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THE PROPERTIES OF FUZZY CONNECTIONS

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Abstract. We investigate the properties of fuzzy connections. We find generating functions which induce fuzzy connections. In particular, we show that their connections relate to fuzzy relations.

1. Introduction

Wille [7] introduced the formal concept lattices by allowing some uncertainty in data as examples as Galois, dual Galois, residuated and dual residuated connections. Formal concept analysis is an important mathematical tool for data analysis and knowledge processing [1-4,7]. Orlowska and Rewitzky [5] investigated the algebraic structures of operators of Galois-style connections. Bělohlávek [1-2] introduced the formal concept lattices with respect to fuzzy Galois connections on a complete residuated lattice. Fuzzy Galois connections are developed many directions [3,4,8]

In this paper, we investigate the properties of fuzzy connections (Galois, dual Galois, residuated and dual residuated connections) on a complete residuated lattice. We find generating functions which induce fuzzy connections (Galois, dual Galois, residuated and dual residuated connections). In particular, we show that their connections relate to fuzzy relations.

2. Preliminaries

Definition 2.1. [1,2,6] A triple $(X, \leq, \odot)$ is called a *complete residuated lattice* iff it satisfies the following properties:

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(L1) $(X, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(X, \odot, 1)$ is a commutative monoid;

(L3) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma} (x_i \odot y), \ \forall x_i, y \in L.$$

Let $(L, \leq, \odot)$ be a complete residuated lattice. For each $x, y \in L$, we define

$$x \to y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$(x \odot y) \leq z$ iff $x \leq (y \to z)$.

Remark 2.2. [1,2,6](1) A completely distributive lattice is a complete residuated lattice. Moreover, the unit interval $([0, 1], \leq, \vee, \wedge, 0, 1)$ is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm t, $([0, 1], \leq, t)$, is a complete residuated lattice.

Let $(L, \leq, \odot)$ be a complete residuated lattice. An order reversing map $^* : L \to L$ defined by $a^* = a \to 0$ is called a *strong negation* if $a^{**} = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, ^*)$ is a complete residuated lattice with a strong negation *. For $\alpha \in L, A, 1_x \in L^X$, we denote

$$(\alpha \odot A)(x) = \alpha \odot A(x), \ \ (\alpha \to A)(x) = \alpha \to A(x).$$

$$1_x(x) = 1, \ 1_x(y) = 0, \ \forall y \in X - \{x\}.$$

Lemma 2.3. [6] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(2) $x \odot y \leq x \wedge y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(4) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$.

(5) $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.

(6) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(7) $x \odot (x \to y) \le y$ and $x \to y \le (y \to z) \to (x \to z)$.
(8) $y \odot z \le x \to (x \odot y \odot z)$ and $x \odot (x \odot y \to z) \le y \to z$.
(9) $x \to y \le (x \odot z) \to (y \odot z)$.
(10) $(x \to y) \odot (y \to z) \le x \to z$.
(11) $x \le y \to z$ iff $y \le x \to z$.
(12) $x \to y = 1$ iff $x \le y$.
(13) $x \to y = y^* \to x^*$.
(14) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

Definition 2.4. [8] Let X be a set. A function $e_X : X \times X \to L$ is called a *fuzzy partially order* on X if it satisfies the following conditions:
(E1) $e_X(x,x) = 1$ for all $x \in X$,
(E2) $e_X(x,y) \odot e_X(y,z) \le e_X(x,z)$, for all $x,y,z \in X$,
(E3) if $e_X(x,y) = e_X(y,x) = 1$, then $x = y$.
The pair (X, e_X) is a *fuzzy partially order set* (simply, fuzzy poset).

We define a function $e_{L^X} : L^X \times L^X \to L$ as

$$e_{L^X}(A,B) = \bigwedge_{x\in X} (A(x) \to B(x)).$$

Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 2.3 (10-11).
We define fuzzy connections in a sense [8].

Definition 2.5. Let (X, e_X) and (Y, e_Y) be fuzzy posets and $f : X \to Y$ and $g : Y \to X$ maps.
(1) (e_X, f, g, e_Y) is called a *Galois connection* if

$$e_Y(y, f(x)) = e_X(x, g(y)),\ \forall x \in X, y \in Y.$$

(2)(e_X, f, g, e_Y) is called a *dual Galois connection* if

$$e_Y(f(x), y) = e_X(g(y), x),\ \forall x \in X, y \in Y.$$

(3) (e_X, f, g, e_Y) is called a *residuated connection* if

$$e_Y(f(x), y) = e_X(x, g(y)),\ \forall x \in X, y \in Y.$$

(4) (e_X, f, g, e_Y) is called a *dual residuated connection* if

$$e_Y(y, f(x)) = e_X(g(y), x),\ \forall x \in X, y \in Y.$$

3. The properties of fuzzy connections

THEOREM 3.1. *The following statements hold:*

(1) *There exists a Galois connection* (e_{L^X}, F, G, e_{L^Y}) *iff there exists a function* $F : L^X \to L^Y$ *with* $F(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i)$ *and* $F(\alpha\odot A) = \alpha \to F(A)$ *and* $F(1_x)(y) = G(1_y)(x)$, $\forall x \in X, y \in Y$.

(2) *There exists a residuated connection* (e_{L^X}, F, G, e_{L^Y}) *iff there exists a function* $F : L^X \to L^Y$ *with* $F(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F(A_i)$ *and* $F(\alpha \odot A) = \alpha \odot F(A)$ *and* $F(1_x)^*(y) = G(1^*_y)(x)$, $\forall x \in X, y \in Y$.

