_μ^{ext} in the commutation space and $F_{\mu\nu}$ represents small dynamic fluctuations) to the Lagrangian (3.7). Maintaining terms up to the second order with respect to *F*, (3.7) is rewritten as:

$$\mathcal{L}_{NC} = \mathcal{L}_{QED} + \frac{1}{4} iq\theta^{\alpha\beta} f_{\alpha\mu}\bar{\psi}\gamma^\mu \vec{D}_\beta \psi + \frac{1}{4} iq\theta^{\alpha\beta}\left(\partial_\mu A_\alpha^{ext}\right)\bar{\psi}\gamma^\mu \vec{D}_\beta\psi - \frac{1}{2} q\theta^{\alpha\beta} A_\beta\left(\partial_\alpha A_\mu^{ext}\right)\bar{\psi}\gamma^\mu\psi$$

$$-\frac{1}{2} q\theta^{\alpha\beta} f_{\alpha\mu} F_{\beta\nu} F^{\mu\nu} - \frac{1}{2} q\theta^{\alpha\beta} f^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} + \frac{1}{8} q\theta^{\alpha\beta} f_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} q\theta^{\alpha\beta} f_{\mu\nu} F_{\alpha\beta} F^{\mu\nu} \tag{3.8}$$

Comparing (3.8) with the Lagrangian written in QEDE, from the SME below:

$$\mathcal{L}_{LIV} = \frac{1}{2} i\bar{\psi}\gamma^\mu \vec{D}_\mu\psi - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \alpha_\mu \bar{\psi}\gamma^\mu\psi + \frac{1}{2} ic_{\mu\nu}\bar{\psi}\gamma^\mu \vec{D}^\nu\psi - \frac{1}{4}\left(k_F\right)_{\alpha\beta\mu\nu} F^{\alpha\beta}F^{\mu\nu} \tag{3.9}$$

we obtain the Lorentz violation coefficients as follows:

$$c_{\mu\nu} = -\frac{1}{2} qf_\mu^\lambda \theta_{\lambda\nu} + \frac{1}{2} q\left(\partial_\mu A_\lambda^{ext}\right)\theta_\nu^\lambda \tag{3.10}$$

$$\left(k_F\right)_{\alpha\beta\mu\nu} = -4q\theta_{\lambda\alpha} f^\lambda \eta_{\beta\nu} + 2q\theta_{\alpha\mu} f_{\beta\nu} - q\theta_{\alpha\beta} f_{\mu\nu} \tag{3.11}$$

$$a_\mu = \frac{1}{2} q\theta^{\alpha\beta} A_\beta^{ext}\left(\partial_\alpha A_\mu^{ext}\right) \tag{3.12}$$

The electric charge that appears in (3.9) will not change, it means $q_{eff} = q$.

Similarly to what we done in the first approach, the Lorentz violation coefficients have been calculated in the second approach. In this state, the NC fields are extended as:

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\alpha\beta} A_\alpha\left(\partial_\beta A_\mu + F_{\beta\mu}\right)$$

$$\hat{\psi} = \psi - \frac{1}{2}\theta^{\alpha\beta} A_\alpha \partial_\beta \psi \tag{3.13}$$

By inserting them into (3.1), the corrections obtained in this approach are as the form below:

$$\mathcal{L}_{NC}^{LIV} = \frac{1}{2} i\bar{\psi}\gamma^\mu \vec{D}_\mu\psi - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{8} iq\theta^{\alpha\beta} F_{\alpha\beta}\bar{\psi}\gamma^\mu \vec{D}_\mu\psi + \frac{1}{4} iq\theta^{\alpha\beta} F_{\alpha\mu}\bar{\psi}\gamma^\mu \vec{D}_\beta\psi$$

$$+\frac{1}{4} mq\theta^{\alpha\beta} F_{\alpha\beta}\bar{\psi}\psi - \frac{1}{2} q\theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} F^{\mu\nu} + \frac{1}{8} q\theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \tag{3.14}$$

The same as the first approach, this Lagrangian has gauge invariance. Similarly, by applying the background field and comparing with (3.9), the following Lorentz violation coefficients are achieved [12]:

$$c_{\mu\nu} = -\frac{1}{2} qf_\mu^\lambda \theta_{\lambda\nu} \tag{3.15}$$

$$\begin{aligned}(k_F)_{\alpha\beta\mu\nu} &= -qf_{\alpha}^{\lambda}\theta_{\lambda\mu}\eta_{\beta\nu} \\ &+\frac{1}{2}qf_{\alpha\mu}\theta_{\beta\nu}-\frac{1}{4}qf_{\alpha\beta}\theta_{\mu\nu}-(\alpha\leftrightarrow\beta)-(\mu\leftrightarrow\nu)+(\alpha\beta\leftrightarrow\mu\nu)\end{aligned} \tag{3.16}$$

In (3.9), the electric charge that appears in the covariant derivation in this approach is replaced by the effective electric charge, $q_{eff}=(1+(1/4)qf^{\mu\nu}\theta_{\mu\nu})q$.

