

MODAL, NECESSITY, SUFFICIENCY AND CO-SUFFICIENCY OPERATORS

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ABSTRACT. We investigate the properties of modal, necessity, sufficiency and co-sufficiency operators. We show that their operations induce various relations, respectively.

1. Introduction

Pawlak [5] introduced rough set theory to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation about information. An information consists of (X, A) where X is a set of objects and A is a set of attributes, a map $a : X \rightarrow P(A_a)$ where A_a is the value set of the attribute a . Recently, intensional modal-like logics with the propositional operators induced by relations are important mathematical tools for data analysis and knowledge processing [1-3, 6-9].

In this paper, we investigate the properties of modal, necessity, sufficiency and co-sufficiency operators. We show that their operations induce various relations, respectively.

2. Preliminaries

DEFINITION 2.1. [2] Let $P(X), P(Y)$ be the families of subsets on X and Y , respectively. Then a map $F : P(X) \rightarrow P(Y)$ is called

- (1) *modal operator* if $F(\bigcup_{i \in \Gamma} A_i) = \bigcup_{i \in \Gamma} F(A_i)$, $F(\emptyset) = \emptyset$,
- (2) *necessity operator* if $F(\bigcap_{i \in \Gamma} A_i) = \bigcap_{i \in \Gamma} F(A_i)$, $F(X) = Y$,
- (3) *sufficiency operator* if $F(\bigcup_{i \in \Gamma} A_i) = \bigcap_{i \in \Gamma} F(A_i)$, $F(\emptyset) = Y$,
- (4) *co-sufficiency operator* if $F(\bigcap_{i \in \Gamma} A_i) = \bigcup_{i \in \Gamma} F(A_i)$, $F(X) = \emptyset$.

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(5) a dual operator F^∂ is defined by $F^\partial(A) = F(A^c)^c$. Moreover, we define $F^c(A) = (F(A))^c$ and $F^*(A) = F(A^c)$.

DEFINITION 2.2. [2,4] Let $R \subset P(X \times Y)$ be a relation. For each $A \in P(X)$, we define operations $(y, x) \in R^{-1}$ iff $(x, y) \in R$ and $[R], [[R]], \langle R \rangle, \langle\langle R \rangle\rangle, [R]^*, \langle R \rangle^* : P(X) \rightarrow P(Y)$ as follows:

$$\begin{aligned} [R](A) &= \{y \in Y \mid (\forall x)((x, y) \in R \rightarrow x \in A)\}, \\ [[R]](A) &= \{y \in Y \mid (\forall x \in X)(x \in A \rightarrow (x, y) \in R)\} \\ \langle R \rangle(A) &= \{y \in Y \mid (\exists x \in X)((x, y) \in R, x \in A)\} \\ \langle\langle R \rangle\rangle(A) &= \{y \in Y \mid (\exists x \in X)((x, y) \in R^c, x \in A^c)\}, \\ [R]^*(A) &= \{y \in Y \mid (\forall x \in X)((x, y) \in R \rightarrow x \in A^c)\} \\ \langle R \rangle^*(A) &= \{y \in Y \mid (\exists x \in X)((x, y) \in R, x \in A^c)\}. \end{aligned}$$

THEOREM 2.3. [2] Let $R \subset P(X \times Y)$ be a relation.

(1) $\langle R \rangle$ is a modal operator and $[R]$ is a necessity operator with $\langle R \rangle(A) = ([R](A^c))^c = [R]^\partial(A)$, for each $A \in P(X)$.

(2) If $F : P(X) \rightarrow P(Y)$ is a modal operator on $P(X)$, there exists a unique relation $R_F \subset P(X \times Y)$ such that $\langle R_F \rangle = F$ and $[R_F] = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\})$.

(3) $R_{\langle R \rangle} = R$.

3. Modal, necessity, sufficiency and co-sufficiency operators

LEMMA 3.1. Let $F, G : P(X) \rightarrow P(Y)$ be operators. Then the following properties hold:

(1) $(F^\partial)^\partial = F$, $(F^c)^c = F$ and $(F^*)^* = F$.

(2) $(F^\partial)^* = (F^*)^\partial$, $(F^\partial)^c = (F^c)^\partial$ and $(F^*)^c = (F^c)^* = F^\partial$.

(3) $(F \cup G)^\partial = F^\partial \cap G^\partial$, $(F \cup G)^* = F^* \cup G^*$ and $(F \cup G)^c = F^c \cap G^c$.

