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TECHNIQUES FOR COMPUTATIONS OF THE AUTOMORPHISM GROUP OF DOMAIN

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Abstract. In this paper, we introduce techniques for computations of the automorphism group of special doamins, for example the Kohn-Nirenberg domain, Fornaess domain and Cartan Hartogs domain.

1. Introduction

Let D be the domain in the complex euclidean space $\mathbb{C}^n$. The automorphism group $\mathrm{Aut}(D)$ be the set of all injective holomorphic maps from D onto itself. This group is a topological group under usual fucntion composition and compact-open topology. H. Cartan show that $\mathrm{Aut}(D)$ is a real Lie group whenever the domain D is bounded.

From the viewpoint of geometry, transitive automorphism group is one of main stream of complex differential geometry. Also, the classfication problem for domains with noncompact automorphism group is deep histrory. For more detailed information, see the book written by R. Greene, K.-T. Kim and S. Krantz([9]). In this paper, we focus on the computations of automorphism group of special domain in $\mathbb{C}^n$. In general, we cannot compute the automorphism group of given domain D. In case of domains with noncompact automorphism group, the computation is easier than domain with compact automorphism group. We introduce known techniques for in Section 2. In Section 2, we deal with domain with noncompact automorphism group. In Section 3, we introduce techniques for domains with compact autormophism group, especially unbounded domains. In Section 4, we introduce new invariants to determine holomorphic equivalence of strongly pseudoconvex domains.

2. Domains with Noncompact Automorphism Group

First, we introduce Cartan Uniqueness Theorem.

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Theorem 2.1 (H. Cartan)**.** *Let D be a bounded domain in $\mathbb{C}^n$ containing the origin o. If f is homorphich map with $f(o) = o$ and $df_o = Id$, then f is equal to the identity map.*

Lee proved that Cartan Uniqueness Theorem for domain in the almost complex manifold in [12]. Next, we will introduce Linearization Theorem.

Definition 1. Let D be a domain in $\mathbb{C}^n$. We say that D is *circular* if D is invariant under the circular action

$$(z_1, z_2, \dots, z_n) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2, \dots, e^{i\theta} z_n)$$

for all $\theta \in \mathbb{R}$.

Theorem 2.2. *Let D be a bounded circular domain in $\mathbb{C}^n$ containing the origin o. If $f \in \mathrm{Aut}(D)$ and $f(o) = o$, then f is linear.*

Proof. Let $R_\theta(z_1, z_2, \dots, z_n) = (e^{i\theta} z_1, e^{i\theta} z_2, \dots, e^{i\theta} z_n)$. The automorphism

$$F := f^{-1} \circ R_{-\theta} \circ f \circ R_\theta$$

is the automorphism preserving the orgin o. Moreover, the differential at the origin dF_o is the identity matrix. By Cartan Uniqueness Theorem, F is equal to the identity map. That means

$$f(e^{i\theta} z_1, e^{i\theta} z_2, \dots, e^{i\theta} z_n) = e^{i\theta} f(z_1, z_2, \dots, z_n)$$

is satisfied for all θ. This implies the conclusion. □

Using this theorem, we can compute the automorphism group of unit ball and Thullen domain

$$E_m = \left\{ (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^{2m} < 1 \right\}.$$

Let f be an automorphism group of the unit ball $\mathbb{B} := E_1$ in $\mathbb{C}^2$. The point $f(0,0) \in \mathbb{B}$ is denoted by (α, β). We know the Möbius tranformation

$$M_a(z, w) = \left(\frac{z - a}{1 - \overline{a} z}, \frac{\sqrt{1 - |a|^2}}{1 - \overline{a} z} w \right)$$

is the automorphism of the unit ball $\mathbb{B}$ for $|a| < 1$. Let $f_1 = M_\alpha \circ f$. Then f_1 is the automorphism with $f_1(0,0) = M_\alpha(\alpha, \beta) = \left(0, \frac{\beta}{\sqrt{1-|\alpha|^2}}\right)$. Clearly, the reflection map $R(z, w) = (w, z)$ is the automomrophism of the unit ball. We consider $F := M_{\frac{\beta}{\sqrt{1-|a|^2}}} \circ R \circ f_1$. The map F is the automorphism with $F(0,0) = (0,0)$. By the Cartan Linearization Theorem, we obtain that F is linear map. The linear map F preseve the unit ball. So, we conclude that F is an unitary map. Hence every automorphism f is equal to composition of the Möbius tranform and unitary maps.

Remark 1. Note that we already know that the automorphism group of the unit ball is transitive. The unitary map and Möbius transformationation move the origin to any point in the unit ball. Key idea of the above computation is based on this fact. Given automorphism $f \in \mathrm{Aut}(\mathbb{B})$ and the point $f(0,0) = (\alpha, \beta)$, apply the Mobius transformation and unitary map to (α, β) such that this point is moved to the origin. But the Thullen domain is not transitive. How to find all automorphisms of the Thullen domain E_m.