Comparing these two approaches, it can be seen in the determination of Lorentz violation coefficients that in the first approach and without extending NC fields, the Lorentz violation coefficients $c_{\mu\nu}$ and k_F correlate the NC parameter, $\theta_{\mu\nu}$, and background fields through (3.10) and (3.11), respectively. Moreover, in this approach, there is the Lorentz violation coefficient a_μ as well (see (3.12)) whereas in the second approach, the noncommutative corrections containing Lorentz violation appear in $c_{\mu\nu}$ and k_F through extending the NC fields according to the Seiberg–Witten map. These coefficients are shown by (3.15) and (3.16).

4. NONCOMMUTATIVE SCALE

Many studies have been carried out in the framework of noncommutative effects. Some of them satisfactorily predicted the NC scale [19]. Here we investigate the NC scale resulting from Lorentz symmetry violation. As mentioned before, many tests that contain Lorentz and CPT symmetry violation are conducted in the SME framework. Thanks to advances in laboratory equipment, such tests give robust and more acceptable limits on Lorentz violation coefficients. Because both Lorentz violation coefficients have CPT symmetry, we only take tests containing Lorentz symmetry violation into consideration. The cosmic microwave background (CMB) on which many studies have been performed is characterized as a natural source of photons and electromagnetic waves. Based on these investigations, the Lorentz-violating parameter k_F is obtained of the order of 10^{-31} [20]. Considering the relationship between this coefficient and the NC parameter based on both noncommutative approaches $k_F \sim q\theta B$, the NC scale is approximately of the order of 200 GeV. In comparison to the investigations conducted in the past, this limit increases the order of NC effects by two.

However, in the Fermion part of the QEDE, there are many limits put on $c_{\mu\nu}$ and its combinations [5]. In this regard, atomic clock comparison can be noted. In such experiments, the hyperfine frequencies of atoms and pure ions are precisely measured. With empirical data obtained from the atom comparison ${}^{9}Be_{+}/H$ under a 0.8-T magnetic field, the Lorentz violation coefficient is estimated $|\tilde{c}_J|<10^{-25}$ GeV [21]. Considering (3.10) and (3.15), the NC scale is of the order of 22 TeV, which is of the same order as the result in ref. [12].

New limits of each Lorentz violation coefficient are extended in the different parts of the

SME, which are collected in a new version in 2010 [5]. In Table 1, different limits of NC scale are calculated for the results obtained for $c_{\mu\nu}$ and its combinations in various physical systems.

Table 2 provides the results obtained for the NC scale in the special systems sensitive to photon properties.

TABLE 1. Noncommutative scale with regards to $c_{\mu\nu}$ in the Fermion part.

$c_{\mu\nu}$	Empirical data	$\theta_{\mu\nu}$	Λ_{NC}	Magnetic field B	Physical system
$\left\lvert\frac{1}{2}c^e XY\right\rvert$	$< 8\times10^{-15}$	$< (10^{-1}\text{ GeV})^{-2}$	0.1 GeV	~ 1.7T	Optical amplifiers [22]
$\left\lvert\frac{1}{2}c^e XY\right\rvert$	$\sim 2.1\times10^{-16}$	$< (5\times10^{-1}\text{ GeV})^{-2}$	0.5 GeV	$\sim$ T	Optical and microwave amplifiers [23]
$0.83c_{(TX)}$ $+0.51c_{(TY)}$ $+0.22c_{(TZ)}$	$\sim 4\times10^{-11}$	$< (4\times10^{-5}\text{ GeV})^{-2}$	$\gtrsim 4\times10^{-2}$ MeV	$\sim$ mT	Transition $1S-2S$ [24]
$\lvert\tilde{c}_Q^P\rvert$	$\sim 0.3\times10^{-22}$ GeV	$(7-20\text{ GeV})^{-2}$	$7-20$ GeV	20μT-0.2mT	Cs fusion [7]
$\lvert\tilde{c}_-^P\rvert$	$\sim 1.8\times10^{-25}$ GeV	$(8\times10-4\times10^2\text{ GeV})^{-2}$	$8\times10-4\times10^2$ GeV	20μT-0.2mT	Cs fusion [7]
$\lvert\tilde{c}_J^N\rvert$ $J = X,Y$	$< 10^{-25}$ GeV	$< (21\times10^{23}\text{ GeV})^{-2}$	$\gtrsim 22$ TeV	~ 0.81T	$^9Be_+/H$ comparison [21]
$\lvert\tilde{c}_-^N\rvert,\lvert\tilde{c}_Z^N\rvert$	$< 10^{-27}$ GeV	$< (77\text{ GeV})^{-2}$	$\gtrsim 77$ GeV	$\sim$ mG	$Hg/Hg\&Ne/Ne$ comparison [21]