(4) $F, G : P(X) \rightarrow P(Y)$ are modal operators, then $F \cup G$ is a modal operator and its dual operator $F^\partial \cap G^\partial$ is a necessity operator.

(5) $F, G : P(X) \rightarrow P(Y)$ are necessity operators, then $F \cap G$ is a necessity operator and its dual operator $F^\partial \cap G^\partial$ is a modal operator.

- Proof.* (1) $(F^\partial)^\partial(A) = (F^\partial(A^c))^c = F(A)$.
 (2) $(F^\partial)^*(A) = F^\partial(A^c) = F^c(A) = (F^*(A^c))^c = (F^*)^\partial(A)$.
 $(F^\partial)^c(A) = (F^\partial(A))^c = F(A^c) = (F^c(A^c))^c = (F^c)^\partial(A)$.
 (3) $(F \cup G)^\partial(A) = ((F \cup G)(A^c))^c = (F(A^c))^c \cap (G(A^c))^c = F^\partial(A) \cap G^\partial(A)$. Other cases are similarly proved.
 (4) and (5) are easily proved from (3). □

LEMMA 3.2. (1) A map $F : P(X) \rightarrow P(Y)$ is a modal operator iff $F^\partial : P(X) \rightarrow P(Y)$ is a necessity operator.

(2) A map $F : P(X) \rightarrow P(Y)$ is a sufficiency operator iff $F^\partial : P(X) \rightarrow P(Y)$ is a co-sufficiency operator operator.

(3) A map $F : P(X) \rightarrow P(Y)$ is a modal operator iff $F^c : P(X) \rightarrow P(Y)$ is a sufficient operator.

(4) A map $F : P(X) \rightarrow P(Y)$ is a co-sufficiency operator iff $F^c : P(X) \rightarrow P(Y)$ is a necessity operator operator.

(5) A map $F : P(X) \rightarrow P(Y)$ is a sufficiency operator iff $F^* : P(X) \rightarrow P(Y)$ is a necessity operator operator.

(6) A map $F : P(X) \rightarrow P(Y)$ is a modal operator iff $F^* : P(X) \rightarrow P(Y)$ is a co-sufficiency operator.

Proof. (1) Let $F : P(X) \rightarrow P(Y)$ be a modal operator.

$$\begin{aligned} F^\partial\left(\bigcap_{i \in \Gamma} A_i\right) &= \left(F\left(\bigcup_{i \in \Gamma} A_i^c\right)\right)^c = \left(\bigcup_{i \in \Gamma} F(A_i^c)\right)^c \\ &= \bigcap_{i \in \Gamma} (F(A_i^c))^c = \bigcap_{i \in \Gamma} F^\partial(A_i). \\ F^\partial(X) &= \left(F(X^c)\right)^c = \left(F(\emptyset)\right)^c = Y. \end{aligned}$$

Conversely, $(F^\partial)^\partial(A) = (F^\partial(A^c))^c = F(A)$.

$$\begin{aligned} F\left(\bigcup_{i \in \Gamma} A_i\right) &= \left(F^\partial\left(\bigcap_{i \in \Gamma} A_i^c\right)\right)^c = \left(\bigcap_{i \in \Gamma} F^\partial(A_i^c)\right)^c \\ &= \bigcup_{i \in \Gamma} (F^\partial(A_i^c))^c = \bigcup_{i \in \Gamma} F(A_i). \\ F(\emptyset) &= \left(F^\partial((\emptyset)^c)\right)^c = F(X)^c = \emptyset. \end{aligned}$$

(2), (3) and (4) are similarly proved as same in (1). □

THEOREM 3.3. *Let $R \subset P(X \times Y)$ be a relation.*

(1) $\langle\langle R \rangle\rangle^*$ is a modal operator and $[[R]]^*$ is a necessity operator with $\langle\langle R \rangle\rangle^*(A) = ([[R]]^*(A^c))^c = ([[R]]^*)^\partial(A)$ for each $A \in P(X)$.

(2) If $F : P(X) \rightarrow P(Y)$ is a modal operator on $P(X)$, there exists a unique relation $R_F \subset P(X \times Y)$ such that $\langle\langle R_F \rangle\rangle^* = F$ and $[[R_F]]^* = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\})^c$.

(3) $R_{\langle\langle R \rangle\rangle^*} = R$.