(3) *There exists a dual Galois connection* (e_{L^X}, F, G, e_{L^Y}) *iff there exists a function* $F : L^X \to L^Y$ *with* $F(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F(A_i)$, $F(\alpha \to A) = \alpha \odot F(A)$ *and* $F(1^*_x)(y) = G(1^*_y)(x)$, $\forall x \in X, y \in Y$.

(4) *There exists a dual residuated connection* (e_{L^X}, F, G, e_{L^Y}) *iff there exists a function* $F : L^X \to L^Y$ *with* $F(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i)$ *and* $F(\alpha \to A) = \alpha \to F(A)$ *and* $F(1^*_x)(y) = G(1_y)^*(x)$, $\forall x \in X, y \in Y$.

Proof. (1) ($\Rightarrow$) By Lemma 2.3 (3,6), we have

$$\begin{aligned}
e_{L^Y}(B, F(\bigvee_{i\in\Gamma} A_i)) &= e_{L^X}(\bigvee_{i\in\Gamma} A_i, G(B)) = \bigwedge_{i\in\Gamma} e_{L^X}(A_i, G(B)) \\
&= \bigwedge_{i\in\Gamma} e_{L^Y}(B, F(A_i)) = e_{L^Y}(B, \bigwedge_{i\in\Gamma} F(A_i)), \\
e_{L^Y}(B, F(\alpha \odot A)) &= e_{L^X}(\alpha \odot A, G(B)) = \alpha \to e_{L^X}(A, G(B)) \\
&= \alpha \to e_{L^Y}(B, F(A)) = e_{L^Y}(\alpha \odot B, F(A)) \\
&= e_{L^Y}(B, \alpha \to F(A)).
\end{aligned}$$

For $B = 1_y \in L^Y$,

$$\begin{aligned}
F(\bigvee_{i\in\Gamma} A_i)(y) &= e_{L^Y}(1_y, F(\bigvee_{i\in\Gamma} A_i)) = e_{L^Y}(1_y, \bigwedge_{i\in\Gamma} F(A_i)) \\
&= \bigwedge_{i\in\Gamma} F(A_i)(y), \\
F(\alpha \odot A)(y) &= e_{L^Y}(1_y, F(\alpha \odot A)) = e_{L^Y}(1_y, \alpha \to F(A)) \\
&= \alpha \to F(A)(y).
\end{aligned}$$

Moreover, $G(1_y)(x) = e_{L^X}(1_x, G(1_y)) = e_{L^Y}(1_y, F(1_x)) = F(1_x)(y)$.

($\Leftarrow$) Since $C = \bigvee_{x\in X}(C(x)\odot 1_x)$, we have

$$F(C)(y) = F(\bigvee_{x\in X}(C(x)\odot 1_x))(y) = \bigwedge(C(x)\to F(1_x)(y)).$$

We define a function $G : L^Y \to L^X$ with

$$\begin{aligned} G(B)(x) &= \bigvee\{C(x) \mid F(C)\geq B\} \\ &= \bigvee\{C(x) \mid \bigwedge(C(x)\to F(1_x)(y)\geq B(y)\} \\ &\quad \text{(by Lemma 2.3 (11))} \\ &= \bigvee\{C(x) \mid \bigwedge(B(y)\to F(1_x)(y)\geq C(x)\} \\ &= \bigwedge(B(y)\to F(1_x)(y)). \end{aligned}$$

$$\begin{aligned} e_{L^Y}(B, F(A)) &= \bigwedge_{y\in Y}(B(y)\to F(\bigvee_{x\in X}(A(x)\odot 1_x))(y)) \\ &= \bigwedge_{y\in Y}(B(y)\to \bigwedge_{x\in X}(A(x)\to F(1_x)(y))) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}(B(y)\to (A(x)\to F(1_x)(y))) \\ &\quad \text{(by Lemma 2.3 (6))} \\ &= \bigwedge_{x\in X}(A(x)\to \bigwedge_{y\in Y}(B(y)\to F(1_x)(y))) \\ &= \bigwedge_{x\in X}(A(x)\to G(B)(x)) = e_{L^X}(A, G(B)). \end{aligned}$$

(2)($\Rightarrow$)

$$\begin{aligned} e_{L^Y}(F(\bigvee_{i\in\Gamma}A_i), B) &= e_{L^X}(\bigvee_{i\in\Gamma}A_i, G(B)) = \bigwedge_{i\in\Gamma}e_{L^X}(A_i, G(B)), \\ &= \bigwedge_{i\in\Gamma}e_{L^Y}(F(A_i), B) = e_{L^Y}(\bigvee_{i\in\Gamma}F(A_i), B) \\ e_{L^Y}(F(\alpha\odot A), B) &= e_{L^X}(\alpha\odot A, G(B)) = \alpha\to e_{L^X}(A, G(B)) \\ &= \alpha\to e_{L^Y}(F(A), B) = e_{L^Y}(\alpha\odot F(A), B). \end{aligned}$$

For $B = 1_y^* \in L^Y$, by Lemma 2.3 (13), $F(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F(A_i)$ from:

$$e_{L^Y}(F(\bigvee_{i\in\Gamma} A_i), 1_y^*) = e_{L^Y}(1_y, F(\bigvee_{i\in\Gamma} A_i)^*) = F(\bigvee_{i\in\Gamma} A_i)^*(y)$$
$$e_{L^Y}(\bigvee_{i\in\Gamma} F(A_i), 1_y^*) = (\bigvee_{i\in\Gamma} F(A_i))^*(y).$$

Similarly, $F(\alpha \odot A) = \alpha \odot F(A)$ for all $\alpha \in L$. Since $G(1_y^*) \in L^X$ and $F(1_x) \in L^Y$, $F(1_x)^*(y) = e_{L^Y}(1_y, F(1_x)^*) = e_{L^Y}(F(1_x), 1_y^*) = e_{L^X}(1_x, G(1_y^*)) = G(1_y^*)(x)$.

