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SEVERAL TYPES OF BIPOLAR FUZZY HYPER BCK-IDEALS IN HYPER BCK-ALGEBRAS

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Abstract. Several types of a bipolar fuzzy hyper BCK-ideal in a hyper BCK-algebra are introduced, and their relations are investigated.

1. Introduction

Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets, which are introduced by Lee ([8], [9]), are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Using the notion of bipolar-valued fuzzy sets, Jun et al. [1] introduced the notion of a bipolar fuzzy implicatiive hyper BCK-ideal in hyper BCK-algebras, and investigated several properties. They established a relationship between bipolar fuzzy implicative hyper BCK-ideal and bipolar fuzzy hyper BCK-ideal, and provided conditions for a bipolar fuzzy hyper BCK-ideal to be a bipolar fuzzy implicative hyper BCK-ideal. Also, they discussed bipolar fuzzy (weak, s-weak, strong) hyper BCK-ideals in hyper BCK-algebras (see [2] and [3]).

In this paper, we introduce several types of a bipolar fuzzy hyper BCK-ideal, and investigate relations between each type.

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2. Preliminaries

Let X be the universe of discourse. A *bipolar-valued fuzzy set* φ in H is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in H\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : H \to [0, 1]$ are mappings. The positive membership degree $\varphi^+(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in H\}$, and the negative membership degree $\varphi^-(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in H\}$. If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in H\}$. If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, it is the situation that x does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in H\}$ but somewhat satisfies the counter-property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in H\}$. It is possible for an element x to be $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [9]). For the sake of simplicity, we shall use the symbol $\varphi = (H; \varphi^-, \varphi^+)$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in H\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [5], [6], and [7].

Let H be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall x, y \in H)(x \circ y \ll \{x\}). \tag{2.1}$$

In any hyper BCK-algebra H, the following hold:

(a1) $x \circ 0 \ll \{x\},\ 0 \circ x \ll \{0\}\ 0 \circ 0 \ll \{0\}$,
(a2) $(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}$,
(a3) $0 \circ 0 = \{0\}$,
(a4) $0 \ll x$,
(a5) $x \ll x$,
(a6) $A \ll A$,
(a7) $A \subseteq B \ \Rightarrow\ A \ll B$,
(a8) $0 \circ x = \{0\}$,
(a9) $0 \circ A = \{0\}$,
(a10) $A \ll \{0\} \ \Rightarrow\ A = \{0\}$,
(a11) $A \circ B \ll A$,
(a12) $x \in x \circ 0$,
(a13) $x \circ 0 \ll \{y\} \ \Rightarrow\ x \ll y$,
(a14) $y \ll z \ \Rightarrow\ x \circ z \ll x \circ y$,
(a15) $x \circ y = \{0\} \ \Rightarrow\ (x \circ z) \circ (y \circ z) = \{0\},\ x \circ z \ll y \circ z$,
(a16) $A \circ \{0\} = \{0\} \ \Rightarrow\ A = \{0\}$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

A nonempty subset I of a hyper BCK-algebra H is said to be a *hyper BCK-ideal* of H if it satisfies

(I1) $0 \in I$,
(I2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

We now review some fuzzy logic concepts. A *fuzzy set* in a set H is a function $\mu : X \to [0, 1]$, and the complement of μ, denoted by $\bar{\mu}$, is the fuzzy set in H given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in H$. For any $t \in [0, 1]$ and a fuzzy set μ in a nonempty set H, the set

$$U(\mu; t) = \{x \in H \mid \mu(x) \geq t\} \ \text{(resp. } L(\mu; t) = \{x \in H \mid \mu(x) \leq t\})$$

is called an *upper* (resp. *lower*) *level set* of μ.

A fuzzy set μ in a hyper BCK-algebra H is called a *fuzzy hyper BCK-ideal* of H if it satisfies

$$(\forall x, y \in H)(x \ll y \ \Rightarrow\ \mu(y) \leq \mu(x)), \tag{2.2}$$

$$(\forall x, y \in H)(\mu(x) \geq \min\Big\{\inf_{a \in x \circ y} \mu(a), \mu(y)\Big\}). \tag{2.3}$$

3. Several types of bipolar fuzzy hyper BCK-ideals

In what follows let H denote a hyper BCK-algebra unless otherwise specified.