Proof. (1) We have $\langle\langle R \rangle\rangle^*(A) = ([[R]]^*(A^c))^c = ([[R]]^*)^\partial(A)$ from:

$$\begin{aligned} y \in ([[R]]^*(A^c))^c &\text{ iff } \left((\forall x \in X)(X \in A \rightarrow (x, y) \in R) \right)^c \\ &\text{ iff } \left((\forall x \in X)((x, y) \in R^c, x \in A)^c \right)^c \\ &\text{ iff } (\exists x \in X)((x, y) \in R^c, x \in A) \\ &\text{ iff } y \in \langle\langle R \rangle\rangle^*(A). \end{aligned}$$

(2) Since $A = \bigcup_{x \in A} \{x\}$ and $F(A) = \bigcup_{x \in A} F(\{x\})$, we have

$$\begin{aligned} y \in \langle\langle R_F \rangle\rangle^*(A) &\text{ iff } (\exists x \in X)((x, y) \in R_F^c \ \& \ x \in A) \\ &\text{ iff } (\exists x \in X)(y \in F(\{x\}) \ \& \ x \in A) \\ &\text{ iff } y \in \bigcup_{x \in A} F(\{x\}) = F\left(\bigcup_{x \in A} \{x\}\right) = F(A). \end{aligned}$$

$$\begin{aligned} y \in [[R_F]]^*(A) &\text{ iff } (\forall x \in X)(x \in A^c \rightarrow (x, y) \in R_F) \\ &\text{ iff } (\forall x \in X)((x, y) \in R_F^c \rightarrow x \in A) \\ &\text{ iff } (\forall x \in X)(y \in F(\{x\}) \rightarrow x \in A) \\ &\text{ iff } \left((\exists x \in X)(y \in F(\{x\}) \ \& \ x \in A^c) \right)^c \\ &\text{ iff } y \in \left(\bigcup_{x \in A^c} F(\{x\}) \right)^c \\ &\text{ iff } y \in \left(F\left(\bigcup_{x \in A^c} \{x\}\right) \right)^c = (F(A^c))^c = F^\partial(A). \end{aligned}$$

(3)

$$\begin{aligned}
 (x, y) \in R_{\langle\langle R \rangle\rangle^*} &\text{ iff } y \in \langle\langle R \rangle\rangle^* (\{x\})^c \\
 &\text{ iff } \left((\exists z \in X) ((z, y) \in R^c \ \& \ z \in \{x\}) \right)^c \\
 &\text{ iff } (x, y) \in (R^c)^c = R.
 \end{aligned}$$

□

THEOREM 3.4. *Let $R \in P(X \times Y)$ be a relation.*

(1) $[[R]]$ is a sufficiency operator and $\langle\langle R \rangle\rangle$ is a co-sufficiency operator with $\langle\langle R \rangle\rangle(A) = ([[R]](A^c))^c = [[R]]^\partial(A)$ for each $A \in P(X)$.

(2) If $F : P(X) \rightarrow P(Y)$ is a sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $[[R_F]] = F$ and $\langle\langle R_F \rangle\rangle = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\})$.

(3) $R_{[[R]]} = R$.

Proof. (1) We have $\langle\langle R \rangle\rangle(A) = ([[R]](A^c))^c = [[R]]^\partial(A)$ from:

$$\begin{aligned}
 x \in ([[R]](A^c))^c &\text{ iff } \left((\forall y \in X) (y \in A^c \rightarrow (x, y) \in R) \right)^c \\
 &\text{ iff } \left((\forall y \in X) ((x, y) \in R^c \ \& \ y \in A^c) \right)^c \\
 &\text{ iff } (\exists y \in X) ((x, y) \in R^c \ \& \ y \in A^c) \\
 &\text{ iff } x \in \langle\langle R \rangle\rangle(A).
 \end{aligned}$$