($\Leftarrow$) Since $C = \bigvee_{x\in X}(C(x) \odot 1_x)$, we have

$$F(C)(y) = F(\bigvee_{x\in X}(C(x) \odot 1_x))(y) = \bigvee_{x\in X}(C(x) \odot F(1_x)(y)).$$

We define a function $G : L^Y \to L^X$ with

$$\begin{aligned} G(B)(x) &= \bigvee\{C(x) \mid F(C) \le B\} \\ &= \bigvee\{C(x) \mid \bigvee(C(x) \odot F(1_x)(y) \le B(y)\} \\ &= \bigwedge_{y\in Y}(F(1_x)(y) \to B(y)). \end{aligned}$$

$$\begin{aligned} e_{L^Y}(F(A), B) &= \bigwedge_{y\in Y}(F(\bigvee_{x\in X}(A(x) \odot 1_x))(y) \to B(y)) \\ &= \bigwedge_{y\in Y}(\bigvee_{x\in X}(A(x) \odot F(1_x)(y) \to B(y)) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}(A(x) \to (F(1_x)(y) \to B(y))) \\ &= \bigwedge_{x\in X}(A(x) \to \bigwedge_{y\in Y}(F(1_x)(y) \to B(y))) \\ &= \bigwedge_{x\in X}(A(x) \to G(B)(x)) = e_{L^X}(A, G(B)). \end{aligned}$$

(3)($\Rightarrow$)

$$\begin{aligned}
e_{L^Y}(F(\bigwedge_{i\in\Gamma} A_i), B) &= e_{L^X}(G(B), \bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} e_{L^X}(G(B), A_i) \\
&= \bigwedge_{i\in\Gamma} e_{L^Y}(F(A_i), B) = e_{L^Y}(\bigvee_{i\in\Gamma} F(A_i), B) \\
e_{L^Y}(F(\alpha\to A), B) &= e_{L^X}(G(B), \alpha\to A) = \alpha\to e_{L^X}(G(B), A), \\
&= \alpha\to e_{L^X}(F(A), B) = e_{L^X}(\alpha\odot F(A), B).
\end{aligned}$$

For $B = 1_y^* \in L^Y$, by a similar method as in (2), we have $F(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F(A_i)$ and $F(\alpha\to A) = \alpha\odot F(A)$.

$$\begin{aligned}
F(1_x^*)^*(y) &= \bigwedge_{w\in Y} (1_y(w) \to F(1_x^*)^*(w)) \\
&= e_{L^Y}(F(1_x^*), 1_y^*) = e_{L^X}(G(1_y^*), 1_x^*) \\
&= G(1_y^*)^*(x).
\end{aligned}$$

($\Leftarrow$) For $A(x) = \bigwedge_{z\in X}(1_z(x)\to A(z)) = \bigwedge_{z\in X}(A^*(z)\to 1_z^*(x))$, we have $F(A)(y) = F(\bigwedge_{z\in X}(A^*(z)\to 1_z^*))(y) = \bigvee_{z\in X}(A^*(z)\odot F(1_z^*)(y))$, we define

$$\begin{aligned}
G(B)(x) &= \bigwedge\{C(x) \mid F(C)\le B\} \\
&= \bigwedge\{C(x) \mid F(\bigwedge_{x\in X}(C^*(x)\to 1_x^*))\le B\} \\
&= \bigvee\{C(x) \mid \bigvee_{x\in X}(C^*(x)\odot F(1_x^*)(y) \le B(y)\} \\
&= \bigvee\{C(x) \mid C^*(x) \le \bigwedge_{y\in Y}(F(1_x^*)(y)\to B(y))\} \\
&= \bigvee_{y\in Y}(F(1_x^*)(y)\odot B^*(y)).
\end{aligned}$$

$$
\begin{aligned}
e_{L^Y}(F(A),B) &= \bigwedge_{y\in Y}(\bigvee_{x\in X}(A^*(x)\odot F(1^*_x)(y)\to B(y))\\
&= \bigwedge_{y\in Y}\bigwedge_{x\in X}((B^*(y)\odot F(1^*_x)(y)\to A(x))\\
&= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B^*(y)\odot F(1^*_x)(y)\to A(x))\\
&= \bigwedge_{x\in X}(G(B)(x)\to A(x)) = e_{L^X}(G(B),A).
\end{aligned}
$$