For a bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H, consider the following conditions:

(b1) $(\forall x, y \in H)$ $(x \ll y \Rightarrow \varphi^-(x) \leq \varphi^-(y),\ \varphi^+(x) \geq \varphi^+(y))$.

(b2) $(\forall x, y \in H)$ $\left(\varphi^-(x) \leq \max\left\{\sup_{b \in x \circ y} \varphi^-(b),\ \varphi^-(y)\right\}\right)$.

(b3) $(\forall x, y \in H)$ $\left(\varphi^-(x) \leq \max\left\{\inf_{b \in x \circ y} \varphi^-(b),\ \varphi^-(y)\right\}\right)$.

(b4) $(\forall x, y \in H)$ $\left(\varphi^+(x) \geq \min\left\{\inf_{b \in x \circ y} \varphi^+(b),\ \varphi^+(y)\right\}\right)$.

(b5) $(\forall x, y \in H)$ $\left(\varphi^+(x) \geq \min\left\{\sup_{b \in x \circ y} \varphi^+(b),\ \varphi^+(y)\right\}\right)$.

Definition 3.1. A bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H is called a *bipolar fuzzy hyper BCK-ideal* of

(1) *type 1* if it satisfies (b1), (b2) and (b4).
(2) *type 2* if it satisfies (b1), (b2) and (b5).
(3) *type 3* if it satisfies (b1), (b3) and (b4).
(4) *type 4* if it satisfies (b1), (b3) and (b5).

Example 3.2. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the hyper operation "$\circ$" which is given by the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Define a bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H by

	0	a	b
φ^-	-0.7	-0.4	-0.2
φ^+	0.8	0.3	0.2

It is easily verified that $\varphi = (H; \varphi^-, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 1. Note that

$$\varphi^+(b) = 0.2 \ngeq 0.3 = \min\{\sup_{x \in b \circ a} \varphi^+(x), \varphi^+(a)\}$$

and

$$\varphi^-(b) = -0.2 \nleq -0.4 = \max\{\inf_{x \in b \circ a} \varphi^-(x), \varphi^-(a)\}.$$

Hence $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is neither a bipolar fuzzy hyper BCK-ideal of type 2 nor a bipolar fuzzy hyper BCK-ideal of type 3. Moreover, $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is not a bipolar fuzzy hyper BCK-ideal of type 4.

Example 3.3. Consider a hyper BCK-algebra $H = \{0, a, b\}$ in Example 3.2 and let $\psi = (H;\, \psi^-,\, \psi^+)$ be a bipolar fuzzy set in H defined by

	0	a	b
ψ^-	-0.7	-0.4	-0.2
ψ^+	0.9	0.3	0.3

It is easily verified that $\psi = (H;\, \psi^-,\, \psi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 2. Since $\psi^-(b) = -0.2 \not\leq -0.4 = \max\{\inf\limits_{x \in b \circ a} \psi^-(x), \psi^-(a)\}$, we know that $\psi = (H;\, \psi^-,\, \psi^+)$ is not a bipolar fuzzy hyper BCK-ideal of type 3, and hence not of type 4.

Example 3.4. Let $H = \{0, a, b\}$ be a hyper BCK-algebra which is given in Example 3.2. Define a bipolar fuzzy set $\psi = (H;\, \psi^-,\, \psi^+)$ in H by

	0	a	b
ψ^-	-0.7	-0.4	-0.4
ψ^+	0.9	0.3	0.2

Then $\psi = (H;\, \psi^-,\, \psi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 3, but neither of type 2 nor of type 4 since

$$\psi^+(b) = 0.2 \not\geq 0.3 = \min\Big\{\sup_{x \in b \circ a} \psi^+(x), \psi^+(a)\Big\}.$$

Example 3.5. Let $H = \{0, a, b\}$ be a hyper BCK-algebra which is given in Example 3.2. Define a bipolar fuzzy set $\psi = (H;\, \psi^-,\, \psi^+)$ in H by

	0	a	b
ψ^-	-0.8	-0.4	-0.4
ψ^+	0.9	0.3	0.3

Then $\psi = (H;\, \psi^-,\, \psi^+)$ is a bipolar fuzzy hyper BCK-ideal of types 1, 2, 3 and 4.

