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DETERMINANT AND SPECTRUM PRESERVING MAPS ON $\mathbf{M_n}$

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Abstract. Let M_n be the algebra of all complex $n \times n$ matrices and $\phi : M_n \to M_n$ a surjective map (not necessarily additive or multiplicative) satisfying one of the following equations:

$$\det(\phi(A)\phi(B) + \phi(X)) = \det(AB + X),\ \ A, B, X \in M_n,$$

$$\sigma(\phi(A)\phi(B) + \phi(X)) = \sigma(AB + X),\ \ A, B, X \in M_n.$$

Then it is an automorphism, where $\sigma(A)$ is the spectrum of $A \in M_n$. We also show that if $\mathfrak{A}$ be a standard operator algebra, $\mathfrak{B}$ is a unital Banach algebra with trivial center and if $\phi : \mathfrak{A} \to \mathfrak{B}$ is a multiplicative surjection preserving spectrum, then ϕ is an algebra isomorphism.

1. Introduction

The study of linear operators on algebras or vector spaces that leave certain functions, subsets or relations invariant is now commonly referred to as the linear preserver problems. The first result on linear preservers is due to Frobenius [4] who studied the linear maps on matrix algebras preserving determinant. Let M_n be the algebra of all complex n x n matrices. If $A \in M_n$, then A^t denote its transpose. Frobenius proved that if $\phi : M_n \to M_n$ is a bijective linear map satisfying $\det A = \det \phi(A), A \in M_n$ then either ϕ is of the form $\phi(A) = MAN, A \in M_n$ or ϕ is of the form $\phi(A) = MA^tN, A \in M_n$, where $M, N \in M_n$ are nonsingular matrices with $\det(MN) = 1$. Recently, additive preserver problems are also active topics in the study of matrix algebras or infinite dimensional operator algebras. These are problems similar to the linear preserver problems but they consider only the addition of the algebras or vector spaces. In this direction, only a few results concerning the

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preserver problem have been obtained (see, for example, [2, 5, 9, 10] and the references therein). In [3] Dolinar and Šemrl showed the following result holds true without the linearity or additivity of the map.

Theorem A [3, Theorem 1.1] Let $\phi : M_n \to M_n$ be a surjective mapping satisfying

$$\det(A + \lambda B) = \det(\phi(A) + \lambda\phi(B)),\ A, B \in M_n, \lambda \in \mathbb{C}.$$

Then there exist $M, N \in M_n$ with $\det(MN) = 1$ such that either

$$\phi(A) = MAN,\ A \in M_n,$$

or

$$\phi(A) = MA^tN,\ A \in M_n.$$

It is proved in [11] that Theorem A holds true without the surjectivity condition.

Theorem A concerns with the determinant preserving maps with respect to the vector space structure of the underlying algebra.

The purpose of this note is to consider the same problem as in Theorem A with respect to the ring structure of the algebra. More precisely, we will show that if $\phi : M_n \to M_n$ is a surjective map satisfying

$$\det(\phi(A)\phi(B) + \phi(X)) = \det(AB + X),\ A, B, X \in M_n,$$

or

$$\sigma(\phi(A)\phi(B) + \phi(X)) = \sigma(AB + X),\ A, B, X \in M_n,$$

then it is an automorphism, where $\sigma(A)$ is the spectrum of $A \in M_n$. We also show that if $\mathfrak{A}$ be a standard operator algebra, $\mathfrak{B}$ is a unital Banach algebra with trivial center and if $\phi : \mathfrak{A} \to \mathfrak{B}$ is a multiplicative surjection preserving spectrum, then ϕ is an algebra isomorphism.

2. Results

The following lemma plays a key role in the proof of Theorem 3, so we list its proof for reader's convenience.

