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COMPLICATED BCC-ALGEBRAS AND ITS DERIVATION

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Abstract. Any BCK-ideal of a BCC-algebra can be decomposed into the union of some sets. The notion of a complicatedness and a derivation for a BCC-algebra is introduced and some related properties are obtained.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras : BCK-algebras and BCI-algebras ([5,6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The class of all BCK-algebras is a quasi-variety. K. Iséki posed an interesting problem (solved by A. Wronski [8]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori [7] introduced a notion of BCC-algebras, W. A. Dudek [1,2] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [4], J. Hao introduced the concept of ideals in a BCC-algebra and studied some related properties.

In this paper, we show that any BCK-ideal of a BCC-algebra can be decomposed into the union of some sets. We also introduce the notion of a complicated BCC-algebra and a derivation on a BCC-algebra and investigate some related properties.

2. Preliminaries

By a *BCC-algebra* ([7]) we mean a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms: for all $x, y, z \in X$,

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(a_1) $((x*y)*(z*y))*(x*z)=0$,
(a_2) $0*x=0$,
(a_3) $x*0=x$,
(a_4) $x*y=0$ and $y*x=0$ imply $x=y$.

For brevity, we also call X a BCC-algebra. In X we can define a binary operation " $\leq$ " by $x\leq y$ if and only if $x*y=0$. " $\leq$ " is called the *BCC-order* on X. Then $\leq$ is a partial ordering on X. A non-empty subset S of a BCC-algebra X is called a *subalgebra* of X if $x*y\in S$ for all $x,y\in S$.

In a BCC-algebra, the following hold: for any $x,y,z\in X$,

(b_1) $x*x=0$,
(b_2) $(x*y)*x=0$,
(b_3) $x\leq y\Rightarrow x*z\leq y*z$,
(b_4) $x\leq y\Rightarrow z*y\leq z*x$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [2]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(b_5) $(x*y)*z=(x*z)*y$.

Definition 2.1 ([4])**.** Let X be a BCC-algebra and $\emptyset\neq I\subseteq X$. I is called an *ideal* (or a *BCK-ideal*) of X if it satisfies the following conditions:

(i) $0\in I$;
(ii) $x*y, y\in I$ imply $x\in I$, for any $x,y\in X$.

Theorem 2.2 ([4])**.** *In a BCC-algebra X, every ideal of X is a subalgebra of X.*

Definition 2.3 ([3]). Let X be a BCC-algebra and $\emptyset\neq I\subseteq X$. I is called a *BCC-ideal* of X if it satisfies the following conditions:

(i) $0\in I$;
(ii) $(x*y)*z\in I$ and $y\in I$ imply $x*z\in I$.

Lemma 2.4 ([3]). *In a BCC-algebra X, any BCC-ideal of X is an ideal of X.*

Corollary 2.5 ([3]). *Any BCC-ideal of a BCC-algebra X is a subalgebra of X.*

Remark. *In a BCC-algebra, a subalgebra need not be an ideal and an ideal need not be a BCC-ideal, in general (see* [3,4]*).*

Lemma 2.6. *An ideal I of a BCC-algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Complicated BCC-algebras

For any BCC-algebra X and $x, y \in X$, we denote

$$A(x, y) := \{z \in X | (z * x) * y = 0\}.$$

Note that $A(x, y)$ is a non-empty set, since $0, x, y \in A(x, y)$.

Theorem 3.1. *If I is an ideal of a BCC-algebra X, then $I = \cup_{x,y \in I} A(x, y)$.*

Proof. Let I be an ideal of a BCC-algebra X. If $z \in I$, then $(z*z)*0 = 0 * 0 = 0$. Hence $z \in A(z, 0)$. It follows that

$$I \subseteq \cup_{z \in I} A(z, 0) \subseteq \cup_{x,y \in I} A(x, y).$$

Let $z \in \cup_{x,y \in I} A(x, y)$. Then there exist $a, b \in I$ such that $z \in A(a, b)$, so that $(z * a) * b = 0 \in I$. Since I is an ideal of X, we have $z \in I$. Thus $\cup_{x,y \in I} A(x, y) \subseteq I$, and consequently $I = \cup_{x,y \in I} A(x, y)$. □

Corollary 3.2. *If I is an ideal of a BCC-algebra X, then $I = \cup_{x \in I} A(x, 0)$.*

Proof. By Theorem 3.1, we have

$$\cup_{x \in I} A(x, 0) \subseteq \cup_{x,y \in I} A(x, y) = I.$$

If $x \in I$, then $x \in A(x, 0)$ since $(x * x) * 0 = 0 * 0 = 0$. Hence $I \subseteq \cup_{x \in I} A(x, 0)$. This competes the proof. □

We give an example satisfying Theorem 3.1 and Corollary 3.2. See the following example.