(4)($\Rightarrow$) We have:

$$
\begin{aligned}
&e_{L^Y}(B,F(\bigwedge_{i\in\Gamma}A_i)) = e_{L^X}(G(B),\bigwedge_{i\in\Gamma}A_i) = \bigwedge_{i\in\Gamma}e_{L^X}(G(B),A_i)\\
&= \bigwedge_{i\in\Gamma}e_{L^Y}(B,F(A_i)) = e_{L^Y}(B,\bigwedge_{i\in\Gamma}F(A_i))\\
&e_{L^Y}(B,F(\alpha\to A)) = e_{L^X}(G(B),\alpha\to A) = e_{L^X}(\alpha\odot G(B),A)\\
&= \alpha\to e_{L^X}(G(B),A) = \alpha\to e_{L^X}(B,F(A)) = e_{L^X}(B,\alpha\to F(A))
\end{aligned}
$$

For $B = 1_y \in L^Y$, by a similar method as in (1), we have $F(\bigwedge_{i\in\Gamma}A_i) = \bigwedge_{i\in\Gamma}F(A_i)$ and $F(\alpha\to A) = \alpha\to F(A)$.

$$
\begin{aligned}
F(1^*_x)(y) &= \bigwedge_{w\in Y}(1_y(w)\to F(1^*_x)(w))\\
&= e_{L^Y}(1_y,F(1^*_x)) = e_{L^X}(G(1_y),1^*_x)\\
&= G(1_y)^*(x).
\end{aligned}
$$

($\Leftarrow$) For $A(x) = \bigwedge_{z\in X}(1_z(x)\to A(z)) = \bigwedge_{z\in X}(A^*(z)\to 1^*_z(x))$, we have $F(A)(y) = F(\bigwedge_{z\in X}(A^*(z)\to 1^*_z))(y) = \bigwedge_{z\in X}(A^*(z)\to$

$F(1_z^*)(y))$, we define

$$\begin{aligned}
G(B)(x) &= \bigwedge\{C(x) \mid F(C) \geq B\} \\
&= \bigwedge\{C(x) \mid F(\bigwedge_{x\in X}(C^*(x) \to 1_x^*)) \geq B\} \\
&= \bigwedge\{C(x) \mid \bigwedge_{x\in X}(C^*(x) \to F(1_x^*)(y) \geq B(y)\} \\
&= \bigwedge\{C(x) \mid C^*(x) \leq \bigwedge_{y\in Y}(B(y) \to F(1_x^*)(y))\} \\
&= \bigwedge_{y\in Y}(F(1_x^*)^*(y) \odot B(y)).
\end{aligned}$$

$$\begin{aligned}
e_{L^Y}(B, F(A)) &= \bigwedge_{y\in Y}(B(y) \to \bigwedge_{x\in X}(F(1_x^*)^*(y) \to A(x))) \\
&= \bigwedge_{y\in Y}\bigwedge_{x\in X}((B(y) \odot F(1_x^*)^*(y) \to A(x)) \\
&= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B(y) \odot F(1_x^*)^*(y) \to A(x)) \\
&= \bigwedge_{x\in X}(G(B)(x) \to A(x)) = e_{L^X}(G(B), A).
\end{aligned}$$

□

THEOREM 3.2. (1) (e_{L^X}, F, G, e_{L^Y}) *is a Galois connection iff there exists* $R : X \times Y \to L$ *such that*

$$F(A)(y) = \bigwedge_{x\in X}(A(x) \to R(x, y)), \quad G(B)(x) = \bigwedge_{y\in Y}(B(y) \to R(x, y)).$$

(2) (e_{L^X}, F, G, e_{L^Y}) *is a residuated connection iff there exists* $R : X \times Y \to L$ *such that*

$$F(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x, y)), \quad G(B)(x) = \bigwedge_{y\in Y}(R(x, y) \to B(y)).$$

(3) (e_{L^X}, F, G, e_{L^Y}) *is a dual Galois connection iff there exists* $R : X \times Y \to L$ *such that*

$$F(A)(y) = \bigvee_{x \in X} (A^*(x) \odot R(x,y)), \;\; G(B)(x) = \bigwedge_{y \in Y} (R(x,y) \odot B^*(y)).$$

(4) (e_{L^X}, F, G, e_{L^Y}) *is a dual residuated connection iff there exists* $R \subset X \times Y$ *such that*

$$F(A)(y) = \bigwedge_{y \in Y} (R(x,y) \to A(x)), \;\; G(B)(x) = \bigvee_{y \in Y} (B(y) \odot R(x,y)).$$

Proof. (1)($\Rightarrow$)

$$\begin{aligned} F(1_x)(y) &= e_{L^Y}(1_y, F(1_x)) = e_{L^X}(1_x, G(1_y)) \\ &= \bigwedge_{z \in X} \Big(1_x(z) \to G(1_y)(z)\Big) = G(1_y)(x). \end{aligned}$$

Put $R(x,y) = F(1_x)(y) = G(1_y)(x)$. Then

$$\begin{aligned} F(A)(y) &= e_{L^Y}(1_y, F(A)) = e_{L^X}(A, G(1_y)) \\ &= \bigwedge_{x \in X} \Big(A(x) \to G(1_y)(x)\Big) \\ &= \bigwedge_{x \in X} \Big(A(x) \to R(x,y)\Big). \end{aligned}$$

$$\begin{aligned} G(B)(x) &= e_{L^X}(1_x, G(B)) = e_{L^Y}(B, F(1_x)) \\ &= \bigwedge_{y \in Y} \Big(B(y) \to F(1_x)(y)\Big) \\ &= \bigwedge_{y \in Y} \Big(B(y) \to R(x,y)\Big). \end{aligned}$$

