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PARALLEL PERFORMANCE OF MULTISPLITTING METHODS WITH PREWEIGHTING

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Abstract. In this paper, we first study convergence of a special type of multisplitting methods with preweighting, and then we provide some comparison results of those multisplitting methods. Next, we propose both parallel implementation of an SOR-like multisplitting method with preweighting and an application of the SOR-like multisplitting method with preweighting to a parallel preconditioner of Krylov subspace method. Lastly, we provide parallel performance results of both the SOR-like multisplitting method with preweighting and Krylov subspace method with the parallel preconditioner to evaluate parallel efficiency of the proposed methods.

1. Introduction

In this paper, we consider multisplitting methods with preweighting for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse H-matrix.

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ $(x > 0)$ denotes that all components of x are nonnegative (positive), and $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. For two vectors x, $y \in \mathbb{R}^n$, $x \geq y$ $(x > y)$ means that $x - y \geq 0$ $(x - y > 0)$. These definitions carry immediately over to matrices. For a square matrix A, $\mathrm{diag}(A)$ denotes a diagonal matrix whose diagonal part coincides with the diagonal part of A. Let $\rho(A)$ denote the *spectral radius* of a square matrix A. Varga [11] showed that for any two square matrices A and B, $|A| \leq B$ implies $\rho(A) \leq \rho(B)$.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *monotone* if A is nonsingular with $A^{-1} \geq 0$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if it is a monotone

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matrix with $a_{ij} \leq 0$ for $i \neq j$. The *comparison matrix* $\langle A \rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

A matrix A is called an *H-matrix* if $\langle A \rangle$ is an M-matrix.

A representation $A = M - N$ is called a *splitting* of A if M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, and it is called *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$ [1]. It is well known that if $A = M - N$ is a weak regular splitting of A, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$ [1, 11]. A splitting $A = M - N$ is called an *H-compatible splitting* of A if $\langle A \rangle = \langle M \rangle - |N|$. It was shown in [5] that if A is an H-matrix and $A = M - N$ is an H-compatible splitting of A, then $\rho(M^{-1}N) < 1$. A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's, called weighting matrices, are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$.

Lemma 1.1 ([2]). *Let $A^{-1} \geq 0$ and $A = M_1 - N_1 = M_2 - N_2$ be weak regular splittings. In either of the following cases*:

(a) $N_1 \leq N_2$,
(b) ${M_1}^{-1} \geq {M_2}^{-1}, N_1 \geq 0$,
(c) ${M_1}^{-1} \geq {M_2}^{-1}, N_2 \geq 0$,

the inequality $\rho({M_1}^{-1}N_1) \leq \rho({M_2}^{-1}N_2)$ holds.

The *multisplitting method with postweighting* which is usually called the multisplitting method has been extensively studied in the literature, see [3, 4, 6, 7, 10, 12, 14, 15]. However, the *multisplitting method with preweighting* has not been studied extensively, see [4, 13]. This is the main motivation for studying convergence of multisplitting method with preweighting.

This paper is organized as follows. In Section 2, we first study convergence of a special type of multisplitting methods with preweighting, and then we provide some comparison results of those multisplitting methods. In Section 3, we propose both parallel implementation of an SOR-like multisplitting method with preweighting and an application of the SOR-like multisplitting method with preweighting to a parallel preconditioner of Krylov subspace method. In Section 4, we provide parallel performance results of both the SOR-like multisplitting method with preweighting and Krylov subspace method with the parallel preconditioner to evaluate parallel efficiency of the proposed methods. Lastly, some concluding remarks are withdrawn.

2. Convergence of multisplitting methods with preweighting

Let (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, be a multisplitting of A. Then the corresponding multisplitting method with preweighting for solving $Ax = b$ [13] is given by

$$x_{i+1} = H_0 x_i + G_0 b \\ = x_i + G_0(b - Ax_i),\ i = 0, 1, 2, \ldots, \tag{2}$$

where

$$G_0 = \sum_{k=1}^{\ell} {M_k}^{-1} E_k \text{ and } H_0 = I - G_0 A. \tag{3}$$

$H_0 = I - \sum_{k=1}^{\ell} {M_k}^{-1} E_k A$ is called an iteration matrix for the multisplitting method with preweighting. Notice that then $H = I - \sum_{k=1}^{\ell} E_k {M_k}^{-1} A$ is called an iteration matrix for the multisplitting method. By simple calculation, one obtains

$${H_0}^T = A^T \left(I - \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} A^T \right) (A^T)^{-1}.$$

Let $\hat{H} = I - \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} A^T = \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} {N_k}^T$. Then $\hat{H}$ is similar to ${H_0}^T$ and hence $\rho(H_0) = \rho(\hat{H})$. Notice that $\hat{H}$ is an iteration matrix for the multisplitting method corresponding to a multisplitting $({M_k}^T, {N_k}^T, E_k)$, $k = 1, 2, \ldots, \ell$, of A^T. Hence, convergence result of *multisplitting method with preweighting* corresponding to a multisplitting of A can be obtained from that of *multisplitting method* corresponding to a multisplitting of A^T.

The multisplitting method with preweighting associated with a multisplitting (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, of A for solving the linear system (1) is as follows:

ALGORITHM 1: MULTISPLITTING METHOD WITH PREWEIGHTING
Given an initial vector x_0
For $i = 0, 1, \ldots,$ until convergence
    For $k = 1$ to ℓ {parallel execution}
        $M_k y_k = E_k(b - Ax_i)$
    $x_{i+1} = x_i + \sum_{k=1}^{\ell} y_k$

We first consider the multisplitting method with preweighting corresponding to a special type of multisplitting (M_k, N_k, E_k), $k = 1, 2, \ldots, l$, of A which was first introduced by White [13] and studied further by Frommer and Mayer [4]. Let's assume that $\ell = 3$ for simplicity of exposition. Then A is partitioned into

$$A = \begin{pmatrix} A_1 & -C_{12} & -C_{13} & -C_{14} \\ -C_{21} & A_2 & -C_{23} & -C_{24} \\ -C_{31} & -C_{32} & A_3 & -C_{34} \\ -C_{41} & -C_{42} & -C_{43} & A_4 \end{pmatrix}, \tag{4}$$

where A_i's are square matrices. Let $A_k = B_k - C_k$ $(1 \le k \le \ell+1)$ be a splitting of A_k. Let

$$
(5)\quad
\begin{aligned}
M_1 &= \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ -C_{41} & 0 & 0 & B_4 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 I \end{pmatrix}, \\
M_2 &= \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & -C_{42} & 0 & B_4 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_2 I \end{pmatrix}, \\
M_3 &= \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & -C_{43} & B_4 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & e_3 I \end{pmatrix}, \\
N_k &= M_k - A \ \ (1 \le k \le 3),
\end{aligned}
$$

where $\sum_{k=1}^{\ell} e_k = 1$. Using this multisplitting (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, of A, G_0 and H_0 are of the form

$$
(6)\quad
\begin{aligned}
G_0 &= \sum_{k=1}^{\ell} {M_k}^{-1} E_k \\
&= \begin{pmatrix} {B_1}^{-1} & 0 & 0 & 0 \\ 0 & {B_2}^{-1} & 0 & 0 \\ 0 & 0 & {B_3}^{-1} & 0 \\ {B_4}^{-1} C_{41} {B_1}^{-1} & {B_4}^{-1} C_{42} {B_2}^{-1} & {B_4}^{-1} C_{43} {B_3}^{-1} & {B_4}^{-1} \end{pmatrix}, \\
H_0 &= I - G_0 A = \begin{pmatrix} {B_1}^{-1} C_1 & {B_1}^{-1} C_{12} & {B_1}^{-1} C_{13} & {B_1}^{-1} C_{14} \\ {B_2}^{-1} C_{21} & {B_2}^{-1} C_2 & {B_2}^{-1} C_{23} & {B_2}^{-1} C_{24} \\ {B_3}^{-1} C_{31} & {B_3}^{-1} C_{32} & {B_3}^{-1} C_3 & {B_3}^{-1} C_{34} \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_i &= \sum_{k=1, k \ne i}^{\ell} {B_4}^{-1} C_{4,k} {B_k}^{-1} C_{k,i} + {B_4}^{-1} C_{4,i} {B_i}^{-1} C_i \ \text{ for } \ i = 1, 2, \ldots \ell, \\
\beta_4 &= \sum_{k=1}^{\ell} {B_4}^{-1} C_{4,k} {B_k}^{-1} C_{k,4} + {B_4}^{-1} C_4.
\end{aligned}
$$

The following theorems are convergence results of multisplitting method with preweighting corresponding to the multisplitting of the form (5) when A is a monotone matrix or an H-matrix.

