

ON CONVERGENCE THEOREMS OF ITERATIVE SCHEMES FOR PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. In this paper, we establish the strong convergence for the Mann iterative scheme associated with uniformly continuous pseudocontractive mappings in real Banach spaces. Moreover, our technique of proofs is of independent interest.

1. Introduction and preliminaries

Let E be a real Banach space and K be a nonempty convex subset of E . Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 \text{ and } \|f^*\| = \|x\|\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by \hat{j} .

Let $T : D(T) \subset E \rightarrow E$ be a mapping with domain $D(T)$ in E .

Definition 1.1. T is said to be Lipschitzian if there exists $L > 1$ such that for all $x, y \in D(T)$

$$\|Tx - Ty\| \leq L \|x - y\|.$$

Definition 1.2. T is said to be nonexpansive if for all $x, y \in D(T)$, the following inequality holds:

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in D(T).$$

Definition 1.3. T is said to be pseudocontractive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 \text{ for all } x, y \in D(T), \quad n \geq 1.$$

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Remark 1. It is well known that every nonexpansive mapping is pseudocontractive. Indeed if T is nonexpansive mapping then for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\begin{aligned} \langle Tx - Ty, j(x - y) \rangle &\leq \|Tx - Ty\| \|x - y\| \\ &\leq \|x - y\|^2, \quad n \geq 1. \end{aligned}$$

Rhoades in [10] showed that the class of pseudocontractive mappings properly contains the class of nonexpansive mappings.

The class of pseudocontractions is, perhaps, the most important generalization of the class of nonexpansive mappings because of its strong relationship with the class of accretive mappings. A mapping $A : E \rightarrow E$ is accretive if and only if $I - A$ is pseudocontractive.

For a nonempty convex subset K of a normed space E and $T : K \rightarrow K$, the Mann iteration scheme [7] is defined by the following sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - b_n)x_n + b_nTx_n, \quad n \geq 1, \end{cases} \quad (M)$$

where $\{b_n\}$ is a sequence in $[0, 1]$.

In the last few years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz *strongly* pseudocontractive mappings using the *Mann iteration scheme* (see for example, [7]). Results which had been known only in *Hilbert spaces* and only for *Lipschitz mappings* have been extended to more general Banach spaces (see for example [1-2, 4, 12] and the references cited therein).

In [2-4], Chidume extended the results of Schu [11] from Hilbert spaces to the much more general class of real Banach spaces and approximate the fixed points of pseudocontractive mappings.

In [5], Haiyun and Yuting gave the answer of the question raised by Chidume [1] and proved: If X is a real Banach space with a uniformly convex dual X^* , K is a nonempty bounded closed convex subset of X , and $T : K \rightarrow K$ is a continuous strongly pseudocontractive mapping, then the Ishikawa iteration scheme converges strongly to the unique fixed point of T .

In this paper, we establish the strong convergence for the Mann iterative scheme associated with uniformly continuous pseudocontractive mappings in real Banach spaces. We also generalize the results of Schu [11] from Hilbert spaces to more general Banach spaces and improve the results of Chidume [1-2], Chidume and Udomene [3], Chidume and Zegeye [4] and Haiyun and Yuting [5].

2. Main results

We will need the following results.

Lemma 2.1. ([13]) *Let $J : E \rightarrow 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.2. ([12]) *If there exists a positive integer N such that for all $n \geq N, n \in \mathbb{N}$,*

$$\rho_{n+1} \leq (1 - \theta_n)\rho_n + b_n,$$

then

$$\lim_{n \rightarrow \infty} \rho_n = 0,$$

where $\theta_n \in [0, 1)$, $\sum_{n=1}^{\infty} \theta_n = \infty$, and $b_n = o(\theta_n)$.

Lemma 2.3. ([9]) *If there exists a positive integer N such that for all $n \geq N, n \in \mathbb{N}$,*

$$\rho_{n+1} \leq (1 - \delta_n^2)\rho_n + b_n,$$

then

$$\lim_{n \rightarrow \infty} \rho_n = 0,$$

where $\delta_n \in [0, 1)$, $\sum_{n=1}^{\infty} \delta_n^2 = \infty$, and $b_n = o(\delta_n)$.

We now prove our main results.

Theorem 2.4. *Let K be a nonempty closed convex subset of a real Banach space E , $T : K \rightarrow K$ a uniformly continuous pseudocontractive mapping such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n \geq 1} \in [0, 1]$ be such that*

- (i) $\sum_{n \geq 1} \alpha_n^2 = \infty$,
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$.

For arbitrary $x_0 \in K$, let $\{x_n\}_{n \geq 1}$ be iteratively defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 1. \quad (2.1)$$

Then the following conditions are equivalent:

- (a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the fixed point p of T ,
- (b) $\lim_{n \rightarrow \infty} Tx_n = p$,
- (c) $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Proof. Let p be a fixed point of T . Suppose that $\lim_{n \rightarrow \infty} x_n = p$, then the uniform continuity of T yields that

$$\lim_{n \rightarrow \infty} Tx_n = p.$$

Therefore $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Set

$$M_1 = \|x_0 - p\| + \sup_{n \geq 1} \|Tx_n - p\|.$$

Obviously $M_1 < \infty$.

It is clear that $\|x_0 - p\| \leq M_1$. Suppose that $\|x_n - p\| \leq M_1$. Next we will prove that $\|x_{n+1} - p\| \leq M_1$.

Consider

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - p\| \\ &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \\ &\leq (1 - \alpha_n + \alpha_n)M_1 \\ &= M_1. \end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n \geq 1}$ is bounded. Let $M_2 = \sup_{n \geq 1} \|x_n - p\|$.