$(\Leftarrow)$

$$\begin{aligned} e_{L^Y}(B, F(A)) &= \bigwedge_{y\in Y}(B(y) \to F(A)(y)) \\ &= \bigwedge_{y\in Y}(B(y) \to \bigwedge_{x\in X}(A(x) \to R(x,y))) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}(B(y) \to (A(x) \to R(x,y))) \\ &= \bigwedge_{x\in X}(A(x) \to \bigwedge_{y\in Y}(B(y) \to R(x,y))) \\ &= \bigwedge_{x\in X}(A(x) \to G(B)(x)) = e_{L^X}(A, G(B)). \end{aligned}$$

$(2)(\Rightarrow)$

$$\begin{aligned} F(1_x)^*(y) &= e_{L^Y}(F(1_x), 1_y^*) = e_{L^X}(1_x, G(1_y^*)) \\ &= G(1_y^*)(x). \end{aligned}$$

Put $R(x,y) = F(1_x)(y) = (G(1_y^*))^*$, then

$$\begin{aligned} F(A)^*(y) &= e_{L^Y}(F(A), 1_y^*) = e_{L^X}(A, G(1_y^*)) \\ &= \bigwedge_{x\in X}(A(x) \to G(1_y^*)(x)) \\ &= \Big(\bigvee_{x\in X}(A(x) \odot G(1_y^*)^*(x))\Big)^* \\ F(A)(y) &= \bigvee_{x\in X}(A(x) \odot R(x,y)). \end{aligned}$$

$$\begin{aligned} G(B)(x) &= e_{L^X}(1_x, G(B)) = e_{L^Y}(F(1_x), B) \\ &= \bigwedge_{y\in Y}(F(1_x)(y) \to B(y)) \\ &= \bigwedge_{y\in Y}(R(x,y) \to B(y)). \end{aligned}$$

(⇐)

$$\begin{aligned}
e_{L^Y}(F(A),B) &= \bigwedge_{y\in Y}(\bigvee_{x\in X}(A(x)\odot R(x,y))\to B(y))\\
&= \bigwedge_{y\in Y}\bigwedge_{x\in X}(A(x)\to(R(x,y)\to B(y)))\\
&= \bigwedge_{x\in X}(A(x)\to\bigwedge_{y\in Y}(R(x,y)\to B(y)))\\
&= \bigwedge_{x\in X}(A(x)\to G(B)(x)) = e_{L^X}(A,G(B)).
\end{aligned}$$

(3)(⇒)

$$\begin{aligned}
F(1_x^*)^*(y) &= e_{L^Y}(F(1_x^*),1_y^*) = e_{L^X}(G(1_y^*),1_x^*)\\
&= G(1_y^*)^*(x).
\end{aligned}$$

Put $R(x,y) = F(1_x^*)(y) = G(1_y^*)(x)$, then

$$\begin{aligned}
F(A)^*(y) &= e_{L^Y}(F(A),1_y^*) = e_{L^X}(G(1_y^*),A)\\
&= \bigwedge_{x\in X}(G(1_y^*)(x)\to A(x))\\
&= \Big(\bigvee_{x\in X}(A^*(x)\odot G(1_y^*)(x))\Big)^*\\
F(A)(y) &= \bigvee_{x\in X}(A^*(x)\odot R(x,y)).
\end{aligned}$$

$$\begin{aligned}
G(B)^*(x) &= e_{L^X}(G(B),1_x^*) = e_{L^Y}(F(1_x^*),B)\\
&= \bigwedge_{y\in Y}(F(1_x^*)(y)\to B(y))\\
&= \Big(\bigvee_{y\in Y}(B^*(y)\odot F(1_x^*)(y))\Big)^*\\
G(B)(x) &= \bigvee_{y\in Y}(B^*(y)\odot R(x,y)).
\end{aligned}$$

(⇐)

$$\begin{aligned} e_{L^Y}(F(A), B) &= \bigwedge_{y\in Y}(\bigvee_{x\in X}(A^*(x)\odot R(x,y))\to B(y)) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}((B^*(y)\odot R(x,y))\to A(x)) \\ &= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B^*(y)\odot R(x,y))\to A(x)) \\ &= \bigwedge_{x\in X}(G(B)(x)\to A(x)) = e_{L^X}(G(B), A). \end{aligned}$$

(4)(⇒)

$$\begin{aligned} F(1_x^*)(y) &= \bigwedge_{w\in Y}(1_y(w)\to F(1_x^*)(w)) \\ &= e_{L^Y}(1_y, F(1_x^*)) = e_{L^X}(G(1_y), 1_x^*) \\ &= G(1_y)^*(x). \end{aligned}$$

Put $R(x,y) = F(1_x^*)^*(y) = G(1_y)$. We have

$$\begin{aligned} F(A)(y) &= e_{L^Y}(1_y, F(A)) = e_{L^X}(G(1_y), A) \\ &= \bigwedge_{x\in X}(G(1_x)(y)\to A(x)) \\ &= \bigwedge_{x\in X}(R(x,y)\to A(x)). \end{aligned}$$

$$\begin{aligned} G(B)^*(x) &= e_{L^X}(G(B), 1_x^*) = e_{L^Y}(B, F(1_x^*)) \\ &= \bigwedge_{y\in Y}(B(y)\to F(1_x^*)(y)) \\ &= \Big(\bigvee_{y\in Y}(B(y)\odot F(1_x^*)^*(y))\Big)^* \\ G(B)(x) &= \bigvee_{y\in Y}(R(x,y)\odot B(y)). \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned}
e_{L^Y}(B,F(A)) &= \bigwedge_{y\in Y}(B(y)\to \bigwedge_{x\in X}(R(x,y)\to A(x))) \\
&= \bigwedge_{y\in Y}\bigwedge_{x\in X}((B(y)\odot R(x,y))\to A(x)) \\
&= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B(y)\odot R(x,y))\to A(x)) \\
&= \bigwedge_{x\in X}(G(B)(x)\to A(x)) = e_{L^X}(G(B),A).
\end{aligned}$$