(2) Since $F(\bigcup_{x \in A} \{x\}) = \bigcap_{x \in A} F(\{x\})$, we have

$$\begin{aligned}
 y \in [[R_F]](A) &\text{ iff } (\forall x \in X) (x \in A \rightarrow (x, y) \in R_F) \\
 &\text{ iff } (\forall x \in X) (x \in A \rightarrow y \in F(\{x\})) \\
 &\text{ iff } y \in \bigcap_{x \in A} F(\{x\}) = F\left(\bigcup_{x \in A} \{x\}\right) = F(A).
 \end{aligned}$$

$$\begin{aligned}
 y \in \langle\langle R_F \rangle\rangle(A) &\text{ iff } (\exists x \in X) ((x, y) \in R_F^c \ \& \ x \in A^c) \\
 &\text{ iff } (\exists x \in X) (y \in F(\{x\})^c \ \& \ x \in A^c) \\
 &\text{ iff } y \in \bigcup_{x \in A^c} F(\{x\})^c = \left(\bigcap_{x \in A^c} F(\{x\}) \right)^c \\
 &\text{ iff } y \in \left(F\left(\bigcup_{x \in A^c} \{x\}\right) \right)^c = (F(A^c))^c = F^\partial(A).
 \end{aligned}$$

(3)

$$\begin{aligned} (x, y) \in R_{[[R]]} &\text{ iff } (\forall z \in X)(z \in \{x\} \rightarrow (z, y) \in R) \\ &\text{ iff } (x, y) \in R. \end{aligned}$$

□

THEOREM 3.5. *Let $R \in P(X \times Y)$ be a relation.*

(1) $[R]^*$ is a sufficiency operator and $\langle R \rangle^*$ is a co-sufficiency operator with $[R]^*(A) = (\langle R \rangle^*(A^c))^c$.

(2) If $F : P(X) \rightarrow P(Y)$ is a sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $[R_F]^* = F$ and $\langle R_F \rangle^* = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\})^c$.

(3) $R_{[R]^*} = R$.

Proof. (1)

$$\begin{aligned} y \in (\langle R \rangle^*(A^c))^c &\text{ iff } \left((\exists x \in X)(x \in A \ \& \ (x, y) \in R) \right)^c \\ &\text{ iff } (\forall x \in X)((x, y) \in R \rightarrow x \in A^c) \\ &\text{ iff } y \in [R]^*(A). \end{aligned}$$

(2)

$$\begin{aligned} y \in [R_F]^*(A) &\text{ iff } (\forall x \in X)((x, y) \in R_F \rightarrow x \in A^c) \\ &\text{ iff } (\forall x \in X)(x \in A \rightarrow y \in F(\{x\})) \\ &\text{ iff } y \in \bigcap_{x \in A} F(\{x\}) = F\left(\bigcup_{x \in A} \{x\}\right) = F(A). \end{aligned}$$

$$\begin{aligned} y \in \langle R_F \rangle^*(A) &\text{ iff } (\exists x \in X)((x, y) \in R_F \ \& \ x \in A^c) \\ &\text{ iff } (\exists x \in X)(y \in F(\{x\})^c \ \& \ x \in A^c) \\ &\text{ iff } \left((\forall x \in X)(x \in A^c \rightarrow y \in F(\{x\})) \right)^c \\ &\text{ iff } y \in \left(\bigcap_{x \in A^c} (F(\{x\})) \right)^c \\ &\text{ iff } y \in \left(F\left(\bigcup_{x \in A^c} \{x\}\right) \right)^c = (F(A^c))^c \\ &\text{ iff } y \in F^\partial(A). \end{aligned}$$

(3)

$$\begin{aligned}
 (x, y) \in R_{[R]^*} & \text{ iff } y \in [R]^*(\{x\}^c)^c \\
 & \text{ iff } \left((\forall z \in X)((z, y) \in R \rightarrow z \in \{x\}^c) \right)^c \\
 & \text{ iff } (x, y) \in R.
 \end{aligned}$$

□

THEOREM 3.6. *Let $R \subset P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \rightarrow P(Y)$ is a necessity operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $[R_F] = F$ and $\langle R_F \rangle = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)^c$.*

(2) $R_{[R]} = R$.

Proof. (1)

$$\begin{aligned}
 y \in [R_F](A) & \text{ iff } (\forall x \in X)((x, y) \in R_F \rightarrow x \in A) \\
 & \text{ iff } (\forall x \in X)(y \in F(\{x\}^c)^c \rightarrow x \in A) \\
 & \text{ iff } (\forall x \in X)(x \in A^c \rightarrow y \in F(\{x\}^c)) \\
 & \text{ iff } y \in \bigcap_{x \in A^c} F(\{x\}^c) = F\left(\bigcap_{x \in A^c} \{x\}^c\right) = F(A).
 \end{aligned}$$