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

Now from Lemma 1 for all $n \geq 1$, we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - p\|^2 \\ &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\|^2 \\ &\leq (1 - \alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\langle Tx_n - p, j(x_{n+1} - p) \rangle \\ &= (1 - \alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\langle Tx_{n+1} - p, j(x_{n+1} - p) \rangle \\ &\quad + 2\alpha_n\langle Tx_n - Tx_{n+1}, j(x_{n+1} - p) \rangle \tag{2.2} \\ &\leq (1 - \alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\|x_{n+1} - p\|^2 \\ &\quad + 2\alpha_n\|Tx_n - Tx_{n+1}\|\|x_{n+1} - p\| \\ &\leq (1 - \alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\|x_{n+1} - p\|^2 \\ &\quad + 2\alpha_n\lambda_n, \end{aligned}$$

where

$$\lambda_n = M\|Tx_n - Tx_{n+1}\|. \tag{2.3}$$

Using (2.1) we have

$$\begin{aligned} \|x_n - x_{n+1}\| &= \alpha_n\|x_n - Tx_n\| \\ &\leq 2M\alpha_n. \end{aligned} \tag{2.4}$$

From the condition $\lim_{n \rightarrow \infty} \alpha_n = 0$ and (2.4), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0$$

and the uniform continuity of T leads to

$$\lim_{n \rightarrow \infty} \|Tx_n - Tx_{n+1}\| = 0,$$

thus we have

$$\lim_{n \rightarrow \infty} \lambda_n = 0. \tag{2.5}$$

The real function $f : [0, \infty) \rightarrow [0, \infty)$, defined by $f(t) = t^2$ is increasing and convex. For all $\lambda \in [0, 1]$ and $t_1, t_2 > 0$ we have

$$((1 - \lambda)t_1 + \lambda t_2)^2 \leq (1 - \lambda)t_1^2 + \lambda t_2^2. \tag{2.6}$$

Consider

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - p\|^2 \\
 &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\|^2 \\
 &\leq [(1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\|]^2 \\
 &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|Tx_n - p\|^2 \\
 &\leq (1 - \alpha_n)\|x_n - p\|^2 + M^2\alpha_n.
 \end{aligned} \tag{2.7}$$

Substituting (2.7) in (2.2), we get

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq [(1 - \alpha_n)^2 + 2\alpha_n(1 - \alpha_n)]\|x_n - p\|^2 \\
 &\quad + 2\alpha_n(M^2\alpha_n + \lambda_n) \\
 &= (1 - \alpha_n^2)\|x_n - p\|^2 + \varepsilon_n\alpha_n,
 \end{aligned} \tag{2.8}$$

where $\varepsilon_n = 2(M^2\alpha_n + \lambda_n)$. Now with the help of $\sum_{n \geq 1} \alpha_n^2 = \infty$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, (2.5) and Lemma 3, we obtain from (2.8) that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = 0,$$

completing the proof. \square

Corollary 2.5. *Let K be a nonempty closed convex subset of a real Hilbert space E , $T : K \rightarrow K$ a uniformly continuous pseudocontractive mapping such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n \geq 1} \in [0, 1]$ be such that*

- (i) $\sum_{n \geq 1} \alpha_n^2 = \infty$,
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$.

For arbitrary $x_0 \in K$, let $\{x_n\}_{n \geq 1}$ be iteratively defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 1.$$

Then the following conditions are equivalent:

- (a) $\{x_n\}_{n=1}^\infty$ converges strongly to the fixed point p of T ,
- (b) $\lim_{n \rightarrow \infty} Tx_n = p$,
- (c) $\{Tx_n\}_{n=1}^\infty$ is bounded.

Remark 2. We know the followings

1. We do not need geometry of the Banach space.
2. We do not need the boundedness assumption on K .
3. We remove additional assumptions on the mapping T .
4. We prove our results for the original Ishikawa iteration scheme, which is very simple in comparison to the previously known cumbersome iteration schemes.

References

- [1] C. E. Chidume, *Approximation of fixed points of strongly pseudocontractive mappings*, Proc. Amer. Math. Soc. **120** (2) (1994), 545–551.
- [2] C. E. Chidume, *Iterative approximation of fixed points of Lipschitz pseudocontractive maps*, Proc. Amer. Math. Soc. **129** (2001), 2245–2251.
- [3] C. E. Chidume and A. Udomene, *Strong convergence theorems for uniformly continuous pseudocontractive maps*, J. Math. Anal. Appl. **323** (1) (2006), 88–99.
- [4] C. E. Chidume and H. Zegeye, *Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps*, Proc. Amer. Math. Soc. **132** (2004), 831–840.
- [5] Z. Haiyun and J. Yuting, *Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption*, Proc. Amer. Math. Soc. **125** (6) (1997), 1705–1709.
- [6] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508–520.
- [7] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
- [8] C. H. Morales and J. S. Jung, *Convergence of paths for pseudo-contractive mappings in Banach spaces*, Proc. Amer. Math. Soc. **128** (2000), 3411–3419.
- [9] A. Rafiq, *Some results on asymptotically pseudocontractive mappings*, Fixed Point Theory **11** (2) (2010), 355–360.
- [10] B. E. Rhoades, *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.
- [11] J. Schu, *Approximating fixed points of Lipschitzian pseudocontractive mappings*, Houston J. Math. **19** (1993), 107–115.
- [12] X. Weng, *Fixed point iteration for local strictly pseudocontractive mapping*, Proc. Amer. Math. Soc. **113** (3) (1991), 727–731.
- [13] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (12) (1991), 1127–1138.

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