Example 3.3. Let $X := \{0, 1, 2, 3, 4\}$ be a BCC-algebra ([3]) which is not a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Then $I := \{0, 1\}$ is an ideal of X. Moreover, it is easy to check that $I = A(0, 0) = A(0, 1) = A(1, 0) = A(1, 1)$.

Theorem 3.4. *Let I be a non-empty subset of a BCC-algebra X such that $0 \in I$ and $I = \cup_{x,y \in I} A(x,y)$. Then I is an ideal of X.*

Proof. Let $x * y, y \in I = \cup_{x,y \in I} A(x,y)$. Since $(x * y) * (x * y) = 0$, we have $x \in A(y, x * y)$. Hence I is an ideal of X. $\square$

Combining Theorems 3.1 and 3.4, we have the following corollary.

Corollary 3.5. *Let X be a BCC-algebra. Then I is an ideal of X if and only if $I = \cup_{x,y \in I} A(x,y)$.*

Definition 3.6. Let X be a BCC-algebra. Given $x, y \in X$, we define a set

$$A(x,y) := \{z \in X | (z * x) * y = 0\}.$$

X is said to be *complicated* if for any $x, y \in X$, the set $A(x,y)$ has the greatest element.

The greatest element of $A(x,y)$ is denoted by $x + y$.

Example 3.7. Let $X := \{0, a, b, c\}$ be a BCC-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then X is a complicated BCC-algebra.

Theorem 3.8. *Let X be a complicated BCC-algebra and let $a, b \in X$. Then the set*

$$\mathcal{H}(a,b) := \{x \in X | a \leq b + x\}$$

has the least element, and it is $a * b$.

Proof. The inequality $a * b \leq a * b$ implies that $a \leq b + (a * b)$ and so $a * b \in \mathcal{H}(a,b)$. Let $z \in \mathcal{H}(a,b)$. Then $a \leq b + z$, which implies from (b_3) and Definition 3.6 that $a * b \leq (b + z) * b \leq z$. Hence $a * b$ is the least element of $\mathcal{H}(a,b)$. $\square$

Proposition 3.9. *Let X be a complicated BCC-algebra. Then for any $x, y, z \in X$, the following hold:*

(i) $(\forall a, b \in X)(a \leq a + b, b \leq a + b)$,
(ii) $(\forall a \in X)(a + 0 = 0 = 0 + a)$,
(iii) $(\forall a, b, c \in X)(a \leq b \Rightarrow a + c \leq b + c)$.

Proof. (i) and (ii) are straightforward.
(iii) Let $a, b, c \in X$ with $a \leq b$. It follows from (b_4) that $(a+c) * b \leq (a+c) * a \leq c$. Hence we have $(a+c) * b \leq c$. Thus $a + c \leq b + c$. □

Proposition 3.10. *Let X be a BCC-algebra with $(a * b) * b = a * b$ for any $a, b \in X$. Then*

(i) $(\forall a, b \in X)(a \leq b \Rightarrow a + b = b)$,
(ii) $(\forall a \in X)(a \leq a \Rightarrow a + a = a)$.

Proof. (i) Let $a, b \in X$ with $a \leq b$. Using (b_4), we have $(a+b) * b \leq (a+b) * a \leq b$ and so $(a+b) * b \leq b$. Hence $0 = ((a+b) * b) * b = (a+b) * b$ and so $a + b \leq b$. Since $b \leq a + b$ for all $a, b \in X$, we have $a + b = b$. □

We provide some characterizations of ideals in a complicated BCC-algebra.