Theorem 2.1 ([13]). *Let $A^{-1} \ge 0$ and $(M_k, N_k, E_k), k = 1, 2, \ldots, \ell$, be a multisplitting of A with M_k, N_k and E_k defined as in* (5). *If $A_k = B_k - C_k$*

is a weak regular splitting of A_k *and* $C_{ij} \geq 0$ $(1 \leq i, j, k \leq \ell + 1,\ i \neq j)$*, then* $H_0 \geq 0$ *and* $\rho(H_0) < 1$*, where* $H_0 = I - \sum_{k=1}^{\ell} {M_k}^{-1} E_k A$.

Theorem 2.2 ([4])**.** *Let* $(M_k, N_k, E_k), k = 1, 2, \ldots, \ell$*, be a multisplitting of* A *with* M_k*,* N_k *and* E_k *defined as in* (5)*. If* A *is an* H*-matrix and* $A_k = B_k - C_k$ *is an* H*-compatible splitting of* A_k *for* $k = 1, 2, \ldots, \ell+1$*, then* $\rho(H_0) < 1$*, where* $H_0 = I - \sum_{k=1}^{\ell} {M_k}^{-1} E_k A$.

The following theorem provides a convergence result of the AOR-like multisplitting method with preweighting of the form (5) when A is an H-matrix.

Theorem 2.3. *Assume that* A *is an* H*-matrix with* $A = D - F$*, where* $D = \mathrm{diag}(A)$*. Let* $(M_k, N_k, E_k), k = 1, 2, \ldots, \ell$*, be a multisplitting of* A *with* M_k*,* N_k *and* E_k *defined as in* (5)*, where for* $k = 1, 2, \ldots, \ell + 1$

$$B_k = \frac{1}{\omega}(D_k - \gamma L_k), \;\; C_k = \frac{1}{\omega}\left((1-\omega)D_k + (\omega - \gamma)L_k + \omega V_k\right), \tag{7}$$

$D_k = \mathrm{diag}(A_k)$*,* L_k *is a strictly lower triangular matrix and* V_k *is a general matrix satisfying* $V_k = D_k - L_k - A_k$*. If* $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$ *and* $\langle A_k \rangle = |D_k| - |L_k| - |V_k|$ *for* $k = 1, 2, \ldots, \ell + 1$*, then* $\rho(H_0) < 1$*, where* $H_0 = I - \sum_{k=1}^{\ell} M_k^{-1} E_k A$ *and* $\alpha = \rho(|D|^{-1}|F|)$.

Proof. Since $\langle A_k \rangle = |D_k| - |L_k| - |V_k|$, the corresponding coefficients of $(\omega - \gamma)L_k$ and ωV_k have the same signs for $k = 1, 2, \ldots, \ell+1$. We first consider the case where $0 < \gamma \leq \omega \leq 1$. From (7), one obtains for $k = 1, 2, \ldots, \ell + 1$

$$\begin{aligned}
\langle B_k \rangle - |C_k| &= \langle \frac{1}{\omega}(D_k - \gamma L_k) \rangle - |\frac{1}{\omega}\left((1-\omega)D_k + (\omega-\gamma)L_k + \omega V_k\right)| \\
&= \frac{1}{\omega}(|D_k| - \gamma |L_k|) - \frac{1}{\omega}\left((1-\omega)|D_k| + (\omega - \gamma)|L_k| + \omega |V_k|\right) \\
&= |D_k| - |L_k| - |V_k| = \langle A_k \rangle.
\end{aligned}$$

Hence, $A_k = B_k - C_k$ is an H-compatible splitting of A_k for $k = 1, 2, \ldots, \ell+1$. By Theorem 2.2, $\rho(H_0) < 1$ for $0 < \gamma \leq \omega \leq 1$. Next we consider the case where $1 < \omega < \frac{2}{1+\alpha}$ and $\gamma \leq \omega$. For $k = 1, 2, \ldots, \ell + 1$, let

$$\begin{aligned}
\tilde{C_k} &= \frac{1}{\omega}\left((\omega - 1)D_k + (\omega - \gamma)L_k + \omega V_k\right), \\
\tilde{A_k} &= B_k - \tilde{C_k}.
\end{aligned}$$

Then, it can be easily seen that for $k = 1, 2, \ldots, \ell + 1$,

$$\tilde{A_k} = \frac{2-\omega}{\omega} D_k - L_k - V_k.$$

Let $\tilde{A} = \frac{2-\omega}{\omega} D - F$. Then $\langle \tilde{A} \rangle = \frac{2-\omega}{\omega}|D| - |F|$ is a regular splitting of $\langle \tilde{A} \rangle$. Since $1 < \omega < \frac{2}{1+\alpha}$, $\rho\left(\frac{\omega}{2-\omega}|D|^{-1}|F|\right) = \frac{\omega\alpha}{2-\omega} < 1$. It follows that $\langle \tilde{A} \rangle^{-1} \geq 0$ and thus $\tilde{A}$ is an H-matrix. Since $A_k = D_k - L_k - V_k$, $\tilde{A_k} = \frac{2-\omega}{\omega} D_k - L_k - V_k$, which is a block diagonal component of $\tilde{A}$. Clearly, $\tilde{A}_k$ is an H-matrix for $k = 1, 2, \ldots, \ell + 1$. Notice that for $k = 1, 2, \ldots, \ell + 1$,

$$\begin{aligned}\langle B_k \rangle - |\tilde{C}_k| &= \frac{1}{\omega}(|D_k| - \gamma |L_k|) - \frac{1}{\omega}((\omega - 1)|D_k| + (\omega - \gamma)|L_k| + \omega |V_k|) \\ &= \frac{2 - \omega}{\omega}|D_k| - |L_k| - |V_k| = \langle \tilde{A}_k \rangle.\end{aligned}$$

Note that $\langle \tilde{A} \rangle$ can be written as

$$\langle \tilde{A} \rangle = \begin{pmatrix} \langle \tilde{A}_1 \rangle & -|C_{1,2}| & \cdots & -|C_{1,\ell+1}| \\ -|C_{2,1}| & \langle \tilde{A}_2 \rangle & \cdots & -|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots \\ -|C_{\ell+1,1}| & -|C_{\ell+1,2}| & \cdots & \langle \tilde{A}_{\ell+1} \rangle \end{pmatrix}.$$

For $k = 1, 2, \ldots, \ell$, let

$$\tilde{M}_k = \begin{pmatrix} \langle B_1 \rangle & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & & \langle B_k \rangle & & 0 \\ \vdots & & 0 & \ddots & \vdots \\ 0 & \cdots & -|C_{\ell+1,k}| & \cdots & \langle B_{\ell+1} \rangle \end{pmatrix} \quad \text{and} \quad \tilde{N}_k = \tilde{M}_k - \langle \tilde{A} \rangle.$$

Then $(\tilde{M}_k, \tilde{N}_k, E_k)$, $k = 1, 2, \ldots \ell$, is a multisplitting of $\langle \tilde{A} \rangle$ of the form (5). Since $\langle \tilde{A} \rangle^{-1} \geq 0$ and $\langle \tilde{A}_k \rangle = \langle B_k \rangle - |\tilde{C}_k|$ is a regular splitting of $\langle \tilde{A}_k \rangle$ for $k = 1, 2, \ldots, \ell + 1$, $\rho(\tilde{H}_0) < 1$ from Theorem 2.1, where

$$\tilde{H}_0 = \begin{pmatrix} \langle B_1 \rangle^{-1}|\tilde{C}_1| & \langle B_1 \rangle^{-1}|C_{1,2}| & \cdots & \langle B_1 \rangle^{-1}|C_{1,\ell}| & \langle B_1 \rangle^{-1}|C_{1,\ell+1}| \\ \langle B_2 \rangle^{-1}|C_{2,1}| & \langle B_2 \rangle^{-1}|\tilde{C}_2| & \cdots & \langle B_2 \rangle^{-1}|C_{2,\ell}| & \langle B_2 \rangle^{-1}|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle B_\ell \rangle^{-1}|C_{\ell,1}| & \langle B_\ell \rangle^{-1}|C_{\ell,2}| & \cdots & \langle B_\ell \rangle^{-1}|\tilde{C}_\ell| & \langle B_\ell \rangle^{-1}|C_{\ell,\ell+1}| \\ \tilde{\beta}_1 & \tilde{\beta}_2 & \cdots & \tilde{\beta}_\ell & \tilde{\beta}_{\ell+1} \end{pmatrix},$$

$$\tilde{\beta}_i = \sum_{k=1, k \neq i}^{\ell} \langle B_{\ell+1} \rangle^{-1}|C_{\ell+1,k}|\langle B_k \rangle^{-1}|C_{k,i}| + \langle B_{\ell+1} \rangle^{-1}|C_{\ell+1,i}|\langle B_i \rangle^{-1}|\tilde{C}_i| \quad (1 \leq i \leq \ell),$$

$$\tilde{\beta}_{\ell+1} = \sum_{k=1}^{\ell} \langle B_{\ell+1} \rangle^{-1}|C_{\ell+1,k}|\langle B_k \rangle^{-1}|C_{k,\ell+1}| + \langle B_{\ell+1} \rangle^{-1}|\tilde{C}_{\ell+1}|.$$