LEMMA 1. [3, Lemma 2.1] *Let $A, B \in M_n$ be matrices such that $\det(A + X) = \det(B + X)$ for every $X \in M_n$. Then $A = B$.*

Proof. If we denote $Y = A + X$ and $C = B - A$, then $\det Y = \det(C + Y)$ for every $Y \in M_n$. Denote rank $C = r$. Then there exists Y_0 of rank $n - r$ such that $C + Y_0$ is invertible. Hence $\det Y_0 \neq 0$, or equivalently, $r = 0$. It follows that $C = 0$, as desired. $\square$

For x, y in a Hilbert space $\mathcal{H}$, $x \otimes y$ denote the rank one or zero operator on $\mathcal{H}$ given by $z \mapsto < z, y > x$. The following lemma was proved in [9, Lemma 2.4] for operators on Banach spaces. Here, we give a short proof in case of $\mathbb{C}^n$ by using the idea of [6, Lemma 1] .

LEMMA 2. *Let $A, B \in M_n$. If $\sigma(A + X) = \sigma(B + X)$ for every $X \in M_n$, then $A = B$.*

Proof. Let Y be any element of M_n and $X = -B + Y$. Then $\sigma(A - B + Y) = \sigma(B - B + Y) = \sigma(Y)$. Assume that $A - B \neq 0$, and x be a vector of $\mathbb{C}^n$ such that $(A - B)x = y \neq 0$. There is a vector z such that $< x, z >= 1$ and $< y, z >\neq 0$. If $Y = (x - y) \otimes z$, then $(A - B + Y)x = x$, so $1 \in \sigma(A - B + Y)$. But $\sigma(Y) = \{0, < x - y, z >\}$ and $< x - y, z >= 1- < y, z >\neq 1$. Hence $\sigma(A - B + Y) \neq \sigma(Y)$. This is a contradiction. □

We consider M_n as a ring, that is, we consider the addition and multiplication simultaneously and consider the problem similar to that of Theorem A.

THEOREM 3. *Let M_n be the algebra of all complex $n \times n$ matrices. If $\phi : M_n \to M_n$ is a surjective map satisfying*

$$\det(\phi(A)\phi(B) + \phi(X)) = \det(AB + X), \ A, B, X \in M_n,$$

then there is an invertible $M \in M_n$ such that ϕ is of the form

$$\phi(A) = MAM^{-1}, \ A \in M_n.$$

Proof. We first show that $\phi(0) = 0$. Noting that for every $A, X \in M_n$,

$$\det(\phi(A)\phi(0) + \phi(X)) = \det(X) = \det(\phi(0)\phi(A) + \phi(X)),$$

we have by Lemma 1 that $\phi(0)$ commutes with every element of M_n. Hence $\phi(0) = \lambda I$ for some scalar λ. Then $\det(\lambda\phi(A) + \phi(X)) = \det X$. So, by Lemma 1, $\lambda\phi(A) = \lambda\phi(B)$ for every $A, B \in M_n$, from which it follows that $\lambda = 0$ and $\phi(0) = 0$ and hence

$$\det(\phi(A)) = \det A, \ A \in M_n.$$

Similarly, we have that $\phi(I) = \mu I$ for some scalar μ. Since ϕ is determinant preserving it follows that $\mu^n = 1$. Since $\det(\mu\phi(A) + \phi(X)) = \det(A + X)$ for every $A, X \in M_n$, if we take $X = A$, we have $\det((\mu + 1)\phi(A)) = \det(2A)$ for every $A \in M_n$. Then $(\mu + 1)^n = 2^n$

and we have $|\mu+1| = 2$. Since $|\mu| = 1$ and $|\mu+1|$ is the distance between μ and -1, it follows that $\mu = 1$. This shows that $\phi(I) = I$. From this, it follows that

$$\det(\phi(A) + \phi(X)) = \det(A + X),\ \ A, X \in M_n$$

and hence

$$\begin{aligned}\det(\phi(A)\phi(B) + \phi(X)) &= \det(AB + X) \\ &= \det(\phi(AB) + \phi(X)),\ \ A, B, X \in M_n.\end{aligned}$$