Proposition 3.11. *Let A be a non-empty subset of a complicated BCC-algebra X. If A is an ideal of X, then it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \leq x \Rightarrow y \in A)$.
(ii) $(\forall x, y \in A)(\exists z \in A$ *with* $x \leq z, y \leq z)$.

Proof. Assume that A is an ideal of X. Let $x \in A, y \in X$ with $y \leq x$. Then $y * x = 0$. Since I is an ideal of X, we have $y \in A$. (i) is valid.

Let $x, y \in A$. Since $(x+y) * x \leq y$ and $y \in A$, it follows from (i) that $(x+y) * x \in A$ so that $x + y \in A$ because A is an ideal of X. If we take $z := x + y$, then $x \leq z$ and $y \leq z$ by Proposition 3.9 (i). This completes the proof. □

Theorem 3.12. *Let A be a non-empty subset of a complicated BCC-algebra X. Then A is an ideal of X if and only if it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \leq x \Rightarrow y \in A)$.
(ii) $(\forall x, y \in A)(x, y \in A \Rightarrow x + y \in A)$.

Proof. The necessity follows from Proposition 3.11.

Conversely, let A be a non-empty subset of X satisfying conditions (i) and (ii). Obviously $0 \in A$ by (i) and (a_2). Let $x, y \in X$ satisfying $y \in A$ and $x * y \in A$. Then $y + (x * y) \in A$ by (ii). Since $x \leq y + (x * y)$ by Theorem 3.8, it follows from (i) that $x \in A$. Thus A is an ideal of X. □

4. A derivation in BCC-algebras

We introduce the notion of a derivation in BCC-algebras as follows.

Definition 4.1. Let X be a complicated BCC-algebra. A map $d : X \to X$ is said to be a *derivation* on X if it satisfies the following condition

$$d(x \wedge y) = (dx \wedge y) + (x \wedge dy)$$

where $x \wedge y = y * (y * x)$, for all $x, y \in X$.

We often abbreviate $d(x)$ to dx.

Example 4.2. (1) Let $X := \{0, a, b, c\}$ be a complicated BCC-algebra as Example 3.7. Define a function d on X by

$$dx = \begin{cases} 0 & \text{if } x = 0, b \\ c & \text{if } x = a, c \end{cases}$$

Then d is not a derivation on X since $d(c \wedge b) = d(b * (b * c)) = d(b * 0) = d(b) = 0 \neq b = b + 0 = (b * 0) + (0 * 0) = (b * (b * c)) + (0 * (0 * c)) = c \wedge b + c \wedge 0 = (dc \wedge b) + (c \wedge db)$.
(2) Define a function d on X in Example 3.7 by

$$dx = \begin{cases} 0 & \text{if } x = 0, b \\ a & \text{if } x = a, c \end{cases}$$

Then it is easy to see that d is a derivation on X.

Definition 4.3. A BCC-algebra X is said to be *commutative* if for any $x, y \in X$, $x \wedge y = y \wedge x$.

Example 4.4. (1) Let $X := \{0, a, b, c\}$ be a BCC-algebra as Example 3.7. Then X is a commutative BCC-algebra with $(x * y) * y = x * y$ for all $x, y \in X$.
(2) Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Then X is a BCC-algebra which is not a BCK-algebra satisfying $(x * y) * y = x * y$ for all $x, y \in X$, but not commutative, since $3 * (3 * 2) = 3 * 1 = 3 \neq 2 = 2 * 0 = 2 * (2 * 3)$.

(3) Let $X := \{0, 1, 2, 3\}$ be a BCC-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

Then X is commutative, but not satisfying $(x * y) * y = x * y$ for all $x, y \in X$, since $(3 * 2) * 2 = 1 * 2 = 0 \neq 1 = 3 * 2$.