Since B_k is an H-matrix for $1 \leq k \leq \ell + 1$, one obtains

$$|{B_k}^{-1}| \leq \langle B_k \rangle^{-1} \quad \text{and} \quad |C_k| \leq |\tilde{C}_k|.$$

Using these inequalities, $|H_0| \leq \tilde{H}_0$ is obtained. Thus, $\rho(H_0) < 1$ for $1 < \omega < \frac{2}{1+\alpha}$ and $\gamma \leq \omega$. Therefore, $\rho(H_0) < 1$ for $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$. □

If $\gamma = \omega$ in Theorem 2.3, then Theorem 2.3 reduces to a convergence result of the SOR-like multisplitting method with preweighting of the form (5).

Definition 2.4. $A = M - N$ is called an *SSOR-like splitting* of A if

$$M = \frac{1}{\omega(2-\omega)}(D - \omega L)D^{-1}(D - \omega V),$$
$$N = \frac{1}{\omega(2-\omega)}\left((1-\omega)D + \omega L\right)D^{-1}\left((1-\omega)D + \omega V\right),$$

where $0 < \omega < 2$, $D = \operatorname{diag}(A)$, L is a strictly lower triangular matrix and V is a general matrix satisfying $V = D - L - A$.

The following example shows that the SSOR-like splitting of an H-matrix $A = D - L - V$ such that $\langle A \rangle = |D| - |L| - |V|$ is not an H-compatible splitting of A.

Example 2.5. Let $A = D - L - V$ be a 2×2 matrix defined by

$$A = \begin{pmatrix} 2 & -3 \\ 2 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}.$$

It is clear that $\langle A \rangle = |D| - |L| - |V|$. Since $\langle A \rangle^{-1} \geq 0$, A is an H-matrix. By simple calculation

$$M = (D - L)D^{-1}(D - V) = \begin{pmatrix} 2 & -3 \\ 2 & \frac{5}{2} \end{pmatrix},$$
$$N = LD^{-1}V = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}.$$

It follows that

$$\langle M \rangle = \begin{pmatrix} 2 & -3 \\ -2 & \frac{5}{2} \end{pmatrix}, \quad |N| = \begin{pmatrix} 0 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, \quad \langle M \rangle - |N| = \begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}.$$

Note that $A = M - N$ is an SSOR-like splitting of A with $\omega = 1$. However, $\langle M \rangle - |N| \neq \langle A \rangle$, which shows that $A = M - N$ is not an H-compatible splitting of A.

The following theorem provides a convergence result of the SSOR-like multisplitting method with preweighting of the form (5) when A is an H-matrix.

Theorem 2.6. *Assume that A is an H-matrix with $A = D - F$, where $D = \operatorname{diag}(A)$. Let $(M_k, N_k, E_k), k = 1, 2, \ldots, \ell$, be a multisplitting of A with M_k, N_k and E_k defined as in* (5)*, where*

$$\begin{aligned} B_k &= \frac{1}{\omega(2-\omega)}(D_k - \omega L_k){D_k}^{-1}(D_k - \omega V_k), \\ C_k &= \frac{1}{\omega(2-\omega)}\big((1-\omega)D_k + \omega L_k\big){D_k}^{-1}\big((1-\omega)D_k + \omega V_k\big), \end{aligned} \tag{8}$$

$D_k = \text{diag}(A_k)$, L_k is a strictly lower triangular matrix and V_k is a general matrix satisfying $V_k = D_k - L_k - A_k$. If $0 < \omega < \frac{2}{1+\alpha}$ and $\langle A_k \rangle = |D_k| - |L_k| - |V_k|$, then $\rho(H_0) < 1$, where $H_0 = I - \sum_{k=1}^{\ell} M_k^{-1} E_k A$ and $\alpha = \rho(|D|^{-1}|F|)$.

Proof. We consider the first case where $0 < \omega \le 1$. From the assumption, one obtains for $k = 1, 2, \ldots, \ell + 1$,

$$\begin{aligned}\langle A_k \rangle =& \frac{1}{\omega(2-\omega)}(|D_k| - \omega|L_k|)|D_k|^{-1}(|D_k| - \omega|V_k|)\\ &- \frac{1}{\omega(2-\omega)}\big((1-\omega)|D_k| + \omega|L_k|\big)|D_k|^{-1}\big((1-\omega)|D_k| + \omega|V_k|\big).\end{aligned}$$

For $k = 1, 2, \ldots, \ell + 1$, let

$$\begin{aligned}\tilde{B}_k &= \frac{1}{\omega(2-\omega)}(|D_k| - \omega|L_k|)|D_k|^{-1}(|D_k| - \omega|V_k|),\\ \tilde{C}_k &= \frac{1}{\omega(2-\omega)}\big((1-\omega)|D_k| + \omega|L_k|\big)|D_k|^{-1}\big((1-\omega)|D_k| + \omega|V_k|\big).\end{aligned}$$

Then $\langle A_k \rangle = \tilde{B}_k - \tilde{C}_k$ is a regular splitting of $\langle A_k \rangle$ for $k = 1, 2, \ldots, \ell + 1$. Let for $k = 1, 2, \ldots, \ell$,

$$\tilde{M}_k = \begin{pmatrix} \tilde{B}_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & & \tilde{B}_k & & 0 \\ \vdots & & 0 & \ddots & \vdots \\ 0 & \cdots & -|C_{\ell+1,k}| & \cdots & \tilde{B}_{\ell+1} \end{pmatrix} \quad \text{and} \quad \tilde{N}_k = \tilde{M}_k - \langle A \rangle.$$

Then $(\tilde{M}_k, \tilde{N}_k, E_k)$, $k = 1, 2, \ldots, \ell$, is a multisplitting of $\langle A \rangle$ of the form (5). Since $\langle A \rangle^{-1} \ge 0$, $\rho(\tilde{H}_0) < 1$ from Theorem 2.1, where

$$\tilde{H}_0 = \begin{pmatrix} \tilde{B}_1^{-1}\tilde{C}_1 & \tilde{B}_1^{-1}|C_{1,2}| & \cdots & \tilde{B}_1^{-1}|C_{1,\ell}| & \tilde{B}_1^{-1}|C_{1,\ell+1}| \\ \tilde{B}_2^{-1}|C_{2,1}| & \tilde{B}_2^{-1}\tilde{C}_2 & \cdots & \tilde{B}_2^{-1}|C_{2,\ell}| & \tilde{B}_2^{-1}|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{B}_\ell^{-1}|C_{\ell,1}| & \tilde{B}_\ell^{-1}|C_{\ell,2}| & \cdots & \tilde{B}_\ell^{-1}\tilde{C}_\ell & \tilde{B}_\ell^{-1}|C_{\ell,\ell+1}| \\ \tilde{\beta}_1 & \tilde{\beta}_2 & \cdots & \tilde{\beta}_\ell & \tilde{\beta}_{\ell+1} \end{pmatrix}, \tag{9}$$

$$\begin{aligned}\tilde{\beta}_i &= \sum_{k=1, k\neq i}^{\ell} \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,i}| + \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,i}|\tilde{B}_i^{-1}\tilde{C}_i \ (1 \le i \le \ell),\\ \tilde{\beta}_{\ell+1} &= \sum_{k=1}^{\ell} \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,\ell+1}| + \tilde{B}_{\ell+1}^{-1}\tilde{C}_{\ell+1}.\end{aligned}$$