Definition 3.6. A bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H is called a *bipolar fuzzy hyper BCK-ideal* of type $(\forall, \exists)$ (briefly, *bipolar fuzzy* $(\forall, \exists)$-*hyper BCK-ideal* of H) if it satisfies (b1) and

$$(3.1)\qquad \begin{aligned} &(\forall x, y \in H)\ (\exists a \in x \circ y)\ (\varphi^-(x) \leq \max\{\varphi^-(a), \varphi^-(y)\}),\\ &(\forall x, y \in H)\ (\exists b \in x \circ y)\ (\varphi^+(x) \geq \min\{\varphi^+(b), \varphi^+(y)\}). \end{aligned}$$

Definition 3.7. A bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H is called a *bipolar fuzzy hyper BCK-ideal* of type $(\forall, \forall)$ (briefly, *bipolar fuzzy* $(\forall, \forall)$-*hyper BCK-ideal* of H) if it satisfies (b1) and

$$(3.2)\qquad \begin{aligned} &(\forall x, y \in H)\ (\forall a \in x \circ y)\ (\varphi^-(x) \leq \max\{\varphi^-(a), \varphi^-(y)\}),\\ &(\forall x, y \in H)\ (\forall b \in x \circ y)\ (\varphi^+(x) \geq \min\{\varphi^+(b), \varphi^+(y)\}). \end{aligned}$$

Obviously, every bipolar fuzzy $(\forall, \forall)$-hyper BCK-ideal of H is a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H, but the converse is not true as seen in the following example.

Example 3.8. Let $H = \{0, a, b\}$ be a hyper BCK-algebra which is given in Example 3.2. Define a bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H by

	0	a	b
φ^-	-0.6	-0.4	-0.3
φ^+	0.7	0.5	0.3

Then $\varphi = (H; \varphi^-, \varphi^+)$ is a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H. But $\varphi = (H; \varphi^-, \varphi^+)$ is not a bipolar fuzzy $(\forall, \forall)$-hyper BCK-ideal of H since

$$\varphi^-(b) \nleq \max\{\varphi^-(a), \varphi^-(a)\} \text{ for } a \in b \circ a$$

and/or

$$\varphi^+(b) \ngeq \min\{\varphi^+(a), \varphi^+(a)\} \text{ for } a \in b \circ a.$$

A bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H is said to satisfy the **sup-inf** *property* if for any nonempty subset T of H there exist $x_0, y_0 \in T$ such that $\varphi^-(x_0) = \sup_{x \in T} \varphi^-(x)$ and $\varphi^+(y_0) = \inf_{y \in T} \varphi^+(y)$.

Theorem 3.9. *For a bipolar fuzzy hyper BCK-ideal $\varphi = (H; \varphi^-, \varphi^+)$ of type 1, we have*

(1) $(\forall x \in H)\ (\varphi^-(0) \leq \varphi^-(x),\ \varphi^+(0) \geq \varphi^+(x))$.
(2) *If $\varphi = (H; \varphi^-, \varphi^+)$ satisfies the* **sup-inf** *property, then $\varphi = (H; \varphi^-, \varphi^+)$ is a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H.*

Proof. (1) Since $0 \ll x$ for all $x \in H$, it follows from (b1) that $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$.

(2) Let $x, y \in H$. Since $\varphi = (H;\ \varphi^-,\ \varphi^+)$ satisfies the **sup-inf** property, there exist $x_0, y_0 \in T$ such that $\varphi^-(x_0) = \sup\limits_{a \in x \circ y} \varphi^-(a)$ and $\varphi^+(y_0) = \inf\limits_{b \in x \circ y} \varphi^+(b)$. Thus

$$\varphi^-(x) \leq \max\{\sup_{a \in x \circ y} \varphi^-(a), \varphi^-(y)\} = \max\{\varphi^-(x_0), \varphi^-(y)\}$$

and

$$\varphi^+(x) \geq \min\{\inf_{b \in x \circ y} \varphi^+(b), \varphi^+(y)\} = \min\{\varphi^+(y_0), \varphi^+(y)\}.$$

Therefore $\varphi = (H;\ \varphi^-,\ \varphi^+)$ is a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H. □

Corollary 3.10. *For a bipolar fuzzy hyper BCK-ideal $\varphi = (H;\ \varphi^-,\ \varphi^+)$ of type 2, 3 or 4, we have*

(1) $(\forall x \in H)$ $(\varphi^-(0) \leq \varphi^-(x),\ \varphi^+(0) \geq \varphi^+(x))$.
(2) *If $\varphi = (H;\ \varphi^-,\ \varphi^+)$ satisfies the* **sup-inf** *property, then $\varphi = (H;\ \varphi^-,\ \varphi^+)$ is a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H.*

Proof. Straightforward. □

In Theorem 3.9, if $\varphi = (H;\ \varphi^-,\ \varphi^+)$ does not satisfy the **sup-inf** property, then $\varphi = (H;\ \varphi^-,\ \varphi^+)$ is not a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H as seen in the following example.