Then by Lemma 1, it follows that

$$\phi(A)\phi(B) = \phi(AB),\ \ A, B \in M_n.$$

Next, we show that ϕ is injective. Suppose that $\phi(A) = \phi(C)$ for some $A, C \in M_n$. Then $\det(A+X) = \det(\phi(A)+\phi(X)) = \det(\phi(C)+\phi(X)) = \det(C + X)$ for every X. Then by Lemma 1, $A = C$ and ϕ is injective. Hence $\phi : M_n \to M_n$ is a bijective multiplicative map. Since M_n is prime, that is, $AM_nB = \{0\}$ for $A, B \in M_n$ implies either $A = 0$ or $B = 0$, it is additive by [8]. So, ϕ is a determinant preserving additive map. Then by [11, Theorem 3], $\phi(A) = MAN,\ A \in M_n$ or $\phi(A) = MA^tN,\ A \in M_n$, where M, N are nonsingular elements of M_n. Since $\phi(I) = I$, there is an invertible M such that $\phi(A) = MAM^{-1}$ or $\phi(A) = MA^tM^{-1}$ for $A \in M_n$. Assume, on the contrary, that $\phi(A) = MA^tM^{-1}$ for $A \in M_n$. Then

$$\phi(AB) = \phi(A)\phi(B) = (MA^tM^{-1})(MB^tM^{-1}) = M(BA)^tM^{-1} = \phi(BA)$$

for every A, B. From this, it follows that $AB = BA$ for every $A, B \in M_n$. This is a contradiction. Hence ϕ is an automorphism, completing the proof. $\square$

Next, we consider maps about the spectrum as in Theorem 3. Here, the spectrum is just the set of eigenvalues and we do not necessarily count the eigenvalues according to multiplicity. Note that it was shown in [7, Theorem 3] that if $\phi : M_n \to M_n$ is a linear map which preserves the set of eigenvalues counting multiplicities, then ϕ is either an automorphism or an antiautomorphism.

Theorem 4. *Let M_n be the algebra of all complex $n \times n$ matrices. If $\phi : M_n \to M_n$ is a surjective map satisfying*

$$\sigma(\phi(A)\phi(B) + \phi(X)) = \sigma(AB + X),\ \ A, B, X \in M_n,$$

then there is an invertible $M \in M_n$ such that $\phi : M_n \to M_n$ is of the form

$$\phi(A) = MAM^{-1},\ A \in M_n.$$

Proof. The proof is very similar to that of Theorem 3. We first show that $\phi(0) = 0$. Noting that for every $A, X \in M_n$,

$$\sigma(\phi(A)\phi(0) + \phi(X)) = \sigma(X) = \sigma(\phi(0)\phi(A) + \phi(X)),$$

we have by Lemma 2 that $\phi(0)$ commutes with every element of M_n. Hence $\phi(0) = \lambda I$ for some scalar λ. Then $\sigma(\lambda\phi(A) + \phi(X)) = \sigma(X)$. So, by Lemma 2, $\lambda\phi(A) = \lambda\phi(B)$ for every $A, B \in M_n$, from which it follows that $\lambda = 0$ and hence $\phi(0) = 0$. It also follows that

$$\sigma(\phi(A)) = \sigma(A),\ A \in M_n.$$

Next, we show that $\phi(I) = I$. Since $\sigma(A+X) = \sigma(\phi(A)\phi(I)+\phi(X)) = \sigma(\phi(I)\phi(A) + \phi(X))$, $\phi(I) = \mu I$ for some scalar μ. Then $\sigma(A + X) = \sigma(\mu\phi(A) + \phi(X))$ for every $A, X \in M_n$, from which it follows that $\sigma(\phi(A)) = \sigma(A) = \sigma(\mu\phi(A)) = \mu\sigma(\phi(A))$. Hence $\mu = 1$ and $\phi(I) = I$. From this, it follows that for every $A, X \in M_n$

$$\sigma(A + X) = \sigma(\phi(A) + \phi(X))$$

and hence

$$\sigma(\phi(AB) + \phi(X)) = \sigma(AB + X) = \sigma(\phi(A)\phi(B) + \phi(X)).$$

By Lemma 2, it follows that ϕ is multiplicative.