Since A_k is an H-matrix, $D_k - \omega L_k$ and $D_k - \omega V_k$ are H-matrices for $k = 1, 2, \ldots, \ell+1$. Hence one obtains

$$|(D_k - \omega L_k)^{-1}| \le (|D_k| - \omega |L_k|)^{-1},$$
$$|(D_k - \omega V_k)^{-1}| \le (|D_k| - \omega |V_k|)^{-1},$$
$$|{B_k}^{-1}| \le \tilde{B}_k^{-1} \text{ and } |C_k| \le \tilde{C}_k.$$

Using these inequalities, $|H_0| \le \tilde{H}_0$ is obtained. Therefore, $\rho(H_0) < 1$ for $0 < \omega \le 1$. Next we consider the case where $1 < \omega < \frac{2}{1+\alpha}$. Let

$$\hat{C}_k = \frac{1}{\omega(2-\omega)}\big((\omega-1)|D_k| + \omega|L_k|\big)|D_k|^{-1}\big((\omega-1)|D_k| + \omega V_k\big).$$

Then one obtains for $k = 1, 2, \ldots, \ell+1$,

$$\tilde{B}_k - \hat{C}_k = \frac{\omega}{2-\omega}\left(\frac{2-\omega}{\omega}|D_k| - |L_k| - |V_k|\right).$$

Let $\tilde{A} = |D| - \frac{\omega}{2-\omega}|F|$ and $\tilde{A}_k = \frac{2-\omega}{\omega}|D_k| - |L_k| - |V_k|$ for $k = 1, 2, \ldots, \ell+1$. Then $\tilde{A} = |D| - \frac{\omega}{2-\omega}|F|$ is a regular splitting of $\tilde{A}$ and $\frac{\omega}{2-\omega}\tilde{A}_k = \tilde{B}_k - \hat{C}_k$. Since $1 < \omega < \frac{2}{1+\alpha}$, $\rho\left(|D|^{-1}\frac{\omega}{2-\omega}|F|\right) = \frac{\omega}{2-\omega}\rho(|D|^{-1}|F|) = \frac{\omega\alpha}{2-\omega} < 1$. Thus, $\tilde{A}^{-1} \ge 0$. Note that $\tilde{A}$ can be written as

$$\tilde{A} = \begin{pmatrix} \frac{\omega}{2-\omega}\tilde{A}_1 & -\frac{\omega}{2-\omega}|C_{1,2}| & \cdots & -\frac{\omega}{2-\omega}|C_{14}| \\ -\frac{\omega}{2-\omega}|C_{21}| & \frac{\omega}{2-\omega}\tilde{A}_2 & \cdots & -\frac{\omega}{2-\omega}|C_{24}| \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\omega}{2-\omega}|C_{\ell+1,1}| & -\frac{\omega}{2-\omega}|C_{\ell+1,2}| & \cdots & \frac{\omega}{2-\omega}\tilde{A}_{\ell+1} \end{pmatrix}.$$

Let for $k = 1, 2, \ldots, \ell$,

$$M_k^\star = \begin{pmatrix} \tilde{B}_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & & \tilde{B}_k & & 0 \\ \vdots & & 0 & \ddots & \vdots \\ 0 & \cdots & -\frac{\omega}{2-\omega}|C_{\ell+1,k}| & \cdots & \tilde{B}_{\ell+1} \end{pmatrix} \quad \text{and} \quad N_k^\star = M_k^\star - \tilde{A}.$$

Then $(M_k^\star, N_k^\star, E_k)$, $k = 1, 2, \ldots, \ell$, is a multisplitting of $\tilde{A}$ of the form (5). Since $\frac{\omega}{2-\omega}\tilde{A}_k = \tilde{B}_k - \hat{C}_k$ is a regular splitting of $\frac{\omega}{2-\omega}\tilde{A}_k$ for $k = 1, 2, \ldots, \ell+1$, $\rho(H_0^\star) < 1$ from Theorem 2.1, where

$$H_0^\star = \frac{\omega}{2-\omega}\begin{pmatrix} \frac{2-\omega}{\omega}\tilde{B}_1^{-1}\hat{C}_1 & \tilde{B}_1^{-1}|C_{1,2}| & \cdots & \tilde{B}_1^{-1}|C_{1,\ell}| & \tilde{B}_1^{-1}|C_{1,\ell+1}| \\ \tilde{B}_2^{-1}|C_{2,1}| & \frac{2-\omega}{\omega}\tilde{B}_2^{-1}\hat{C}_2 & \cdots & \tilde{B}_2^{-1}|C_{2,\ell}| & \tilde{B}_2^{-1}|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{B}_\ell^{-1}|C_{\ell,1}| & \tilde{B}_\ell^{-1}|C_{\ell,2}| & \cdots & \frac{2-\omega}{\omega}\tilde{B}_\ell^{-1}\hat{C}_\ell & \tilde{B}_\ell^{-1}|C_{\ell,\ell+1}| \\ \beta_1^\star & \beta_2^\star & \cdots & \beta_\ell^\star & \beta_{\ell+1}^\star \end{pmatrix},$$

$$\beta_i^\star = \frac{\omega}{2-\omega} \sum_{k=1,k\neq i}^{\ell} \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,i}| + \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,i}|\tilde{B}_i^{-1}\hat{C}_i \ (1 \le i \le l),$$

$$\beta_{\ell+1}^\star = \frac{\omega}{2-\omega} \sum_{k=1}^{\ell} \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,\ell+1}| + \frac{2-\omega}{\omega}\tilde{B}_{\ell+1}^{-1}\hat{C}_{\ell+1}.$$

Since $|B_k^{-1}| \le \tilde{B}_k^{-1}$, $|C_k| \le \hat{C}_k$ and $\frac{\omega}{2-\omega} > 1$, $|H_0| \le H_0^\star$ is obtained. Thus, $\rho(H_0) < 1$ for $1 < \omega < \frac{2}{1+\alpha}$. Therefore, $\rho(H_0) < 1$ for all $0 < \omega < \frac{2}{1+\alpha}$. □

We next provide comparison results for multisplitting methods with preweighting of the form (5) when A is an M-matrix.

Theorem 2.7. *Assume that A is an M-matrix. Let (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, be a multisplitting of A with M_k, N_k and E_k defined as in* (5)*, where $B_k = \frac{1}{\omega}(D_k - rL_k)$, $C_k = \frac{1}{\omega}\big((1-\omega)D_k + (\omega - r)L_k + \omega V_k\big)$, $D_k = \operatorname{diag}(A_k)$, L_k is a nonnegative strictly lower triangular matrix and V_k is a nonnegative matrix satisfying $V_k = D_k - L_k - A_k$ for $k = 1, 2, \ldots, \ell + 1$. Let*

$$M_{AOR} = \frac{1}{\omega}(D - rL), \quad N_{AOR} = \frac{1}{\omega}\big((1-\omega)D + (\omega - r)L + \omega U\big),$$

$$M_J = \frac{1}{\omega}D \ \text{ and } \ N_J = \frac{1}{\omega}\big((1-\omega)D + \omega L + \omega U\big),$$

where $D = \operatorname{diag}(A)$, $-L$ is a strictly lower triangular part of A and $-U$ is a strictly upper triangular part of A. If $0 < r \le \omega \le 1$, then

$$\rho({M_{AOR}}^{-1}N_{AOR}) \le \rho(H_0) \le \rho({M_J}^{-1}N_J) < 1,$$

where $G_0 = \sum_{k=1}^{\ell} M_k^{-1}E_k$ and $H_0 = I - G_0A$.