□

THEOREM 3.3. (1) (e_{L^X},F,G,e_{L^Y}) *is a Galois connection iff there exists* $F:L^X\to L^Y$ *with* $F(1_x)(y)=\Gamma_x(y)$ *such that* $F(\bigvee_{i\in\Gamma}A_i)=\bigwedge_{i\in\Gamma}F(A_i)$ *and* $F(\alpha\odot A)=\alpha\to F(A)$.

(2) (e_{L^X},F,G,e_{L^Y}) *is a residuated connection iff there exists* $F:L^X\to L^Y$ *with* $F(1_x)(y)=\Gamma_x(y)$ *such that* $F(\bigvee_{i\in\Gamma}A_i)=\bigvee_{i\in\Gamma}F(A_i)$ *and* $F(\alpha\odot A)=\alpha\odot F(A)$.

(3) (e_{L^X},F,G,e_{L^Y}) *is a dual Galois connection iff there exists* $F:L^X\to L^Y$ *with* $F(1^*_x)(y)=\Delta_x(y)$ *such that* $F(\bigwedge_{i\in\Gamma}A_i)=\bigvee_{i\in\Gamma}F(A_i)$ *and* $F(\alpha\to A)=\alpha\odot F(A)$.

(4) (e_{L^X},F,G,e_{L^Y}) *is a dual residuated connection iff there exists* $F:L^X\to L^Y$ *with* $F(1^*_x)(y)=\Delta_x(y)$ *such that* $F(\bigwedge_{i\in\Gamma}A_i)=\bigwedge_{i\in\Gamma}F(A_i)$ *and* $F(\alpha\to A)=\alpha\to F(A)$.

Proof. (1) ($\Rightarrow$) It follows from Theorem 3.1(1).

($\Leftarrow$) Since $C=\bigvee_{x\in X}(C(x)\odot 1_x)$, $F(C)(y)=F(\bigvee_{x\in X}(C(x)\odot 1_x))(y)=\bigwedge(C(x)\to F(1_x)(y))$. Thus,

$$\begin{aligned}
G(B)(x) &= \bigvee\{C(x)\mid F(C)\geq B\} \\
&= \bigvee\{C(x)\mid \bigwedge(C(x)\to F(1_x)(y))\geq B(y)\} \\
&= \bigvee\{C(x)\mid \bigwedge(B(y)\to F(1_x)(y))\geq C(x)\} \\
&= \bigwedge(B(y)\to F(1_x)(y)) = \bigwedge(B(y)\to \Gamma_x(y)),
\end{aligned}$$

$$\begin{aligned}
e_{L^Y}(B, F(A)) &= \bigwedge_{y\in Y} (B(y) \to F(\bigvee_{x\in X} (A(x) \odot 1_x))(y)) \\
&= \bigwedge_{y\in Y} (B(y) \to \bigwedge_{x\in X} (A(x) \to F(1_x)(y))) \\
&= \bigwedge_{y\in Y} \bigwedge_{x\in X} (B(y) \to (A(x) \to F(1_x)(y))) \\
&= \bigwedge_{x\in X} (A(x) \to \bigwedge_{y\in Y} (B(y) \to \Gamma_x(y))) \\
&= \bigwedge_{x\in X} (A(x) \to G(B)(x)) = e_{L^Y}(A, G(B)).
\end{aligned}$$

(2)(⇒) It follows from Theorem 3.1(2).
(⇐)

$$\begin{aligned}
G(B)(x) &= \bigvee\{C(x) \mid F(C) \le B\} \\
&= \bigvee\{C(x) \mid \bigvee(C(x) \odot F(1_x))(y) \le B(y)\} \\
&= \bigwedge_{y\in Y} (F(1_x))(y) \to B(y)) = \bigwedge_{y\in Y} (\Gamma_x(y) \to B(y)).
\end{aligned}$$

$$\begin{aligned}
e_{L^Y}(F(A), B) &= \bigwedge_{y\in Y} (F(\bigvee_{x\in X} (A(x) \odot 1_x)))(y) \to B(y)) \\
&= \bigwedge_{y\in Y} (\bigvee_{x\in X} (A(x) \odot F(1_x))(y) \to B(y)) \\
&= \bigwedge_{y\in Y} \bigwedge_{x\in X} (A(x) \to (F(1_x))(y) \to B(y))) \\
&= \bigwedge_{x\in X} (A(x) \to \bigwedge_{y\in Y} (\Gamma_x(y) \to B(y))) \\
&= \bigwedge_{x\in X} (A(x) \to G(B)(x)) = e_{L^X}(A, G(B)).
\end{aligned}$$

(3)(⇒) It follows from Theorem 3.1(3).