Example 3.11. Let $H = \mathbb{N} \cup \{0, \alpha\}$, where $\alpha(\neq 0) \notin \mathbb{N}$. Define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0\} & \text{if } \ x = 0, \\ \{0, x\} & \text{if } \ (x \leq y,\ x \in \mathbb{N}) \text{ or } (x \in \mathbb{N},\ y = \alpha), \\ \{x\} & \text{if } \ x > y,\ x \in \mathbb{N}, \\ \{0\} \cup \mathbb{N} & \text{if } \ x = y = \alpha, \\ \mathbb{N} & \text{if } \ x = \alpha,\ y \in \mathbb{N}, \\ \{\alpha\} & \text{if } \ x = \alpha,\ y = 0. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algebra. Define a bipolar fuzzy set $\varphi = (H;\ \varphi^-,\ \varphi^+)$ in H by

	0	1	2	3	$\cdots$	α
φ^-	$-4+3$	$-4+3.1$	$-4+3.14$	$-4+3.141$	$\cdots$	$-4+\pi$
φ^+	$4-3$	$4-3.1$	$4-3.14$	$4-3.141$	$\cdots$	$4-\pi$

Then $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 1, and $\varphi = (H;\, \varphi^-,\, \varphi^+)$ does not satisfy the **sup-inf** property. Note that $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is not a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H since

$$\varphi^-(\alpha) = -4 + \pi \nleq \max\{\varphi^-(a), \varphi^-(3)\} = \begin{cases} \varphi^-(3) & \text{if } a < 3, \\ \varphi^-(a) & \text{if } a \geq 3 \end{cases}$$

and/or

$$\varphi^+(\alpha) = 4 - \pi \ngeq \min\{\varphi^+(a), \varphi^+(3)\} = \begin{cases} \varphi^+(3) & \text{if } a < 3, \\ \varphi^+(a) & \text{if } a \geq 3 \end{cases}$$

for $a \in \alpha \circ 3 = \mathbb{N}$.

Theorem 3.12. *Every bipolar fuzzy hyper BCK-ideal of type 4 is a bipolar fuzzy $(\forall, \forall)$-hyper BCK-ideal of H.*

Proof. Let $\varphi = (H;\, \varphi^-,\, \varphi^+)$ be a bipolar fuzzy hyper BCK-ideal of type 4. Since $\inf\limits_{a \in x \circ y} \varphi^-(a) \leq \varphi^-(x_a)$ and $\sup\limits_{b \in x \circ y} \varphi^+(b) \geq \varphi^+(x_b)$ for all $x_a, x_b \in x \circ y$, it follows from (b3) and (b5) that

$$\varphi^-(x) \leq \max\{\inf_{a \in x \circ y} \varphi^-(a), \varphi^-(y)\} \leq \max\{\varphi^-(x_a), \varphi^-(y)\}$$

and

$$\varphi^+(x) \geq \min\{\sup_{b \in x \circ y} \varphi^+(b), \varphi^+(y)\} \geq \min\{\varphi^+(x_b), \varphi^+(y)\}$$

for all $x_a, x_b \in x \circ y$. Hence $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy $(\forall, \forall)$-hyper BCK-ideal of H. □

Corollary 3.13. *Every bipolar fuzzy hyper BCK-ideal of type 4 is a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H.*

Proof. Straightforward. □

Now we consider the converse of Theorem 3.12. Let $\varphi = (H;\, \varphi^-,\, \varphi^+)$ be a bipolar fuzzy $(\forall, \forall)$-hyper BCK-ideal of H. For any $x, y \in H$, we divide into four cases as follows:

(c1) $\varphi^-(a) \geq \varphi^-(y)$ for all $a \in x \circ y$, and $\varphi^+(b) \leq \varphi^+(y)$ for all $b \in x \circ y$.