$$\begin{aligned}
 y \in \langle R_F \rangle(A) & \text{ iff } (\exists x \in X)((x, y) \in R_F \ \& \ x \in A) \\
 & \text{ iff } (\exists x \in X)(y \in F(\{x\}^c)^c \ \& \ x \in A) \\
 & \text{ iff } \left((\forall x \in X)(x \in A \rightarrow y \in F(\{x\}^c)) \right)^c \\
 & \text{ iff } y \in \left(\bigcap_{x \in A} F(\{x\}^c) \right)^c = \left(F\left(\bigcap_{x \in A} \{x\}^c\right) \right)^c \\
 & \text{ iff } y \in F(A^c)^c \text{ iff } y \in F^\partial(A).
 \end{aligned}$$

(2)

$$\begin{aligned}
 (x, y) \in R_{[R]} & \text{ iff } y \in [R](\{x\}^c)^c \\
 & \text{ iff } \left((\forall z \in X)((z, y) \in R \rightarrow z \in \{x\}^c) \right)^c \\
 & \text{ iff } (x, y) \in R.
 \end{aligned}$$

□

THEOREM 3.7. *Let $R \in P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \rightarrow P(Y)$ is a co-sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $\langle\langle R_F \rangle\rangle = F$ and $[[R_F]] = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)$.*

(2) $R_{\langle\langle R_F \rangle\rangle} = R$.

Proof. (1)

$$\begin{aligned} y \in \langle\langle R_F \rangle\rangle(A) &\text{ iff } (\exists x \in X)((x, y) \in R_F^c \ \& \ x \in A^c) \\ &\text{ iff } (\exists x \in X)(y \in F(\{x\}^c) \ \& \ x \in A^c) \\ &\text{ iff } y \in \bigcup_{x \in A^c} F(\{x\}^c) = F\left(\bigcap_{x \in A^c} \{x\}^c\right) = F(A). \end{aligned}$$

$$\begin{aligned} y \in [[R_F]](A) &\text{ iff } (\forall x \in X)(x \in A \rightarrow (x, y) \in R_F) \\ &\text{ iff } (\forall x \in X)(x \in A \rightarrow y \in F(\{x\}^c)^c) \\ &\text{ iff } \left((\exists x \in X)(x \in A \ \& \ y \in F(\{x\}^c)) \right)^c \\ &\text{ iff } y \in \left(\bigcup_{x \in A} F(\{x\}^c) \right)^c \\ &\text{ iff } y \in \left(F\left(\bigcap_{x \in A} \{x\}^c\right) \right)^c \\ &\text{ iff } y \in F(A^c)^c = F^\partial(A). \end{aligned}$$

(2)

$$\begin{aligned} (x, y) \in R_{\langle\langle R \rangle\rangle} &\text{ iff } y \in \langle\langle R \rangle\rangle(\{x\}^c)^c \\ &\text{ iff } \left((\exists z \in X)((z, y) \in R^c \ \& \ z \in \{x\}^c) \right)^c \\ &\text{ iff } (x, y) \in R. \end{aligned}$$

□

THEOREM 3.8. *Let $R \in P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \rightarrow P(Y)$ is a necessity operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $[[R_F]]^* = F$ and $\langle\langle R_F \rangle\rangle^* = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)$.*

(2) $R_{[[R_F]]^*} = R$.

Proof. (1)

$$\begin{aligned}
 y \in [[R_F]]^*(A) & \text{ iff } (\forall x \in X)(x \in A^c \rightarrow (x, y) \in R_F) \\
 & \text{ iff } (\forall x \in X)(x \in A^c \rightarrow y \in F(\{x\}^c)) \\
 & \text{ iff } y \in \bigcap_{x \in A^c} F(\{x\}^c) = F\left(\bigcap_{x \in A^c} \{x\}^c\right) = F(A).
 \end{aligned}$$

$$\begin{aligned}
 x \in \langle\langle R_F \rangle\rangle^*(A) & \text{ iff } (\exists x \in X)((x, y) \in R_F^c \ \& \ x \in A) \\
 & \text{ iff } (\exists x \in X)(y \in F(\{x\}^c)^c \ \& \ x \in A) \\
 & \text{ iff } \left((\forall x \in X)(x \in A \rightarrow y \in F(\{x\}^c)) \right)^c \\
 & \text{ iff } y \in \left(\bigcap_{x \in A} F(\{x\}^c) \right)^c = \left(F\left(\bigcap_{x \in A} \{x\}^c\right) \right)^c \\
 & \text{ iff } y \in F(A^c)^c \text{ iff } y \in F^\partial(A).
 \end{aligned}$$