We know that the Möbius transformation

$$M_a(z,w) = \left(\frac{z-a}{1-\bar{a}z}, \frac{\sqrt[2m]{1-|a|^2}}{\sqrt[m]{1-\bar{a}z}} w \right)$$

for $|a| < 1$ is the automorphism of Thullen domain E_m. The automorphism $f \in \mathrm{Aut}(E_m)$ preseving the origin is only rotation with respcet to each coordinates. More precisely, maps $(z,w) \mapsto (e^{i\theta}z, e^{i\eta}w)$ are the automorphism of E_m. We want to prove that every automorphism $f \in \mathrm{Aut}(E_m)$ is equal to the composition of the above Möbius transformation and rotations.

To achieve this goal, we consider the orbit of the origin. The Möbius transformation move the orgin to the set $\{(z,0) \mid |z| < 1\}$. Also, the rotation has the same phenomenon. We should first show that every automorphism preseves the disc $\{(z,0) \mid |z| < 1\}$. For this, we introduce remarkable result for extension theorem in [2]

Theorem 2.3 (Bell and Ligocka)**.** *If D_1 and D_2 satisfy Bell's codition (R), then any biholomorphic mapping between D_1 and D_2 smoothly to the boundary.*

The Bell's condition (R) is explained in [2].

Note that all point in $\{(z,0) \mid |z| = 1\}$ are not strongly pseudoconvex boundary points. Equivalently, the D'Angelo type of these points is greater than 2. All boundary points of E_m except $\{(z,0) \mid |z| = 1\}$ are strongly pseudoconvex point. Hence every automrophism $f \in \mathrm{Aut}(E_m)$ preserves the disc $\{(z,0) \mid |z| < 1\}$ by maximum principle.

Therefore, if $f \in \mathrm{Aut}(E_m)$, by the above observation, then $f(0,0) = (\alpha, 0)$. Define $F := M_\alpha \circ f$. Then $F \in \mathrm{Aut}(E_m)$ and $F(0,0) = (0,0)$. By Cartan Linearization Theorem, F is linear preserving E_m. Hence $F(z,w) = (e^{i\theta}z, e^{i\eta}w)$ for some θ, $\eta \in \mathbb{R}$.

As a natural generalization of Thullen domains, the Hartogs type domain $\widehat{\Omega}_m$ is defined by

$$\widehat{\Omega}_m = \{\, (z,\zeta) \in \Omega \times \mathbb{C}^m : \rho(z,\zeta) := \|\zeta\|^{2\mu} - N_\Omega(z,z) < 0 \,\}, \tag{1}$$

where Ω is an irreducible bounded classical symmetric domain in complex vector space and $N_\Omega(z,w)$ denotes the generic norm of Ω. We will call $\widehat{\Omega}_m$ the *Hartogs domain* over a bounded classical symmetric domain Ω. The computation of $\mathrm{Aut}(\widehat{\Omega}_m)$ is based on the similar idea. But the difficulty comes from facts that $\widehat{\Omega}_m$ has singular boundary. There is no result about smooth extension of biholomorphic maps. They proved that all boundary points except $b\Omega \times \{0\}$ are strongly pseudoconvex in [1]. Using this fact, they prove that every automorphism preserves $\Omega \times \{0\}$. Therefore, the same technique is working for Hartogs type domains.

3. Domains with compact automorhism group

There are several bounded examples with compact automorphism group.

Let k, m be positive integers greater than 1. Then domain

$$E_{k,m} = \left\{(z,w) \in \mathbb{C}^2 \mid |z|^{2k} + |w|^{2m} < 1\right\}$$

has real-analytic boundary and is a bounded domain with compact autormophism group. More precisely, it has only rotation to each coordinate. For detailed information, see [9]. This example is good because it is bounded domain with real-analytic boundary. So we can apply good results to this domain, for example, Bell-Ligocka Theorem and so on.

The Kohn-Nirenberg domain([10]) and Fornæss domain ([4]) are not bounded domain with real-analytic boundary. The Kohn-Nirenberg domain Ω_{KN} is defined by the following inequality :

$$\operatorname{Re} w + |zw|^2 + |z|^8 + \frac{15}{7}|z|^2 \operatorname{Re} z^6 < 0.$$

The Fornæss domain Ω_t is also defined by the following inequality :

$$\operatorname{Re} w + |zw|^2 + |z|^6 + t|z|^2 \operatorname{Re} z^4 < 0$$

for $1 < |t| < \frac{9}{5}$. The computation of these domain is already known in [3]. $\operatorname{Aut}(\Omega_{KN})$ is generated by map $(z,w) \mapsto (e^{i\frac{\pi}{3}} z, w)$. Hence the automorphism group of the Kohn-Nirenberg domain is a cyclic group of order 6. Similarily, $\operatorname{Aut}(\Omega)_t$ is generated by the map $(z,w) \mapsto (e^{i\frac{\pi}{2}} z, w)$.

Recently, J. Han and Zhao ([7]) introduce the domain Ω_k defined by

$$\operatorname{Re} w + |zw|^2 + |z|^{2n} + k_1|z|^{2n-2m} \operatorname{Re} z^{2m} + k_2|z|^{2n-2m} \operatorname{Im} z^{2m} < 0.$$

They also show that $\operatorname{Aut}(\Omega_k)$ is is a cyclic group of odre $2m$.