(c2) $\varphi^-(a) < \varphi^-(y)$ for some $a \in x \circ y$, and $\varphi^+(b) > \varphi^+(y)$ for some $b \in x \circ y$.

(c3) $\varphi^-(a) \geq \varphi^-(y)$ for all $a \in x \circ y$, and $\varphi^+(b) > \varphi^+(y)$ for some $b \in x \circ y$.

(c4) $\varphi^-(a) < \varphi^-(y)$ for some $a \in x \circ y$, and $\varphi^+(b) \leq \varphi^+(y)$ for all $b \in x \circ y$.

Case (c1) implies that

$$\varphi^-(x) \le \max\{\varphi^-(a), \varphi^-(y)\} = \varphi^-(a)$$

and

$$\varphi^+(x) \ge \min\{\varphi^+(b), \varphi^+(y)\} = \varphi^+(b)$$

so that

$$\varphi^-(x) \le \inf_{a\in x\circ y} \varphi^-(a) \le \max\Big\{\inf_{a\in x\circ y} \varphi^-(a), \varphi^-(y)\Big\}$$

and

$$\varphi^+(x) \ge \sup_{b\in x\circ y} \varphi^+(b) \ge \min\Big\{\sup_{b\in x\circ y} \varphi^+(b), \varphi^+(y)\Big\}.$$

For the Case (c2), we have

$$\varphi^-(x) \le \max\{\varphi^-(a), \varphi^-(y)\} = \varphi^-(y)$$

and

$$\varphi^+(x) \ge \min\{\varphi^+(b), \varphi^+(y)\} = \varphi^+(y),$$

and hence

$$\varphi^-(x) \le \varphi^-(y) \le \max\Big\{\inf_{a\in x\circ y} \varphi^-(a), \varphi^-(y)\Big\}$$

and

$$\varphi^+(x) \ge \varphi^+(y) \ge \min\Big\{\sup_{b\in x\circ y} \varphi^+(b), \varphi^+(y)\Big\}.$$

Case (c3) induces $\varphi^-(x) \le \max\{\varphi^-(a), \varphi^-(y)\} = \varphi^-(a)$, which implies that

$$\varphi^-(x) \le \inf_{a\in x\circ y} \varphi^-(a) \le \max\Big\{\inf_{a\in x\circ y} \varphi^-(a), \varphi^-(y)\Big\},$$

and $\varphi^+(x) \ge \min\{\varphi^+(b), \varphi^+(y)\} = \varphi^+(y) \ge \min\Big\{\sup_{b\in x\circ y} \varphi^+(b), \varphi^+(y)\Big\}$.

Finally, Case (c4) implies that

$$\varphi^-(x) \le \max\{\varphi^-(a), \varphi^-(y)\} = \varphi^-(y) \le \max\Big\{\inf_{a\in x\circ y} \varphi^-(a), \varphi^-(y)\Big\}$$

and $\varphi^+(x) \ge \min\{\varphi^+(b), \varphi^+(y)\} = \varphi^+(b)$, which implies that

$$\varphi^+(x) \ge \sup_{b\in x\circ y} \varphi^+(b) \ge \min\Big\{\sup_{b\in x\circ y} \varphi^+(b), \varphi^+(y)\Big\}.$$

Hence $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 4. We state it as a theorem.

Theorem 3.14. *Every bipolar fuzzy $(\forall,\forall)$-hyper BCK-ideal of H is a bipolar fuzzy hyper BCK-ideal of type* 4.

According to Theorems 3.12 and 3.14, we know that the notion of a bipolar fuzzy $(\forall, \forall)$-hyper BCK-ideal coincide with the notion of a bipolar fuzzy hyper BCK-ideal of type 4.

The following example shows that a bipolar fuzzy hyper BCK-ideal of type 2 may not be a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H.

Example 3.15. Let $H = \mathbb{N} \cup \{0, \alpha\}$ be a hyper BCK-algebra which is described in Example 3.11. Define a bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H by

	0	1	2	3	$\cdots$	α
φ^-	$-4+3$	$-4+3.1$	$-4+3.14$	$-4+3.141$	$\cdots$	$-4+\pi$
φ^+	0	0	0	0	$\cdots$	0

Then $\varphi = (H; \varphi^-, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 2. For any $a \in \alpha \circ 3 = \mathbb{N}$, we have

$$\varphi^-(\alpha) = -4 + \pi \not\leq \max\{\varphi^-(a), \varphi^-(3)\} = \begin{cases} \varphi^-(3) & \text{if } a < 3, \\ \varphi^-(a) & \text{if } a \geq 3. \end{cases}$$

Hence $\varphi = (H; \varphi^-, \varphi^+)$ is not a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H.