Proof. Without loss of generality, we can assume that $\ell = 3$. Let $A_k = D_k - \bar{L}_k - \bar{U}_k$, where $\bar{L}_k$ is a strictly lower triangular part of A_k and $\bar{U}_k$ is a strictly upper triangular part of A_k. It can be easily seen that $\bar{L}_k \ge L_k$ and $V_k \ge \bar{U}_k$. Since $G_0 = \sum_{k=1}^{\ell} M_k^{-1}E_k$ is nonsingular and $H_0 = I - G_0A$, $A = {G_0}^{-1} - {G_0}^{-1}H_0$. Let $B = {G_0}^{-1}$ and $C = {G_0}^{-1}H_0$. Then $A = B - C$ and $H_0 = B^{-1}C$. Since $B_k^{-1} \ge 0$, $C_k \ge 0$ and $C_{ij} \ge 0$ for $i, j, k = 1, 2, \ldots, \ell+1$, $i \ne j$, ${M_k}^{-1} \ge 0$ and thus $G_0 \ge 0$. Notice that G_0, B and C can be written as

$$G_0 = \begin{pmatrix} {B_1}^{-1} & 0 & 0 & 0 \\ 0 & {B_2}^{-1} & 0 & 0 \\ 0 & 0 & {B_3}^{-1} & 0 \\ {B_4}^{-1}C_{41}{B_1}^{-1} & {B_4}^{-1}C_{42}{B_2}^{-1} & {B_4}^{-1}C_{43}{B_3}^{-1} & {B_4}^{-1} \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ -C_{41} & -C_{42} & -C_{43} & B_4 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_{12} & C_{13} & C_{14} \\ C_{21} & C_2 & C_{23} & C_{24} \\ C_{31} & C_{32} & C_3 & C_{34} \\ 0 & 0 & 0 & C_4 \end{pmatrix}, \tag{10}$$

where $C_i = \frac{1-\omega}{\omega}D_i + \frac{\omega - r}{\omega}L_i + V_i$ for $1 \le i \le 4$. It can be easily seen that $C \ge 0$. Hence, we obtain $A = B - C$ is regular splitting of A. By hypothesis,

N_{AOR} and N_J can be written as

$$(11)\quad N_{AOR}=\begin{pmatrix}\tilde{D}_1 & C_{12} & C_{13} & C_{14}\\ 0 & \tilde{D}_2 & C_{23} & C_{24}\\ 0 & 0 & \tilde{D}_3 & C_{34}\\ 0 & 0 & 0 & \tilde{D}_4\end{pmatrix},\quad N_J=\begin{pmatrix}\hat{D}_1 & C_{12} & C_{13} & C_{14}\\ C_{21} & \hat{D}_2 & C_{23} & C_{24}\\ C_{31} & C_{32} & \hat{D}_3 & C_{34}\\ C_{41} & C_{42} & C_{43} & \hat{D}_4\end{pmatrix},$$

where $\tilde{D}_i=\frac{1-\omega}{\omega}D_i+\frac{\omega-r}{\omega}\bar{L}_i+\bar{U}_i$ and $\hat{D}_i=\frac{1-\omega}{\omega}D_i+\bar{L}_i+\bar{U}_i$ for $i=1,2,3,4$. By assumptions, $0\le\frac{\omega-r}{\omega}<1$ and $L_i+V_i=\bar{L}_i+\bar{U}_i$ for $1\le i\le 4$. Hence

$$(12)\quad \begin{aligned}\frac{\omega-r}{\omega}\bar{L}_i+\bar{U}_i&=\frac{\omega-r}{\omega}(L_i+V_i-\bar{U}_i)+\bar{U}_i\\ &=\frac{\omega-r}{\omega}L_i+\frac{\omega-r}{\omega}(V_i-\bar{U}_i)+\bar{U}_i\\ &\le\frac{\omega-r}{\omega}L_i+(V_i-\bar{U}_i)+\bar{U}_i=\frac{\omega-r}{\omega}L_i+V_i\\ &\le L_i+V_i=\bar{L}_i+\bar{U}_i.\end{aligned}$$

From (10), (11) and (12), one obtains

$$(13)\quad N_{AOR}\le C\le N_J.$$

From (13) and Lemma 1.1, one obtains,

$$\rho({M_{AOR}}^{-1}N_{AOR})\le\rho(H_0)\le\rho({M_J}^{-1}N_J)<1.\qquad\square$$

In Theorem 2.7, ${M_{AOR}}^{-1}N_{AOR}$ and ${M_J}^{-1}N_J$ are the iteration matrices for the AOR method and the relaxed Jacobi method, respectively. Also notice that H_0 is an iteration matrix for the AOR-like multisplitting method with preweighting of the form (5).

Theorem 2.8. *Assume that A is an M-matrix. Let (M_k,N_k,E_k), $k=1,2,\dots,\ell$, be a multisplitting of A with M_k, N_k and E_k defined as in* (5)*, where*

$$(14)\quad B_k=\frac{1}{\omega}(D_k-\omega L_k),\quad C_k=\frac{1}{\omega}\left((1-\omega)D_k+\omega V_k\right),$$

$D_k=\mathrm{diag}(A_k)$*, L_k is a nonnegative strictly lower triangular matrix and V_k is a nonnegative matrix satisfying $V_k=D_k-L_k-A_k$ for $k=1,2,\dots,\ell+1$. Let $(\tilde{M}_k,\tilde{N}_k,E_k), k=1,2,\dots,\ell$, be a multisplitting of A with $\tilde{M}_k$, $\tilde{N}_k$ and E_k defined as in* (5)*, except that $\tilde{B}_k$ and $\tilde{C}_k$ are used instead of B_k and C_k,*

$$(15)\quad \begin{aligned}\tilde{B_k}&=\frac{1}{\omega(2-\omega)}(D_k-\omega L_k)D_k^{-1}(D_k-\omega V_k),\\ \tilde{C_k}&=\frac{1}{\omega(2-\omega)}\left((1-\omega)D_k+\omega L_k\right)D_k^{-1}\left((1-\omega)D_k+\omega V_k\right).\end{aligned}$$

If $0<\omega\le 1$, then

$$\rho(\tilde{H}_0)\le\rho(H_0)<1,$$

where $G_0=\sum_{k=1}^{\ell}M_k^{-1}E_k$, $H_0=I-G_0A$, $\tilde{G}_0=\sum_{k=1}^{\ell}\tilde{M}_k^{-1}E_k$ and $\tilde{H}_0=I-\tilde{G}_0A$.

Proof. Without loss of generality, we can assume that $\ell = 3$. From (14) and (15), it is easy to show that $B_k^{-1} \geq 0$ and $\tilde{B}_k^{-1} \geq 0$ for $k = 1, 2, \ldots, \ell + 1$. Hence, $G_0 \geq 0$ and $\tilde{G}_0 \geq 0$. Since G_0 and $\tilde{G}_0$ are nonsingular,

$$A = G_0^{-1} - G_0^{-1} H_0 = \tilde{G}_0^{-1} - \tilde{G}_0^{-1} \tilde{H}_0. \tag{16}$$

Then we have

$$G_0 = \begin{pmatrix} B_1^{-1} & 0 & 0 & 0 \\ 0 & B_2^{-1} & 0 & 0 \\ 0 & 0 & B_3^{-1} & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & B_4^{-1} \end{pmatrix}, \quad \tilde{G}_0 = \begin{pmatrix} \tilde{B}_1^{-1} & 0 & 0 & 0 \\ 0 & \tilde{B}_2^{-1} & 0 & 0 \\ 0 & 0 & \tilde{B}_3^{-1} & 0 \\ \tilde{\alpha}_1 & \tilde{\alpha}_2 & \tilde{\alpha}_3 & \tilde{B}_4^{-1} \end{pmatrix}, \tag{17}$$

where $\alpha_i = B_4^{-1} C_{4,i} B_i^{-1}$ and $\tilde{\alpha}_i = \tilde{B}_4^{-1} C_{4,i} \tilde{B}_i^{-1}$ for $i = 1, 2, 3$. Since $D_k - \omega V_k$ is an M-matrix and $D_k - \omega V_k \leq D_k$, $D_k^{-1} \leq (D_k - \omega V_k)^{-1}$. Hence, $I \leq (D_k - \omega V_k)^{-1} D_k$. It follows that

$$\begin{aligned} B_k^{-1} &\leq \omega (D_k - \omega V_k)^{-1} D_k (D_k - \omega L_k)^{-1} \\ &\leq \omega (2 - \omega)(D_k - \omega V_k)^{-1} D_k (D_k - \omega L_k)^{-1} = \tilde{B}_k^{-1}. \end{aligned} \tag{18}$$

From (17) and (18), $G_0 \leq \tilde{G}_0$. Since (16) are regular splittings of A, from Lemma 1.1 $\rho(\tilde{H}_0) \leq \rho(H_0) < 1$. □

In Theorem 2.8, $\tilde{H}_0$ and H_0 are the iteration matrices for the SSOR-like multisplitting method with preweighting and the SOR-like multisplitting method with preweighting of the form (5), respectively.