The above three example has similar structure in $\mathbb{C}^2$. These domain have an automorphism as rotation of z-axis and contain left half plane $\{(0,w) \mid \operatorname{Re} w < 0\}$. The origin is the only weakly pseudoconvex boundary point of these domain. The half of proof is devoted to automorphism f preserving the origin. Since the origin lies in the boundary, the Cartan Uniqueness Theorem is useless for this case. The other half is devoted to automorphism f with $f(\mathcal{O}) = \infty$ (following notation in [3]). Exactly, they proved that there are no such automorphisms. If exists, by the existence of rotational symmetry at the origin, there exists rotational symmetry at the infinite. This is impossible by using algebraic technique and rotational angle.

4. Diffeomorphic deformations of the structure

Let Ω and Ω' are bounded strongly pseudoconvex domains in $\mathbb{C}^n$. We define the following invariant :

$$d_\Omega(\Omega') := \inf_{f\in\mathcal{F}} \|f^* J_{st} - J_{st}\|_{\mathcal{C}^2(\bar\Omega)} ,$$

where $\mathcal{F} = \{f : \overline{\Omega} \longrightarrow \overline{\Omega'} \mid \ C^2 \text{ diffeomorphism}\}$ and J_{st} is the standard complex structure for $\mathbb{C}^n$.

Remark 2. (i) Since all the structures under consideration are smooth of class $\mathcal{C}^\infty$ then every element in $\mathcal{F}$ is a smooth $\mathcal{C}^\infty$ diffeomorphism from Ω to Ω'.

(ii) The number $d_\Omega(\Omega')$ is invariant under the action of biholomorphisms. More precisely, if Ω_1 (resp. Ω'_1) is biholomorphic to Ω (resp. Ω'), then $d_{\Omega_1}(\Omega'_1) = d_\Omega(\Omega')$.

Theorem 4.1. *Let Ω and Ω' are bounded strongly pseudoconvex domains in $\mathbb{C}^n$. Then $d_\Omega(\Omega') = 0$ if and only if there is a biholomorphism from Ω to Ω'.*

Proof. Let f be a biholomorphism from Ω to Ω'. Then f can be extended to a smooth $\mathcal{C}^\infty$ diffeomorphism between $\overline{\Omega}$ and $\overline{\Omega}'$ (see Remark 2, (i)). Therefore, $d_\Omega(\Omega') = 0$.

Conversely, assume that $d_\Omega(\Omega') = 0$. Then there is a sequence $\{f_n\}$ of C^∞ diffeomorphisms (in view of Remark 2) from $\bar\Omega$ to $\bar\Omega'$ such that $\|f_n^* J_{st} - J_{st}\|_{\mathcal{C}^2(\bar\Omega)} \to 0$ as n goes to infinity. Let $g_n := f_n^{-1}$ and $J_n := f_n^* J_{st}$.

Case 1. There is a point q' in Ω' such that the sequence $\{g_n(q')\}$ converges to a point in $\partial\Omega$. It follows from [6, 13] that there exists a linear almost complex structure J_0 on $\mathbb{R}^{2n}$ such that Ω' is (J_{st}, J_0) biholomorphic to the Siegel half space $\mathbb{H} := \{(x, y) \in \mathbb{R}^{2n} : x_1 + \sum_{j=2}^n (x_j^2 + y_j^2) < 0\}$. The automorphism group $Aut(\mathbb{H}, J_0)$ being transitive by [13], the domain Ω' is a homogeneous domain. Hence the domain Ω is homogeneous for every almost complex J_n. In particular, picking a point q in Ω, there is an automorphism α_n of (Ω, J_n) such that $(\alpha_n \circ g_n)(q') = q$. Note that $\alpha_n^* J_n = J_n$.

Since $(J_n)_n$ converges to J_{st} and Ω is strictly pseudoconvex, by the result of [11], it follows from that the sequence $(\beta_n := \alpha_n \circ g_n)_n$ converges to a holomorphic map G from Ω' to Ω. Moreover, for every n, β_n^{-1} is a (J_n, J_{st}) biholomorphism from Ω to Ω' and satisfies $\beta_n^{-1}(q) = q'$. Then $(\beta_n)_n$ converges to a holomorphic map F from Ω to Ω'. It is easy to prove that $G = F^{-1}$.

Case 2. For every point q' in Ω' the sequence $(g_n(q'))_n$ is relatively compact in Ω. It follows from that $(g_n)_n$ converges to a holomorphic map from Ω' to Ω. If d_Ω^J denotes the Kobayashi-Royden integrated distance, it follows from that :

$$\forall p \in \Omega,\ \exists\, c_p > 0 : d_\Omega^{J_n}(g_n(q'), p) \leq c_p,$$

for sufficiently large n. Hence $d_{\Omega'}^{J_{st}}(q', f_n(p)) \leq c_p$ for sufficiently large n. Since Ω' is strongly pseudoconvex, the sequence $(f_n(p))_n$ is relatively compact in Ω' according to [5]. We can now conclude by [11]. $\square$

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