The following example shows that a bipolar fuzzy hyper BCK-ideal of type 3 may not be a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H.

Example 3.16. Let $H = \mathbb{N} \cup \{0, \alpha\}$ be a hyper BCK-algebra which is described in Example 3.11. Define a bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H by

	0	1	2	3	$\cdots$	α
φ^-	0	0	0	0	$\cdots$	0
φ^+	$4-3$	$4-3.1$	$4-3.14$	$4-3.141$	$\cdots$	$4-\pi$

Then $\varphi = (H; \varphi^-, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 3. For any $a \in \alpha \circ 3 = \mathbb{N}$, we have

$$\varphi^+(\alpha) = 4 - \pi \not\geq \min\{\varphi^+(a), \varphi^+(3)\} = \begin{cases} \varphi^+(3) & \text{if } a < 3, \\ \varphi^+(a) & \text{if } a \geq 3, \end{cases}$$

and so $\varphi = (H; \varphi^-, \varphi^+)$ is not a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H.

The following example shows that any bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal is neither a bipolar fuzzy hyper BCK-ideal of type 2 nor a bipolar fuzzy hyper BCK-ideal of type 3.

Example 3.17. Consider a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal $\varphi = (H;\, \varphi^-,\, \varphi^+)$ in Example 3.8. Note that

$$\varphi^+(b) = 0.3 < 0.5 = \min\{\sup_{x \in b \circ a} \varphi^+(x), \varphi^+(a)\}$$

and

$$\varphi^-(b) = -0.3 > -0.4 = \max\{\inf_{x \in b \circ a} \varphi^-(x), \varphi^-(a)\}.$$

Therefore $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is neither a bipolar fuzzy hyper BCK-ideal of type 2 nor a bipolar fuzzy hyper BCK-ideal of type 3.

Theorem 3.18. *Every bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H is a bipolar fuzzy hyper BCK-ideal of type 1.*

Proof. Let $\varphi = (H;\, \varphi^-,\, \varphi^+)$ be a bipolar fuzzy $(\forall, \exists)$-hyper BCK-ideal of H and let $x, y \in H$. Then there exist $a, b \in x \circ y$ such that

$$\begin{aligned} \varphi^-(x) &\leq \max\{\varphi^-(a), \varphi^-(y)\}, \\ \varphi^+(x) &\geq \min\{\varphi^+(b), \varphi^+(y)\}. \end{aligned} \tag{3.3}$$

Since $a, b \in x \circ y$, we get $\varphi^-(a) \leq \sup_{x_a \in x \circ y} \varphi^-(x_a)$ and $\varphi^+(b) \geq \inf_{x_b \in x \circ y} \varphi^+(x_b)$. It follows from (3.3) that

$$\varphi^-(x) \leq \max\{\sup_{x_a \in x \circ y} \varphi^-(x_a), \varphi^-(y)\}$$

and

$$\varphi^+(x) \geq \min\{\inf_{x_b \in x \circ y} \varphi^+(x_b), \varphi^+(y)\}$$

so that $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 1. □

Example 3.11 shows that the converse of Theorem 3.18 may not be true.

Lemma 3.19. [4] *Let A be a subset of H. If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I.*

For a bipolar fuzzy set $\varphi = (H;\, \varphi^-,\, \varphi^+)$ in a set H, the *negative level set* and *positive level set* are denoted by $N(\varphi^-; \alpha)$ and $P(\varphi^+; \beta)$, and are defined as follows:

$$N(\varphi^-; \alpha) := \{x \in H \mid \varphi^-(x) \leq \alpha\},\ \alpha \in [-1, 0],$$

$$P(\varphi^+;\beta) := \{x \in H \mid \varphi^+(x) \geq \beta\},\ \beta \in [0,1],$$

respectively.