Proposition 4.5. *Let X be a complicated commutative BCC-algebra with $(x * y) * y = x * y$ for any $x, y \in X$. Then the following hold:*

(i) $dx = dx \wedge x \leq x$ *for all* $x \in X$,
(ii) $d(A(a, b)) \subseteq A(a, b)$,
(iii) *If I is an ideal of a BCC-algebra X, then $d(I) \subseteq I$.*

Proof. (i) Since $dx = d(x \wedge x) = (dx \wedge x) + (x \wedge dx) = dx \wedge x + dx \wedge x = dx \wedge x = x * (x * dx) \leq x$, we have $dx = dx \wedge x \leq x$.
(ii) Let $z \in A(a, b)$. Then $z * a \leq b$. Since $dz \leq z$, using (b_3), we have $dz * a \leq z * a \leq b$. Hence $dz \in A(a, b)$. Therefore $d(A(a, b)) \subseteq A(a, b)$.
(iii) Let I be an ideal of a BCC-algebra X. Let $x \in I$. From (i), we get $dx \leq x$. Hence $dx * x = 0 \in I$. Since I is an ideal of X, $dx \in I$. Thus $d(I) \subseteq I$. □

Corollary 4.6. *Let X be a complicated commutative BCC-algebra with $(x * y) * y = x * y$ for any $x, y \in X$. Then*

(i) $d(a + b) \leq a + b$
(ii) $d^2 x = dx$.

Proof. (i) Let $x := a + b$ in Proposition 4.5 (i).
(ii) Using Proposition 4.5 (i), we obtain $d^2 x = d(dx) = d(dx \wedge x) = (d^2 x \wedge x) + dx \wedge dx = d^2 x + dx = dx$. □

Theorem 4.7. *Let X be a complicated commutative BCC-algebra with $(x * y) * y = x * y$ for any $x, y \in X$. Then*

$$dx \wedge dy \leq d(x \wedge y) \leq dx + dy.$$

Proof. Since $dy \leq y$, by (b_4), we have $dx * y \leq dx * dy$ and hence $dx * (dx * dy) \leq dx * (dx * y)$, i.e., $dx \wedge dy \leq dx \wedge y$. Similarly, we obtain

$dx \wedge dy \leq dy \wedge x$. Hence $dx \wedge dy \leq dx \wedge y + dy \wedge x = d(x \wedge y)$. Since $dx \wedge y = dx * (dx * y) \leq dx$, we obtain

$$d(x \wedge y) = dx \wedge y + x \wedge dy \leq dx + x \wedge dy \leq dx + dy.$$

Therefore we have $dx \wedge dy \leq d(x \wedge y) \leq dx + dy$. □

Definition 4.8. Let X be a complicated BCC-algebra. A derivation d on X is said to be *isotone* if $x \leq y$ implies $dx \leq dy$ for all $x, y \in X$.

Proposition 4.9. *Let X be a complicated commutative BCC-algebra and let d be a derivation on X. Then the following hold: for all $x, y \in X$*

(i) *if $d(x \wedge y) = dx \wedge dy$, then d is an isotone derivation.*
(ii) *if $d(x + y) = dx + dy$, then d is an isotone derivation.*

Proof. (i) Let $x, y \in X$ with $x * y = 0$. Then $x \wedge y = x * (x * y) = x * 0 = x$. Hence $dx = d(x \wedge y) = dx \wedge dy \leq dy$. Thus d is an isotone derivation.
(ii) From Proposition 3.10 (i), we have $x+y = y$. Hence $dy = d(x+y) = dx + dy$ and so $dx \leq dy$. Thus d is an isotone derivation. □

Theorem 4.10. *Let X be a complicated commutative BCC-algebra with $(x*y)*y = x*y$ for any $x, y \in X$. Then the following are equivalent:*

(i) *d is the identity derivation,*
(ii) *d is one-to-one,*
(iii) *d is onto.*

Proof. (i) ⇒ (ii) and (i) ⇒ (iii) are straightforward.
(ii) ⇒ (i): Assume that d is not the identity derivation. Let d be a one-to-one function. If there exists an element $a \in X$ with $da \neq a$, then $da < a$. Denote $a_1 := da$. Then $a_1 < a$. Hence $da_1 = d(a_1 \wedge a) = (da_1 \wedge a) + (a_1 \wedge da) = da_1 \wedge a + a_1 = a_1$, i.e., $da_1 = a_1 = da$, which implies $a_1 = a$, a contradiction. Thus d is the identity derivation.
(iii) ⇒ (i): Assume that d is onto, i.e., $dX = X$. For any $x \in X$, there exists $y \in Y$ with $x = dy$. Hence, using Corollary 4.6, we obtain $dx = d(dy) = d^2y = dy = x$. Thus d is the identity derivation. □

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