($\Leftarrow$) Since

$$F(\bigwedge_{x\in X}(1_x \to C(x))) = F(\bigwedge_{x\in X}(C^*(x) \to 1^*_x) = \bigvee(C^*(x) \odot F(1^*_x)),$$

we define

$$\begin{aligned} G(B)(x) &= \bigwedge\{C(x) \mid F(C) \leq B\} \\ &= \bigwedge\{C(x) \mid F(\bigwedge_{x\in X}(1_x \to C(x))) \leq B\} \\ &= \bigvee\{C(x) \mid \bigvee(C^*(x) \odot F(1^*_x)(y) \leq B(y)\} \\ &= \bigvee\{C(x) \mid C^*(x) \leq \bigwedge_{y\in Y}(F(1^*_x)(y) \to B(y))\} \\ &= \bigvee_{y\in Y}(\Delta_x(y) \odot B^*(y)). \end{aligned}$$

$$\begin{aligned} e_{L^Y}(F(A),B) &= \bigwedge_{y\in Y}(F(\bigwedge_{x\in X}((e_X)_x \to A(x))) \to B(y)) \\ &= \bigwedge_{y\in Y}(\bigvee_{x\in X}(A^*(x) \odot F(1^*_x)(y)) \to B(y)) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}((B^*(y) \odot F(1^*_x)(y)) \to A(x)) \\ &= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B^*(y) \odot \Delta_x(y)) \to A(x)) \\ &= \bigwedge_{x\in X}(G(B)(x) \to A(x)) = e_{L^X}(G(B),A). \end{aligned}$$

(4)($\Rightarrow$) It follows from Theorem 3.1(4).
($\Leftarrow$) Since

$$F(\bigwedge_{x\in X}(1_x \to C(x))) = F(\bigwedge_{x\in X}(C^*(x) \to 1^*_x) = \bigwedge(C^*(x) \to F(1^*_x)(y)),$$

we define

$$\begin{aligned} G(B)(x) &= \bigwedge\{C(x) \mid F(C) \geq B\} \\ &= \bigwedge\{C(x) \mid F(\bigwedge_{x\in X}(1_x \to C(x))) \geq B\} \\ &= \bigvee\{C(x) \mid \bigwedge(C^*(x) \to F(1_x^*)(y) \geq B(y)\} \\ &= \bigvee\{C(x) \mid C^*(x) \leq \bigwedge_{y\in Y}(B(y) \to F(1_x^*)(y))\} \\ &= \bigvee_{y\in Y}(\Delta_x^*(y) \odot B(y))^* \end{aligned}$$

$$\begin{aligned} e_{L^Y}(B, F(A)) &= \bigwedge_{y\in Y}(B(y) \to F(\bigwedge_{x\in X}(A^*(x) \to ((e_X)_x^{-1})^*))) \\ &= \bigwedge_{y\in Y}(B(y) \to \bigwedge_{x\in X}(A^*(x) \to F(1_x)^*)(y))) \\ &= \bigwedge_{y\in Y}(B(y) \to \bigwedge_{x\in X}(F(1_x^*)^*(y) \to A)x))) \\ &= \bigwedge_{y\in Y}(\bigwedge_{x\in X}(B(y) \odot F(1_x^*)^*(y) \to A(x))) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}((B(y) \odot F(1_x)^*)^*(y) \to A(x)) \\ &= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B(y) \odot \Delta_x^*(y))^* \to A(x)) \\ &= \bigwedge_{x\in X}(G(B)(x) \to A(x)) = e_{L^X}(G(B), A). \end{aligned}$$

□

Example 3.4. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ be sets with

$$\begin{aligned} &F_1(1_a)(x) = 1,\ F_1(1_a)(y) = 0.8,\ F_1(1_a)(z) = 0.5, \\ &F_1(1_b)(x) = 0.5,\ F_1(1_b)(y) = 0.6,\ F_1(1_b)(z) = 0.9, \\ &F_1(1_c)(x) = 0.3,\ F_1(1_c)(y) = 1,\ F_1(1_c)(z) = 0.4. \end{aligned}$$

$$F_2(1^*_a)(x) = 1,\ F_2(1^*_a)(y) = 0.8,\ F_2(1^*_a)(z) = 0.5,$$
$$F_2(1^*_b)(x) = 0.5,\ F_2(1^*_b)(y) = 0.6,\ F_2(1^*_b)(z) = 0.9,$$
$$F_2(1^*_c)(x) = 0.3,\ F_2(1^*_c)(y) = 1,\ F_2(1^*_c)(z) = 0.4.$$

Define a binary operation $\odot$ (called Łukasiewicz conjection) on $L = [0,1]$ by

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}.$$

(1) If $F_1(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F_1(A_i)$ and $F_1(\alpha \odot A) = \alpha \to F_1(A)$, then

$$\begin{aligned} F_1(A)(x) &= (A(a) \to F_1(1_a)(x)) \wedge (A(b) \to F_1(1_b)(x)) \\ &\quad \wedge (A(c) \to F_1(1_c)(x)) \\ &= (A(a) \to 1) \wedge (A(b) \to 0.5) \wedge (A(c) \to 0.3), \\ F_1(A)(y) &= (A(a) \to 0.8) \wedge (A(b) \to 0.6) \wedge (A(c) \to 1), \\ F_1(A)(z) &= (A(a) \to 0.5) \wedge (A(b) \to 0.9) \wedge (A(c) \to 0.4), \end{aligned}$$

$$\begin{aligned} G_1(B)(a) &= (B(x) \to F_1(1_a)(x)) \wedge (B(y) \to F_1(1_a)(y)) \\ &\quad \wedge (B(z) \to F_1(1_a)(z)) \\ &= (B(x) \to 1) \wedge (B(y) \to 0.8) \wedge (B(z) \to 0.5), \\ G_1(B)(b) &= (B(x) \to 0.5) \wedge (B(y) \to 0.6) \wedge (B(z) \to 0.9), \\ G_1(B)(c) &= (B(x) \to 0.3) \wedge (B(y) \to 1) \wedge (B(z) \to 0.4). \end{aligned}$$

Thus $(e_{L^X}, F_1, G_1, e_{L^Y})$ is a Galois connection.