5. CONCLUSION

As mentioned, both approaches exist in the construction of noncommutation that each of them leads to different and interesting results in the different parts of the standard model. In this paper, we tried to investigate these two approaches' results in Lorentz symmetry violation. Accordingly, we considered the first approach to construct the NFT (without using the Seiberg–Witten map) and obtained its corrections. This Lagrangian that contained Lorentz violation was compared with the Lagrangian of SME in the QED part and the relationship between the Lorentz violation coefficients and NC parameter was found. Here we saw that by considering NC fields as conventional, the Lorentz violation coefficients $c_{\mu\nu}$, k_F , and a_μ correlate with the NC parameter and background electromagnetic field through (3.10), (3.11), and (3.12) respectively. In comparison to the work done in ref. [12] and adopting the second noncommutative approach (using the Seiberg–Witten map), the Lorentz violation coefficients include $c_{\mu\nu}$ and k_F, which are shown in (3.15) and (3.16).

Moreover, using k_F in both approaches and considering CMB, we got the NC scale up to the order $\Lambda_{NC} \approx 200GeV$. Then, we could extend the NC scale to the order 22 TeV through $c_{\mu\nu}$. In this case, the empirical values of $c_{\mu\nu}$ (in the comparison of transition frequencies $^9Be_+ / H$) were used. The calculated NC scales in Tables 1 and 2 indicate that the limit of the NC parameter obtained from the astrophysical system [26] leads to better results.

TABLE 2. Noncommutative scale with regards to k_F in the photon part.

k_F	Empirical data	$\theta_{\mu\nu}$	Λ_{NC}	Magnetic field B	Physical system
k_F	$\sim 10^{-28}$	$< (3\times 10\ \text{GeV})^{-2}$	$\gtrsim 30$ GeV	~ 10 μG	Cosmic sources [4]
k^a $a = 1,2,...,10$	2×10^{-32}	$< (10^3\ \text{GeV})^{-2}$	$\gtrsim 100$ GeV	~ μG	Astrophysics [25]
k^a (some values of a)	2×10^{-37}	$< (2\times 10^5\ \text{GeV})^{-2}$	$\gtrsim 200$ TeV	~ μG	Astrophysics [26]
$(k_B^4)_{20}$	17×10^{-31}	$< (2\times 10^2\ \text{GeV})^{-2}$	$\gtrsim 200$ GeV	~ 10 μG	CMB [20]
$(\tilde{k}_e)^{XJ}$ $J = Y, Z$	$\sim 0.31\times 10^{-17}$	$(3\times 10^{-2}\ \text{GeV})^{-2}$	$\gtrsim 0.03$ GeV	~ 0.6 G	Revolving optical amplifier [27]

APPENDIX A

The density calculation of the noncommutative Lagrangian without using Seiberg–Witten map and calculating Lorentz violation coefficients is provided here. We begin from the Lagrangian in noncommutative QED:

$$\mathcal{L}_{NC} = \frac{1}{2} i\bar{\hat{\psi}} * \gamma^{\mu} \overleftrightarrow{\hat{D}}_{\mu} \hat{\psi} - m\bar{\hat{\psi}} * \hat{\psi} - \frac{1}{4q^2} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \tag{A.1}$$

By assuming NC fields as conventional, considering the definition of star product in (2.2) up to the first order of noncommutation, and using (3.3) and (3.4), the integration of the first term of (A.1) over the whole space is obtained as:

$$S_{NC}^{Fermion} = \int \left\{ \begin{array}{l} \frac{1}{2} i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \frac{1}{2} i\left(\partial_{\mu}\bar{\psi}\right)\gamma^{\mu}\psi \\ +\bar{\psi}\gamma^{\mu}A_{\mu}\psi + \frac{1}{2} i\theta^{\alpha\beta}\left(\partial_{\alpha}\bar{\psi}\right)\gamma^{\mu}\left(\partial_{\beta}A_{\mu}\right)\psi \end{array} \right\} d^4x \tag{A.2}$$