(2)

$$\begin{aligned}
 (x, y) \in R_{[[R_F]]^*} & \text{ iff } y \in [[R_F]]^*(\{x\}^c) \\
 & \text{ iff } (\forall z \in X)(z \in \{x\} \rightarrow (z, y) \in R) \\
 & \text{ iff } (x, y) \in R.
 \end{aligned}$$

□

THEOREM 3.9. *Let $R \in P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \rightarrow P(Y)$ is a co-sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $\langle R_F \rangle^* = F$ and $[R_F]^* = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)$.*

(2) $R_{\langle R_F \rangle^*} = R$.

Proof. (1)

$$\begin{aligned}
 y \in \langle R_F \rangle^* & \text{ iff } (\exists x \in X)((x, y) \in R_F \ \& \ x \in A^c) \\
 & \text{ iff } (\exists x \in X)(y \in F(\{x\}^c) \ \& \ x \in A^c) \\
 & \text{ iff } y \in \bigcup_{x \in A^c} F(\{x\}^c) = F\left(\bigcap_{x \in A^c} \{x\}^c\right) = F(A).
 \end{aligned}$$

$$\begin{aligned}
y \in [R_F]^*(A) &\text{ iff } (\forall x \in X)(x \in (x, y) \in R_F \rightarrow x \in A^c) \\
&\text{ iff } \left((\exists x \in X)(x \in A \ \& \ (x, y) \in R_F) \right)^c \\
&\text{ iff } \left((\exists x \in X)(x \in A \ \& \ y \in F(\{x\}^c)) \right)^c \\
&\text{ iff } y \in \left(\bigcup_{x \in A} F(\{x\}^c) \right)^c \\
&\text{ iff } y \in \left(F\left(\bigcap_{x \in A} \{x\}^c \right) \right)^c \\
&\text{ iff } y \in F(A^c)^c = F^\partial(A).
\end{aligned}$$

(2)

$$\begin{aligned}
(x, y) \in R_{\langle R_F \rangle^*} &\text{ iff } y \in \langle R_F \rangle^*(\{x\}^c) \\
&\text{ iff } \left((\exists z \in X)((z, y) \in R^c \ \& \ z \in \{x\}) \right)^c \\
&\text{ iff } (x, y) \in R.
\end{aligned}$$

□

EXAMPLE 3.10. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ be a set. Define $F, G : P(X) \rightarrow P(Y)$ as

$$F(\{a\}) = \emptyset, F(\{b\}) = \{x\}, F(\{c\}) = \{y, z\},$$

$$G(\{a\}) = X, G(\{b\}) = \{x, y\}, G(\{c\}) = \{y, z\},$$

$$H(\{b, c\}) = \{x\}, H(\{c, a\}) = \{x, y\}, H(\{a, b\}) = \{z\}.$$

(1) If F is a modal operator, then, by Theorem 2.3,

$$F(A) = \begin{cases} \emptyset, & \text{if } A \in \{\emptyset, \{a\}\}, \\ \{x\}, & \text{if } A \in \{\{b\}, \{a, b\}\}, \\ \{y, z\}, & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ Y, & \text{if } A \in \{\{b, c\}, X\}. \end{cases}$$

Since $(x, y) \in R_F$ iff $y \in F(\{x\})$, we obtain:

$$R_F = \{(b, x), (c, y), (c, z)\}, \langle R_F \rangle = F, [R_F] = F^\partial.$$

(2) If F is a modal operator, then, by Theorem 3.3, we obtain F as same in (1). Since $(x, y) \in R_F$ iff $y \in F(\{x\})^c$, we obtain:

$$R_F = \{(a, x), (a, y), (a, z), (b, y), (b, z), (c, x)\},$$

$$\langle\langle R_F \rangle\rangle^* = F, [[R_F]]^* = F^\partial.$$

(3) If G is a sufficiency operator, then, by Theorem 3.4,

$$G(A) = \begin{cases} Y, & \text{if } A \in \{\emptyset, \{a\}\}, \\ \{x, y\}, & \text{if } A \in \{\{b\}, \{a, b\}\}, \\ \{y, z\}, & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ \{y\}, & \text{if } A \in \{\{b, c\}, X\}. \end{cases}$$