Theorem 3.20. *Let $\varphi = (H;\, \varphi^-,\, \varphi^+)$ be a bipolar fuzzy set in H. Then $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 1 if and only if for every $(\alpha,\beta) \in [-1,0]\times[0,1]$, the nonempty negative level set $N(\varphi^-;\alpha)$ and the nonempty positive level set $P(\varphi^+;\beta)$ are hyper BCK-ideals of H.*

Proof. Assume that $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 1 and $N(\varphi^-;\alpha) \neq \emptyset \neq P(\varphi^+;\beta)$ for any $(\alpha,\beta) \in [-1,0] \times [0,1]$. It is clear that $0 \in N(\varphi^-;\alpha) \cap P(\varphi^+;\beta)$ by Theorem 3.9(1). Let $x, y \in H$ be such that $x \circ y \ll N(\varphi^-;\alpha)$ and $y \in N(\varphi^-;\alpha)$. Then for any $a \in x \circ y$, there exists $a_0 \in N(\varphi^-;\alpha)$ such that $a \ll a_0$. It follows from (b1) that $\varphi^-(a) \leq \varphi^-(a_0) \leq \alpha$ for all $a \in x \circ y$ so that $\sup\limits_{a\in x\circ y} \varphi^-(a) \leq \alpha$. Thus

$$\varphi^-(x) \leq \max\{\sup_{a\in x\circ y} \varphi^-(a),\, \varphi^-(y)\} \leq \alpha,$$

and so $x \in N(\varphi^-;\alpha)$. Therefore $N(\varphi^-;\alpha)$ is a hyper BCK-ideal of H. Now let $x, y \in H$ be such that $x \circ y \ll P(\varphi^+;\beta)$ and $y \in P(\varphi^+;\beta)$. Then $x \circ y \ll P(\varphi^+;\beta)$ implies that for every $b \in x \circ y$ there is $b_0 \in P(\varphi^+;\beta)$ such that $b \ll b_0$, so $\varphi^+(b) \geq \varphi^+(b_0)$ by (b1). It follows that $\varphi^+(b) \geq \varphi^+(b_0) \geq \beta$ for all $b \in x \circ y$ so that $\inf\limits_{b\in x\circ y} \varphi^+(b) \geq \beta$. Then

$$\varphi^+(x) \geq \min\{\inf_{b\in x\circ y} \varphi^+(b),\, \varphi^+(y)\} \geq \beta,$$

which implies that $x \in P(\varphi^+;\beta)$. Consequently, $P(\varphi^+;\beta)$ is a hyper BCK-ideal of H.

Conversely, suppose that for each $(\alpha,\beta) \in [-1,0]\times[0,1]$ the nonempty negative and positive level sets $N(\varphi^-;\alpha)$ and $P(\varphi^+;\beta)$ are hyper BCK-ideals of H. Let $x, y \in H$ be such that $x \ll y$, $\varphi^-(y) = \alpha$ and $\varphi^+(y) = \beta$. Then $y \in N(\varphi^-;\alpha)\cap P(\varphi^+;\beta)$, and so $x \ll N(\varphi^-;\alpha)$ and $x \ll P(\varphi^+;\beta)$. It follows from Lemma 3.19 that $x \in N(\varphi^-;\alpha)$ and $x \in P(\varphi^+;\beta)$ so that $\varphi^-(x) \leq \alpha = \varphi^-(y)$ and $\varphi^+(x) \geq \beta = \varphi^+(y)$. Now for any $x, y \in H$ let $\alpha = \max\{\sup\limits_{d\in x\circ y} \varphi^-(d),\, \varphi^-(y)\}$ and $\beta = \min\{\inf\limits_{c\in x\circ y} \varphi^+(c),\, \varphi^+(y)\}$. Then $y \in N(\varphi^-;\alpha) \cap P(\varphi^+;\beta)$, and for each $a, b \in x \circ y$ we have

$$\varphi^-(a) \leq \sup_{d\in x\circ y} \varphi^-(d) \leq \max\{\sup_{d\in x\circ y} \varphi^-(d),\, \varphi^-(y)\} = \alpha$$

and

$$\varphi^+(b) \geq \inf_{c\in x\circ y} \varphi^+(c) \geq \min\{\inf_{c\in x\circ y} \varphi^+(c),\, \varphi^+(y)\} = \beta.$$