Considering the definition of covariant derivation in the simplified commutative space of (A.2), we will have:

$$\frac{1}{2} i\bar{\psi}\gamma^{\mu}\overleftrightarrow{D}_{\mu}\psi + \frac{1}{2} i\theta^{\alpha\beta}\left(\partial_{\beta}A_{\mu}\right)\left(\partial_{\alpha}\bar{\psi}\right)\gamma^{\mu}\psi \tag{A.3}$$

The other Fermion term in (A.1):

$$\int m\bar{\hat{\psi}} * \hat{\psi} d^4x = \int \left\{ m\bar{\psi}\psi + \frac{1}{2} im\theta^{\alpha\beta}\left(\partial_{\alpha}\bar{\psi}\right)\left(\partial_{\beta}\psi\right) \right\} d^4x \tag{A.4}$$

will be similar to the commutative conventional interaction since the NC parameter is antisymmetric. Therefore, the remaining terms in the Fermion part will be:

$$S_{NC}^{Fermion} = \int \left\{ \frac{1}{2} i\bar{\psi}\gamma^{\mu}\overleftrightarrow{D}_{\mu}\psi - m\bar{\psi}\psi + \frac{1}{2} i\theta^{\alpha\beta}\left(\partial_{\beta}A_{\mu}\right)\left(\partial_{\alpha}\bar{\psi}\right)\gamma^{\mu}\psi \right\} d^4x \tag{A.5}$$

The term of the noncommutative photon part is:

$$S_{NC}^{photon} = -\frac{1}{4q^2} \int \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} d^4x = -\frac{1}{4q^2} \int \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} d^4x \tag{A.6}$$

in which the star product was used as $\int \hat{f} * \hat{g} d^4x = \int \hat{f} . \hat{g} d^4x$.

Now, considering the definition of photon fields in noncommutation based on (3.2), the action is obtained according to the commutative conventional fields in this part.

$$S_{NC}^{photon} = -\frac{1}{4q^2} \int \left\{ F_{\mu\nu}F^{\mu\nu} + 2\theta^{\alpha\beta}F_{\mu\nu}\left(\partial_{\alpha}A_{\mu}\right)\left(\partial_{\beta}A_{\nu}\right) \right\} d^4x \tag{A.7}$$

Hence, the final action of noncommutation without extending fields will be:

$$S_{NC} = S_{QED} + \int \left\{ \begin{array}{l} \frac{1}{2} i\theta^{\alpha\beta}\left(\partial_{\beta}A_{\mu}\right)\left(\partial_{\alpha}\bar{\psi}\right)\gamma^{\mu}\psi \\ -\frac{1}{2q^2}\theta^{\alpha\beta}\left(\partial_{\alpha}A_{\mu}\right)\left(\partial_{\beta}A_{\nu}\right)F^{\mu\nu} \end{array} \right\} d^4x \tag{A.8}$$

where S_{QED} represents the action in the commutative space. Now, because we want to compare this Lagrangian with the Lagrangian related to Lorentz violation, the photon field is defined as $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and the correction terms resulted from noncommutation in (A.8) are rewritten as:

$$\theta^{\alpha\beta}\int\left[\frac{1}{4}i\left(\partial_\alpha A_\mu\right)\bar{\psi}\gamma^\mu\vec{D}_\beta\psi\psi-\frac{1}{2}A_\beta\left(\partial_\alpha A_\mu\right)\bar{\psi}\gamma^\mu\psi\right.$$
$$\left.-\frac{1}{2q^2}\begin{Bmatrix}F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu}+\left(\partial_\mu A_\alpha\right)\left(\partial_\beta A_\nu\right)F^{\mu\nu}\\+\left(\partial_\mu A_\alpha\right)\left(\partial_\beta A_\nu\right)F^{\mu\nu}-\left(\partial_\mu A_\alpha\right)\left(\partial_\nu A_\beta\right)F^{\mu\nu}\end{Bmatrix}\right]d^4x \tag{A.9}$$

Since the fields are removed at infinity based on the divergence theorem and we maintain the Lorentz violation terms, the last four terms in (A.9) are transformed as:

$$\theta^{\alpha\beta}\int-\frac{1}{2q^2}\left\{F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu}-A_\alpha\left(\partial_\beta F_{\mu\nu}\right)F^{\mu\nu}\right\} \tag{A.10}$$

Considering the establishment of:

$$\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}=4\theta^{\alpha\beta}A_\alpha F^{\mu\nu}\left(\partial_\beta F_{\mu\nu}\right) \tag{A.11}$$