3. Parallel implementation and application of multisplitting method with preweighting

In Section 2, we have studied convergence of a special type of multisplitting methods with preweighting for solving the linear system (1). In this section, we only introduce parallel implementation and application of the SOR-like multisplitting method with preweighting of the form (5) since those for other multisplitting methods with preweighting of the form (5) can be done similarly. We first propose a parallel implementation of the SOR-like multisplitting method with preweighting of the form (5). Let ℓ denote the number of processors to be used. For simplicity of exposition, we assume that $\ell = 3$. Then, A is partitioned into a 4×4 block of the form

$$A = \begin{pmatrix} A_1 & A_{12} & A_{13} & A_{14} \\ A_{21} & A_2 & A_{23} & A_{24} \\ A_{31} & A_{32} & A_3 & A_{34} \\ A_{41} & A_{42} & A_{43} & A_4 \end{pmatrix}, \tag{19}$$

where the diagonal blocks A_i of A are square matrices. Let $A_i = D_i - L_i - U_i$ $(1 \leq i \leq 4)$, where $D_i = \text{diag}(A_i)$, and L_i and U_i are strictly lower triangular and strictly upper triangular matrices, respectively. Let

$$M_1 = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ A_{41} & 0 & 0 & B_4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & A_{42} & 0 & B_4 \end{pmatrix},$$

(20)

$$M_3 = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & A_{43} & B_4 \end{pmatrix}, \quad N_k = M_k - A \ \ (1 \le k \le 3),$$

where $B_i = \frac{1}{\omega}(D_i - \omega L_i)$ for $1 \le i \le 4$ and $\omega > 0$. Let

(21)

$$E_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 I \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_2 I \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & e_3 I \end{pmatrix},$$

where $\sum_{i=1}^{\ell} e_i = 1$ and $e_i > 0$ for $1 \le i \le \ell$. Then it is clear that (M_k, N_k, E_k), $k = 1, 2, 3$, is a SOR-like multisplitting of A. Hence, the SOR-like multisplitting method with preweighting is given by

(22)

$$\begin{aligned} x_{i+1} &= H_0 x_i + G_0 b \\ &= x_i + G_0(b - Ax_i), \quad i = 0, 1, 2, \ldots, \end{aligned}$$

where $G_0 = \sum_{k=1}^{\ell} {M_k}^{-1} E_k$ and $H_0 = I - G_0 A$.

Making use of (20) to (22), the multisplitting method with preweighting can be executed in parallel as follows.

ALGORITHM 2: PARALLEL IMPLEMENTATION OF MULTISPLITTING METHOD WITH PREWEIGHTING

Choose an initial vector x_0
For $i = 0, 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ {parallel execution on ℓ processors}
(a1) $r^{(k)} = b^{(k)} - A^{(k)} x_i$
  For $k = 1$ to ℓ {parallel execution on ℓ processors}
(a2) solve $B_k t^{(k)} = r^{(k)}$ for $t^{(k)}$
(a3) solve $B_{\ell+1} t_k^{(\ell+1)} = e_k r^{(\ell+1)} - A_{\ell+1,k} t^{(k)}$ for $t_k^{(\ell+1)}$
Compute $t^{(\ell+1)} = \sum_{k=1}^{\ell} t_k^{(\ell+1)}$ in parallel
For $k = 1$ to ℓ {parallel execution on ℓ processors}
(a4) $x_{i+1}^{(k)} = x_i^{(k)} + t^{(k)}$

In Algorithm 2, the superscript (k) means the kth block of a vector and the kth row block of a matrix. In (a1) and (a4), all vectors and matrices are divided into ℓ blocks of equal sizes. In (a2) and (a3), all vectors and matrices are divided into $(\ell + 1)$ blocks. To obtain good load balancing among the processors, the first ℓ blocks are divided into equal sizes and the size of the last $(\ell+1)$th block is chosen to be small compared to the first ℓ blocks.

Next, we propose an application of the SOR-like multisplitting method with preweighting to a parallel preconditioner of Krylov subspace methods such as BiCG, GMRES, BiCGSTAB and so on [9]. If the SOR-like multisplitting method with preweighting converges, then $\rho(H_0) < 1$ and thus $G_0 A = I - H_0$ is nonsingular. It follows that

$$A^{-1} = (I - H_0)^{-1} G_0 = \sum_{i=0}^{\infty} (H_0)^i G_0.$$

Let $P_s = \sum_{i=0}^{s-1} (H_0)^i G_0$, where $s \geq 1$ is an integer. Since $\lim_{s\to\infty} P_s = A^{-1}$, P_s can be viewed as an approximate matrix for A^{-1}. Hence, P_s can be used as a preconditioner of Krylov subspace methods. In other words, $Ax = b$ can be transformed into either $AP_s y = b$ or $P_s Ax = c$, where $y = {P_s}^{-1} x$ and $c = P_s b$. From now on, the preconditioner P_s is called *the s-step preconditioner* of A.

One of the main computational kernel of Krylov subspace methods with the preconditioner P_s is a preconditioner solver step which is to compute $P_s g$ for a given vector g. The efficient computation of $P_s g$ for a given vector g can be done as follows:

Algorithm 3: Preconditioner solver
$x_0 = 0$
For $i = 0$ to $s - 1$
    $x_{i+1} = x_i + G_0(g - Ax_i)$

where $G_0 = \sum_{k=1}^{\ell} {M_k}^{-1} E_k$. Since M_k and E_k are of the form (20) and (21), Algorithm 3 can be executed completely in parallel as described in Algorithm 2. Hence, P_s becomes a good parallel preconditioner for Krylov subspace method. Since other computational kernels of Krylov subspace methods can be easily parallelized, Krylov subspace method with the preconditioner P_s can be fully parallelized among the ℓ processors.

4. Numerical results

In this section, we provide numerical results of both the SOR-like multisplitting method with preweighting described in Section 3 and Krylov subspace method with the preconditioner P_s proposed in Section 3 for solving $Ax = b$, where A is a large sparse H-matrix. Krylov subspace method used for numerical experiments is the right preconditioned BiCGSTAB method. We also tried numerical experiments for both GMRES with the preconditioner P_s and FGMRES (flexible GMRES) using P_s's as preconditioners, but their parallel performance results are much worse than those for the right preconditioned BiCGSTAB. So, we do not report parallel performance results for both GMRES and FGMRES. The test matrix A arises from five-point discretization of the following elliptic second order PDE

$$-(au_x)_x - (bu_y)_y + (cu)_x + (du)_y + fu = g \tag{23}$$

with $a(x, y) > 0$, $b(x, y) > 0$, $c(x, y), d(x, y)$ and $f(x, y)$ defined on a square region Ω, and with suitable boundary conditions on $\partial\Omega$ which denotes the boundary of Ω.

In all cases, the SOR-like multisplitting method with preweighting and the preconditioned BiCGSTAB was started with zero initial vector, and $e_i = 1/\ell$ is used for each $1 \le i \le \ell$. The multisplitting method with preweighting was stopped when $\|r_i\|_2/\|b\|_2 < 10^{-5}$, and the preconditioned BiCGSTAB was stopped when $\|r_i\|_2/\|b\|_2 < 10^{-8}$, where r_i denote the residual vector at the i-th step of the methods and $\|\cdot\|_2$ refers to L_2-norm. All numerical tests have been carried out using the IBM supercomputer Power6 H system at KISTI (Korean Institute of Science and Technology Information). All parallel codes were written in OpenMP Fortran [8] using 64-bit arithmetics. All nonzero elements of A are stored using the compressed row storage format [9]. For all timing runs, elapsed wall-clock time is measured in seconds using the IBM wall-clock timer **rtc**.

All test problems used in this paper are of the type (23) with the unit square region $\Omega = (0, 1) \times (0, 1)$ and the Dirichlet boundary condition $u(x, y) = 0$ on $\partial\Omega$. Only the discretized matrix A is of importance, so the right-hand side vector b is created artificially. Therefore, the right-hand side function $g(x, y)$ in (23) is not relevant.

Example 4.1. We consider equation (23) with $a(x, y) = b(x, y) = 1, c(x, y) = 10(x + y), d(x, y) = 10(x - y)$ and $f(x, y) = 0$. We have used three uniform meshes of $\Delta x = \Delta y = 1/258$, $\Delta x = \Delta y = 1/386$ and $\Delta x = \Delta y = 1/514$ which lead to three matrices of order $n = 257^2 = 66049$, $n = 385^2 = 148225$ and $n = 513^2 = 263169$, where Δx and Δy refer to the mesh sizes in the x-direction and y-direction, respectively. Once the matrix A is constructed from five-point finite difference discretization of the PDE, the right-hand side vector b is chosen so that $b = A(1, 1, \ldots, 1)^T$. Numerical results for Example 4.1 are listed in Tables 1 to 3.

Example 4.2. This example is the same as Example 4.1 except for $c(x, y) = 10e^{xy}, d(x, y) = 10e^{-xy}$. We have used the same uniform meshes as in Example 4.1 and the right-hand side vector b is chosen so that $b = A(1, 1, \ldots, 1)^T$. Numerical results for Example 4.2 are listed in Tables 4 to 6.