Hence $a \in N(\varphi^-;\alpha)$ and $b \in P(\varphi^+;\beta)$, which imply that $x \circ y \subseteq N(\varphi^-;\alpha)$ and $x \circ y \subseteq P(\varphi^+;\beta)$. Using (a7), we get $x \circ y \ll N(\varphi^-;\alpha)$ and $x \circ y \ll P(\varphi^+;\beta)$. Combining $y \in N(\varphi^-;\alpha) \cap P(\varphi^+;\beta)$ and $N(\varphi^-;\alpha)$ and $P(\varphi^+;\beta)$ being hyper BCK-ideals of H, we conclude that $x \in N(\varphi^-;\alpha) \cap P(\varphi^+;\beta)$, and so

$$\varphi^-(x) \leq \alpha = \max\{\sup_{d \in x \circ y} \varphi^-(d),\, \varphi^-(y)\}$$

and

$$\varphi^+(x) \geq \beta = \min\{\inf_{c \in x \circ y} \varphi^+(c),\, \varphi^+(y)\}.$$

This completes the proof. □

Corollary 3.21. *If $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type* 2, 3, 4, $(\forall, \exists)$ *or* $(\forall, \forall)$, *then the nonempty negative level set $N(\varphi^-;\alpha)$ and the nonempty positive level set $P(\varphi^+;\beta)$ are hyper BCK-ideals of H for every $(\alpha, \beta) \in [-1, 0] \times [0, 1]$.*

Proof. Straightforward. □

Theorem 3.22. *If $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 1, then the set*

$$I := \{x \in H \mid \varphi^-(x) = \varphi^-(0),\ \varphi^+(x) = \varphi^+(0)\}$$

is a hyper BCK-ideal of H.

Proof. Obviously $0 \in I$. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $\varphi^-(y) = \varphi^-(0)$, $\varphi^+(y) = \varphi^+(0)$, and for each $a \in x \circ y$ there exists $z \in I$ such that $a \ll z$. Thus $\varphi^-(a) \leq \varphi^-(z) = \varphi^-(0)$ and $\varphi^+(a) \geq \varphi^+(z) = \varphi^+(0)$ by (b1). It follows from (b2) and (b4) that

$$\varphi^-(x) \leq \max\{\sup_{a \in x \circ y} \varphi^-(a),\, \varphi^-(y)\} = \varphi^-(0)$$

and

$$\varphi^+(x) \geq \min\{\inf_{a \in x \circ y} \varphi^+(a),\, \varphi^+(y)\} = \varphi^+(0)$$

so that $\varphi^-(x) = \varphi^-(0)$ and $\varphi^+(x) = \varphi^+(0)$. Hence $x \in I$, which shows that I is a hyper BCK-ideal of H. □

Corollary 3.23. *If $\varphi = (H;\, \varphi^-,\, \varphi^+)$ is a bipolar fuzzy hyper BCK-ideal of type 2,3, 4,* $(\forall, \exists)$ *or* $(\forall, \forall)$, *then the set*

$$I := \{x \in H \mid \varphi^-(x) = \varphi^-(0),\ \varphi^+(x) = \varphi^+(0)\}$$

is a hyper BCK-ideal of H.

Proof. Straightforward. □

$$\begin{tikzcd}
\text{type } 1 & & \text{type } 2 \arrow[ll] \\
& \text{type } (\forall, \exists) \arrow[ul] & \\
& \text{type } (\forall, \forall) \arrow[u] \arrow[dr, shift left] & \\
\text{type } 3 \arrow[uuu] & & \text{type } 4 \arrow[ll] \arrow[uuu] \arrow[ul, shift left]
\end{tikzcd}$$

FIGURE 1. Relation between each type

The converse of Theorem 3.22 is not true as seen in the following example.

Example 3.24. Consider a hyper BCK-algebra $H = \{0, a, b\}$ as in Example 3.2. Define a bipolar fuzzy set $\varphi = (H; \varphi^-, \varphi^+)$ in H by

	0	a	b
φ^-	-0.3	-0.3	-0.7
φ^+	0.6	0.6	0.8

Then

$$I := \{x \in H \mid \varphi^-(x) = \varphi^-(0),\ \varphi^+(x) = \varphi^+(0)\} = \{0, a\}$$

which is a hyper BCK-ideal of H. But $\varphi = (H; \varphi^-, \varphi^+)$ is not a bipolar fuzzy hyper BCK-ideal of type 1 since it does not satisfy the condition (b1).

Note that every bipolar fuzzy hyper BCK-ideal has relations between each type as seen in Figure 1. None of implications in Figure 1 is reversible in general (see Examples 3.2, 3.3, 3.4, 3.8 and 3.11).

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