Since $(x, y) \in R_G$ iff $y \in G(\{x\})$, we obtain:

$$R_G = \{(a, x), (a, y), (a, z), (b, x), (b, y), (c, y), (c, z)\},$$

$$[[R_G]] = G, \langle\langle R_G \rangle\rangle = G^\partial.$$

(4) If G is a sufficiency operator, then, by Theorem 3.5, we obtain G as same in (3). Since $(x, y) \in R_G$ iff $y \in G(\{x\})^c$, we obtain:

$$R_G = \{(b, z), (c, x)\}, [R_G]^* = G, \langle R_G \rangle^* = G^\partial.$$

(5) If H is a necessity operator, then, by Theorem 3.6,

$$H(A) = \begin{cases} \emptyset, & \text{if } A \in \{\emptyset, \{a\}, \{b\}\}, \\ \{x\}, & \text{if } A \in \{\{c\}, \{b, c\}\}, \\ \{x, y\}, & \text{if } A = \{a, c\}, \\ \{z\}, & \text{if } A = \{a, b\}, \\ Y, & \text{if } A = X. \end{cases}$$

Since $(x, y) \in R_H$ iff $y \in H(\{x\}^c)^c$, we obtain:

$$R_H = \{(a, y), (a, z), (b, z), (c, x), (c, y)\}, [R_H] = H, \langle R_H \rangle = H^\partial.$$

(6) If H is a necessity operator, then, by Theorem 3.8, we obtain H as same in (5). Since $(x, y) \in R_H$ iff $y \in H(\{x\}^c)$, we obtain:

$$R_H = \{(a, x), (b, x), (b, y), (c, z)\}, [[R_H]]^* = H, \langle \langle R_H \rangle \rangle^* = H^\partial.$$

(7) If H is a co-sufficiency operator, then, by Theorem 3.7, we have:

$$H(A) = \begin{cases} \emptyset, & \text{if } A = X, \\ \{x, y\}, & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ \{x, z\}, & \text{if } A = \{b\}, \\ \{z\}, & \text{if } A = \{a, b\}, \\ \{x\}, & \text{if } A = \{b, c\}, \\ Y, & \text{if } A \in \{\emptyset, \{a\}\}. \end{cases}$$

Since $(x, y) \in R_H$ iff $y \in H(\{x\}^c)^c$, we obtain:

$$R_H = \{(a, y), (a, z), (b, z), (c, x), (c, y)\}, \langle \langle R_H \rangle \rangle = H, [[R_H]] = H^\partial.$$

(8) If H is a co-sufficiency operator, then, by Theorem 3.9, we obtain H as same in (7). Since $(x, y) \in R_H$ iff $y \in H(\{x\}^c)$, we obtain:

$$R_H = \{(a, x), (b, x), (b, y), (c, z)\}, \langle R_H \rangle^* = H, [R_H]^* = H^\partial.$$

References

- [1] R. Bělohlávek, *Lattices of fixed points of Galois connections*, Math. Logic Quart. **47** (2001), 111–116.
- [2] I. Düntsch, Ewa. Orłowska, *Boolean algebras arising from information systems*, Ann. Pure Appl. Logic **127** (2004), 77–98.
- [3] J. Järvinen, M. Kondo, J. Kortelainen, *Logics from Galois connections*, Internat. J. Approx. Reason. **49** (2008), 595–606.
- [4] Ewa. Orłowska, I. Rewitzky, *Algebras for Galois-style connections and their discrete duality*, Fuzzy Sets and Systems **161** (2010), 1325–1342.
- [5] Z. Pawlak, *Rough sets*, Int.J. Comput. Sci. Math. **11** (1982), 341–356.

- [6] G. Qi, W. Liu, *Rough operations on Boolean algebras*, Inform. Sci. **173** (2005), 49–63.
- [7] R. Wille, *Restructuring lattice theory; an approach based on hierarchies of concept*, in: *1. Rival(Ed.), Ordered Sets, Reidel, Dordrecht, Boston* (1982).
- [8] W. Yao, L.X. Lu, *Fuzzy Galois connections on fuzzy posets*, Math. Logic Quart. **55** (2009), 105–112.
- [9] Y.Y. Yao, *Two Views of the Theory of Rough Swts in Finite Universes*, Internat. J. Approx. Reason. **15** (1996), 291–317.

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