For all test problems, $e_i = 1/\ell$ is used for each $1 \le i \le \ell$. For $n = m^2$, the first ℓ blocks are divided into equal sizes of $\frac{m(m-1)}{\ell}$ and the last $(\ell + 1)$th block has order m.

In Examples 4.1 and 4.2, the SOR-like multisplitting method with preweighting was carried out for $n = 257^2$, 385^2 and various values of ω. The BiCGSTAB method with the preconditioner P_s was carried out for $n = 513^2$ and various values of s and ω.

Table 1. Numerical results of the SOR-like multisplitting method with preweighting for Example 4.1 with $n = 66049$

$n = 66049$								
	ℓ=2		ℓ=4		ℓ=8		ℓ=16	
ω	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time
0.8	43129	55.0	43172	28.5	43258	15.9	43428	10.7
0.9	37136	47.3	37179	24.5	37265	13.6	37436	9.2
1.0	32342	41.6	32385	21.6	32471	11.9	32642	8.2
1.1	28420	36.3	28463	18.8	28549	10.5	28720	7.1
1.2	25151	32.1	25195	16.6	25281	9.3	25452	6.3
1.3	22385	28.5	22429	14.8	22515	8.2	22687	5.6
1.4	Does not Converge							

Table 2. Numerical results of the SOR-like multisplitting method with preweighting for Example 4.1 with $n = 148225$

$n = 148225$								
	ℓ=2		ℓ=4		ℓ=8		ℓ=16	
ω	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time
0.8	88158	254.4	88216	128.1	88333	67.1	88565	40.2
0.9	75909	222.2	75968	111.7	76085	58.8	76318	34.3
1.0	66110	194.1	66169	98.2	66287	51.4	66519	30.3
1.1	58093	170.0	58152	85.6	58270	44.9	58503	26.2
1.2	51412	150.4	51472	75.8	51589	39.8	51823	23.4
1.3	45759	134.0	45819	67.6	45936	35.5	46171	21.2
1.4	Does not Converge							

In Tables 1 to 6, ℓ stands for the number of processors to be used, *Iter* the number of iterations of two iterative methods, *I-time* parallel execution time of two iterative methods. In Tables 3 and 6, P_s refers to the parallel preconditioner described in Section 3.

The scaling behaviors for I-time of the SOR-like multisplitting method with preweighting when $n = 385^2$ are depicted in Figures 1 and 3 by log-log scale. The scaling behaviors for I-time of the BiCGSTAB method using preconditioner P_2 when $n = 513^2$ are depicted in Figures 2 and 4 by log-log scale.

5. Concluding remarks

In this paper, we have studied convergence of a special type of multisplitting methods with preweighting, and we proposed both parallel implementation of the SOR-like multisplitting method with preweighting and an application of the SOR-like multisplitting method with preweighting to a parallel preconditioner of Krylov subspace method.

For test problems used in this paper, the SOR-like multisplitting method with preweighting performed best on the IBM supercomputer Power6 H system

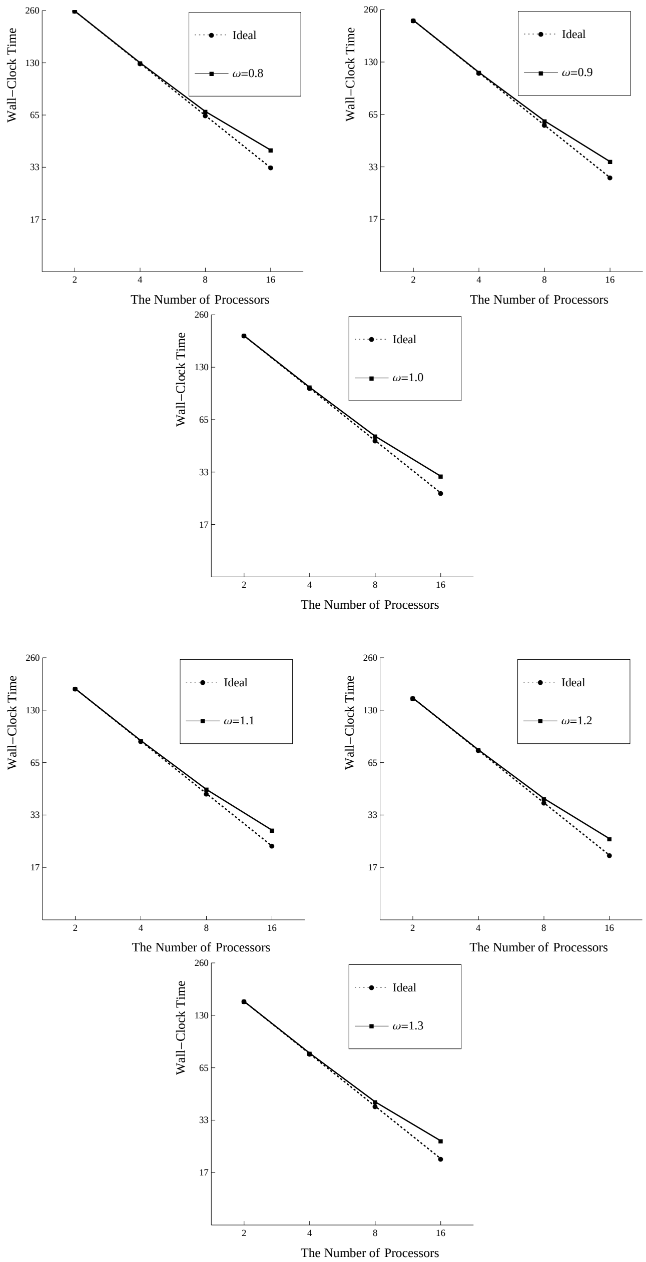

FIGURE 1. Scaling behaviors of the SOR-like multisplitting method with preweighting for Example 4.1 with $n = 148225$

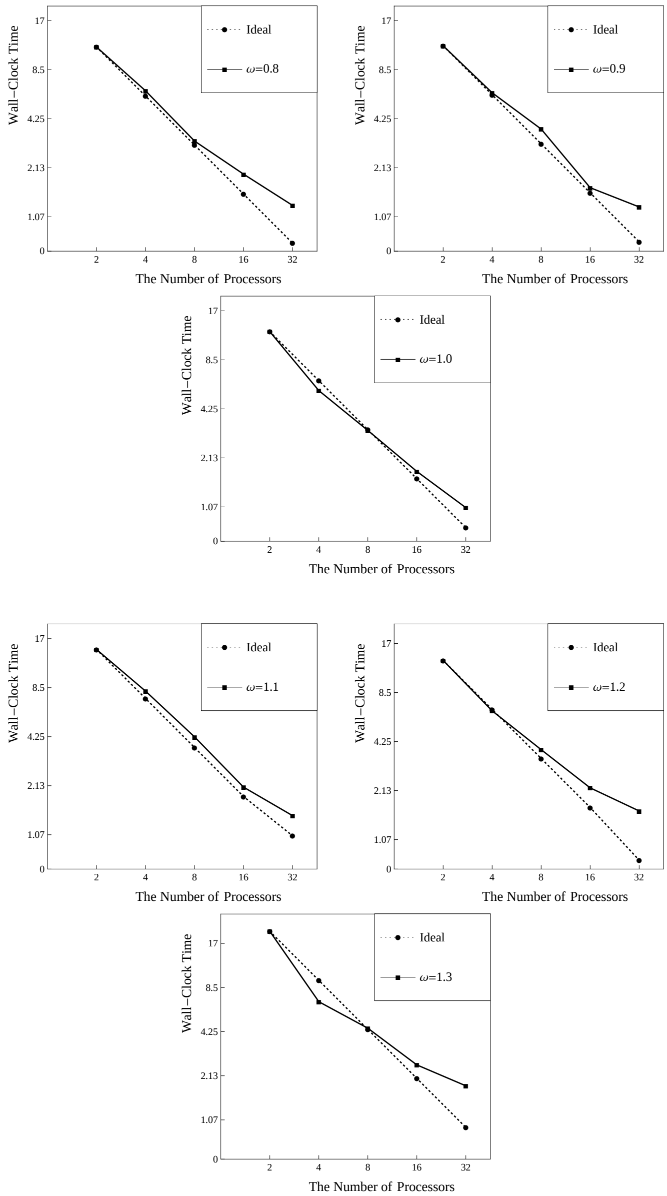

FIGURE 2. Scaling behaviors of BiCGSTAB using preconditioner P_2 for Example 4.1 with $n = 263169$