(2) If $F_1(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F_1(A_i)$ and $F_1(\alpha \odot A) = \alpha \odot F_1(A)$, then

$$\begin{aligned} F_1(A)(x) &= (A(a) \odot F_1(1_a)(x)) \vee (A(b) \odot F_1(1_b)(x)) \\ &\quad \vee (A(c) \odot F_1(1_c)(x)) \\ &= (A(a) \odot 1) \vee (A(b) \odot 0.5) \vee (A(c) \odot 0.3), \\ F_1(A)(y) &= (A(a) \odot 0.8) \vee (A(b) \odot 0.6) \vee (A(c) \odot 1), \\ F_1(A)(z) &= (A(a) \odot 0.5) \vee (A(b) \odot 0.9) \vee (A(c) \odot 0.4), \end{aligned}$$

$$\begin{aligned} G_1(B)(a) &= (F_1(1_a)(x) \to B(x)) \wedge (F_1(1_a)(y) \to B(y)) \\ &\quad \wedge (F_1(1_a)(z) \to B(z)) \\ &= (1 \to B(x)) \wedge (0.8 \to B(y)) \wedge (0.5 \to B(z)), \\ G_1(B)(b) &= (0.5 \to B(x)) \wedge (0.6 \to B(y)) \wedge (0.9 \to B(z)), \\ G_1(B)(c) &= (0.3 \to B(x)) \wedge (1 \to B(y)) \wedge (0.4 \to B(z)). \end{aligned}$$

Thus $(e_{L^X}, F_1, G_1, e_{L^Y})$ is a residuated connection.

(3) If $F_2(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} F_2(A_i)$ and $F_2(\alpha \to A) = \alpha \odot F_2(A)$, then

$$\begin{aligned} F_2(A)(x) &= (A^*(a) \odot F_2(1_a^*)(x)) \vee (A^*(b) \odot F_2(1_b^*)(x)) \\ &\quad \vee (A^*(c) \odot F_2(1_c^*)(x)) \\ &= (A^*(a) \odot 1) \vee (A^*(b) \odot 0.5) \vee (A^*(c) \odot 0.3), \\ F_2(A)(y) &= (A^*(a) \odot 0.8) \vee (A^*(b) \odot 0.6) \vee (A^*(c) \odot 1), \\ F_2(A)(z) &= (A^*(a) \odot 0.5) \vee (A^*(b) \odot 0.9) \vee (A^*(c) \odot 0.4), \end{aligned}$$

$$\begin{aligned} G_2(B)(a) &= (F_2(1_a^*)(x) \odot B^*(x)) \vee (F_2(1_a^*)(y) \odot B^*(y)) \\ &\quad \vee (F_2(1_a^*)(z) \odot B^*(z)) \\ &= (1 \odot B^*(x)) \vee (0.8 \odot B^*(y)) \vee (0.5 \odot B^*(z)), \\ G_2(B)(b) &= (0.5 \odot B^*(x)) \vee (0.6 \odot B^*(y)) \vee (0.9 \odot B^*(z)), \\ G_2(B)(c) &= (0.3 \odot B^*(x)) \vee (1 \odot B^*(y)) \vee (0.4 \odot B^*(z)). \end{aligned}$$

Thus $(e_{L^X}, F_2, G_2, e_{L^Y})$ is a dual Galois connection

(4) If $F_2(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F_2(A_i)$ and $F_2(\alpha \to A) = \alpha \to F_2(A)$, then

$$\begin{aligned} F_2(A)(x) &= (F_2(1_a^*)^*(x) \to A(a)) \wedge (F_2(1_b^*)^*(x) \to A(b)) \\ &\quad \wedge (F_2(1_c^*)(x) \to A(c)) \\ &= (0 \to A(a)) \wedge (0.5 \to A(b)) \wedge (0.3 \to A(c)), \\ F_2(A)(y) &= (0.2 \to A(a)) \wedge (0.4 \to A(b)) \wedge (0.5 \to A(c)), \\ F_2(A)(z) &= (0.5 \to A(a)) \wedge (0.1 \to A(b)) \wedge (0.6 \to A(c)), \end{aligned}$$

$$\begin{aligned} G_2(B)(a) &= (F_2(1_a^*)^*(x) \odot B(x)) \vee (F_2(1_a^*)^*(y) \odot B(y)) \\ &\quad \vee (F_2(1_a^*)^*(z) \odot B(z)) \\ &= (0 \odot B(x)) \vee (0.2 \odot B(y)) \vee (0.5 \odot B(z)), \\ G_2(B)(b) &= (0.5 \odot B(x)) \vee (0.4 \odot B(y)) \vee (0.1 \odot B(z)), \\ G_2(B)(c) &= (0.7 \odot B(x)) \vee (0 \odot B(y)) \vee (0.6 \odot B(z)). \end{aligned}$$

Thus $(e_{L^X}, F_2, G_2, e_{L^Y})$ is a dual residuated connection.

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