Next, we show that ϕ is injective. Suppose that $\phi(A) = \phi(C)$ for some $A, C \in M_n$. Then $\sigma(A + X) = \sigma(\phi(A) + \phi(X)) = \sigma(\phi(C) + \phi(X)) = \sigma(C + X)$ for every X. Then by Lemma 2, $A = C$ and ϕ is injective.

Now, since M_n is prime, that is, $AM_nB = \{0\}$ for $A, B \in M_n$ implies either $A = 0$ or $B = 0$, it is additive by [8]. So, ϕ is a spectrum preserving surjective additive map on M_n. Then by [9], it is either of the form $\phi(A) = MAM^{-1}$ for a linear isomorphism $M : \mathbb{C}^n \to \mathbb{C}^n$ or of the form $\phi(A) = MA^*M^{-1}$ for a linear isomorphism $M : (\mathbb{C}^n)^* \to \mathbb{C}^n$. Assume on the contrary that $\phi(A) = MA^*M^{-1}$, $A \in M_n$. Then $\phi(AB) = M(AB)^*M^{-1} = MB^*A^*M^{-1} = \phi(BA)$. This is a contradiction since ϕ is injective. This completes the proof. $\square$

Finally we consider spectrum preserving maps between Banach algebras. For the linear case, in [1, Corollary 3.4] they showed that if $\mathfrak{A}$ is a semisimple Banach algebra, $\mathfrak{B}$ is a primitive Banach algebra with minimal ideals and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a surjective linear map preserving spectrum, then ϕ is either a homomorphism or an antihomomorphism. Recall that a standard operator algebra on a Banach space X is a closed subalgebra in $\mathcal{B}(X)$ which contains the identity I and the ideal of finite rank operators.

Let X be a Banach space, $x \in X$ and $f \in X^*$. We denote by $x \otimes f$ the rank one or zero operator on X given by $z \mapsto (z, f)x$. Note that standard operator algebras and primitive Banach algebras with unit have trivial centers, that is, scalar multiples of identity element of the algebras.

Theorem 5. *Let $\mathfrak{A}$ be a standard operator algebra on a Banach space X and $\mathfrak{B}$ be a unital Banach algebra with trivial center. If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a multiplicative surjection preserving spectrum, then ϕ is an algebra isomorphism.*

Proof. First we show that ϕ is injective. Let $\phi(A) = \phi(C)$ for $A, C \in \mathfrak{A}$. Then for every $D \in \mathfrak{A}$, we have

$$\sigma(AD) = \sigma(\phi(A)\phi(D)) = \sigma(\phi(C)\phi(D)) = \sigma(CD).$$

Taking D as the operator $x \otimes f$, we have

$$\{0, (Ax, f)\} = \sigma(AD) = \sigma(CD) = \{0, (Cx, f)\},$$

from which it follows that $A = C$. Hence ϕ is a bijective multiplicative map. Since each standard algebra is prime, it is additive by [8]. Next, we show that ϕ is linear. To do this, it suffices to show that it is homogeneous. Let $A \in \mathfrak{A}$ and λ be a scalar. Since $\phi(\lambda I)$ is a central element of $\mathfrak{B}$, there is a scalar μ such that $\phi(\lambda I) = \mu I$. Since

$$\{\lambda\} = \sigma(\lambda I) = \sigma(\phi(\lambda I)) = \sigma(\mu I) = \{\mu\},$$

we have $\phi(\lambda I) = \lambda I$. So, $\phi(\lambda A) = \phi(\lambda I)\phi(A) = \lambda\phi(A)$. Hence ϕ is homogeneous. This completes the proof. □

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