Equation (A.10) can be shown as:

$$-\frac{1}{2q^2}\theta^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu}+\frac{1}{8q^2}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \tag{A.12}$$

By selecting $A_\mu\to qA_\mu$, the charge coupling is added to the problem and the final Lagrangian is expressed as:

$$\mathcal{L}_{NC}=\mathcal{L}_{QED}-\frac{1}{2}q\theta^{\alpha\beta}A_\beta\left(\partial_\alpha A_\mu\right)\bar{\psi}\gamma^\mu\psi+\frac{1}{4}iq\theta^{\alpha\beta}F_{\alpha\mu}\bar{\psi}\gamma^\mu\vec{D}_\beta\psi$$
$$+\frac{1}{4}iq\theta^{\alpha\beta}\left(\partial_\mu A_\alpha\right)\bar{\psi}\gamma^\mu\vec{D}_\beta\psi-\frac{1}{2}q\theta^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu}+\frac{1}{8}q\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \tag{A.13}$$

To obtain the effective Lagrangian under a background electromagnetic field, we apply the transformations $A_\mu\to A_\mu^{ext}+A_\mu$ and $F_{\mu\nu}\to f_{\mu\nu}+F_{\mu\nu}$ to (A.13) and maintain the terms up to the second order of F.

$$\mathcal{L}_{NC}=\mathcal{L}_{QED}-\frac{1}{2}q\theta^{\alpha\beta}A_\beta\left(\partial_\alpha A_\mu^{ext}\right)\bar{\psi}\gamma^\mu\psi+\frac{1}{4}iq\theta^{\alpha\beta}f_{\alpha\mu}\bar{\psi}\gamma^\mu\vec{D}_\beta\psi$$
$$+\frac{1}{4}iq\theta^{\alpha\beta}\left(\partial_\mu A_\alpha^{ext}\right)\bar{\psi}\gamma^\mu\vec{D}_\beta\psi-\frac{1}{2}q\theta^{\alpha\beta}f_{\alpha\mu}F_{\beta\nu}F^{\mu\nu} \tag{A.14}$$
$$-\frac{1}{2}q\theta^{\alpha\beta}f_{\beta\nu}F_{\alpha\mu}F^{\mu\nu}-\frac{1}{2}q\theta^{\alpha\beta}f^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}+\frac{1}{8}q\theta^{\alpha\beta}f_{\mu\nu}F_{\alpha\beta}F^{\mu\nu}$$

Considering the property of Lorentz indexes and because the electromagnetic field tensor is antisymmetric, (A.14) is rewritten as:

$$\mathcal{L}_{NC} = \mathcal{L}_{QED} + \frac{1}{4} iq \left\{ \theta_{\nu}^{\alpha} f_{\alpha\mu} + \theta_{\nu}^{\alpha} \left(\partial_{\mu} A_{\alpha}^{ext} \right) \right\} \bar{\psi} \gamma^{\mu} \vec{D}^{\nu} \psi$$

$$- \frac{1}{2} q\theta^{\alpha\beta} \left\{ A_{\beta} \left(\partial_{\alpha} A_{\mu}^{ext} \right) \right\} \bar{\psi} \gamma^{\mu} \psi + \left\{ \begin{matrix} q\theta_{\alpha}^{\lambda} f_{\lambda\mu} \eta_{\beta\nu} + \frac{1}{8} q\theta^{\alpha\beta} f_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \\ -\frac{1}{2} q\theta_{\alpha\mu} f_{\beta\nu} + \frac{1}{4} q\theta_{\alpha\beta} f_{\mu\nu} \end{matrix} \right\} F^{\alpha\beta} F^{\mu\nu} \tag{A.15}$$

Now, by comparing (A.15) with the SME Lagrangian (3.9), the Lorentz violation coefficients are obtained as follows:

$$\begin{aligned} c_{\mu\nu} &= \frac{1}{2} q \left[\theta_{\alpha\nu} f_{\mu}^{\alpha} + \theta_{\nu}^{\alpha} \left(\partial_{\mu} A_{\alpha}^{ext} \right) \right] \\ a_{\mu} &= \frac{1}{2} q\theta^{\alpha\beta} A_{\beta}^{ext} \left(\partial_{\alpha} A_{\mu}^{ext} \right) \\ \left(k_F \right)_{\alpha\beta\mu\nu} &= -4q\theta_{\alpha}^{\lambda} f_{\lambda\mu} \eta_{\beta\nu} + 2q\theta_{\alpha\mu} f_{\beta\nu} - q\theta_{\alpha\beta} f_{\mu\nu} \end{aligned} \tag{A.16}$$

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