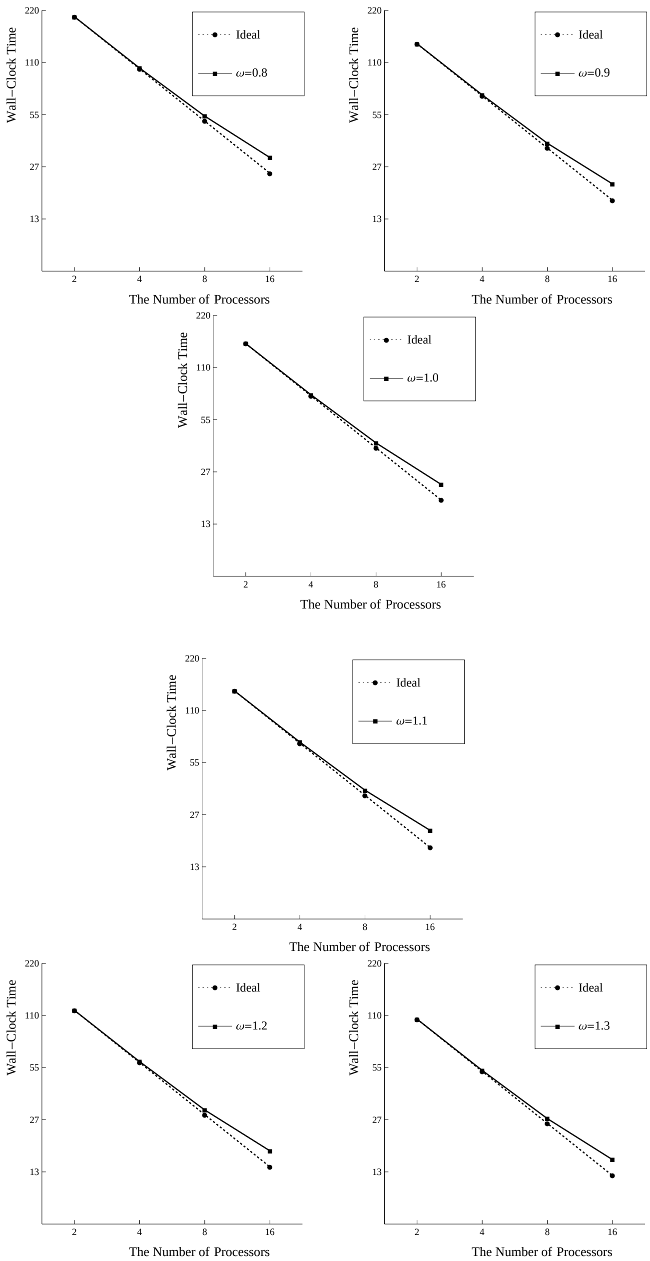

FIGURE 3. Scaling behaviors of the SOR-like multisplitting method with preweighting for Example 4.2 with $n = 148225$

Table 3. Numerical results of BiCGSTAB using preconditioners P_1, P_2, P_3, P_4 for Example 4.1 with $n = 263169$

		$n = 263169$									
		ℓ=2		ℓ=4		ℓ=8		ℓ=16		ℓ=32	
ω	P_s	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time
0.8	1	1180	13.39	1169	6.72	1360	4.07	1231	2.10	1048	1.38
	2	543	11.83	578	6.37	551	3.13	629	1.96	561	1.26
	3	397	12.88	393	6.40	429	3.59	428	1.96	505	1.56
	4	365	15.59	398	8.56	346	3.85	365	2.19	365	1.48
0.9	1	1224	13.84	1300	7.48	1284	3.84	1306	2.22	1182	1.51
	2	555	12.04	523	5.78	607	3.46	470	1.51	517	1.15
	3	423	13.54	452	7.32	415	3.48	413	1.89	404	1.30
	4	344	14.53	308	6.63	353	3.89	325	1.97	386	1.59
1.0	1	1066	12.08	1327	7.62	1120	3.33	1176	1.97	1371	1.66
	2	589	12.77	503	5.55	559	3.16	561	1.77	484	1.06
	3	414	13.21	449	7.26	423	3.52	414	1.89	393	1.23
	4	346	14.61	324	6.94	328	3.61	339	2.02	327	1.33
1.1	1	1133	12.81	1327	7.62	1342	3.99	1199	2.03	1468	1.83
	2	820	17.84	756	8.34	799	4.53	711	2.22	640	1.41
	3	495	15.83	460	7.46	489	4.07	510	2.32	509	1.59
	4	373	15.78	320	6.87	366	4.04	327	1.96	337	1.31
1.2	1	1113	12.61	1070	6.18	1193	3.57	1221	2.08	1186	1.50
	2	1182	25.62	1236	13.58	1230	6.97	1241	3.83	1049	2.60
	3	569	18.23	616	10.20	627	5.22	488	2.25	588	1.89
	4	495	20.93	613	13.13	469	5.15	568	3.38	518	2.09
1.3	1	1560	17.71	1055	6.11	1409	4.20	1474	2.50	1453	1.86
	2	2323	50.39	2127	23.27	1992	11.25	2466	7.60	2330	5.09
	3	561	18.91	761	12.83	772	6.42	816	3.70	759	2.35
	4	1220	53.77	1201	26.64	1128	12.30	1277	7.54	1606	6.34

Table 4. Numerical results of the SOR-like multisplitting method with preweighting for Example 4.2 with $n = 66409$

	$n = 66409$							
	ℓ=2		ℓ=4		ℓ=8		ℓ=16	
ω	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time
0.8	31213	40.7	31247	21.1	31310	11.7	31435	7.8
0.9	26868	35.0	26903	18.2	26966	10.1	27090	6.8
1.0	23392	30.5	23427	15.9	23490	8.8	23615	5.8
1.1	20549	26.8	20583	14.0	20647	7.8	20771	5.2
1.2	18179	23.7	18214	12.3	18277	6.9	18402	4.6
1.3	16174	21.1	16208	11.0	16272	6.1	16397	4.1
1.4	Does not Converge							

when $\omega = 1.3$, and BiCGSTAB method with the parallel preconditioner P_s performed best for almost all cases when $\omega = 0.9$ or 1.0 and $s = 2$. As can be seen in Tables 1 to 6 and Figures 1 to 4, parallel performance of the SOR-like multisplitting method with preweighting is quite efficient, and its application to

TABLE 5. Numerical results of the SOR-like multisplitting method with preweighting for Example 4.2 with $n = 148225$

	$n = 148225$							
	ℓ=2		ℓ=4		ℓ=8		ℓ=16	
ω	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time
0.8	65054	191.0	65102	96.7	65189	51.0	65362	29.4
0.9	56004	164.3	56052	83.3	56140	43.8	56312	25.5
1.0	48764	143.0	48812	72.5	48900	38.2	49073	22.0
1.1	42841	125.5	42889	63.7	42977	33.6	43150	19.7
1.2	37905	111.2	37953	56.3	38041	29.6	38214	17.2
1.3	33728	99.0	33776	50.1	33864	26.4	34038	15.3
1.4	Does not Converge							

TABLE 6. Numerical results of BiCGSTAB using preconditioners P_1, P_2, P_3, P_4 for Example 4.2 with $n = 263169$

		$n = 263169$									
		ℓ=2		ℓ=4		ℓ=8		ℓ=16		ℓ=32	
ω	P_s	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time	Iter	I-time
0.8	1	1171	13.27	1011	5.83	1248	3.73	1250	2.10	1143	1.51
	2	546	11.87	528	5.82	573	3.25	548	1.75	528	1.18
	3	438	14.02	433	7.00	413	3.45	438	1.99	415	1.33
	4	399	16.84	355	7.59	357	3.92	371	2.21	372	1.54
0.9	1	1059	12.01	961	5.53	1037	3.10	1053	1.80	1171	1.52
	2	546	11.84	495	5.45	524	2.98	520	1.64	524	1.17
	3	436	13.93	402	6.51	421	3.51	412	1.88	395	1.26
	4	357	15.04	334	7.14	348	3.83	345	2.06	351	1.44
1.0	1	1166	13.23	1187	6.83	942	2.85	1468	2.46	1169	1.52
	2	547	11.89	623	6.87	536	3.07	544	1.73	492	1.12
	3	423	13.55	409	6.65	414	3.46	446	2.02	381	1.22
	4	332	14.06	321	6.90	323	3.57	364	2.17	327	1.33
1.1	1	1039	11.48	987	5.68	1046	3.12	1398	2.35	1373	1.75
	2	655	14.31	852	9.35	552	3.14	809	2.50	666	1.46
	3	421	13.38	431	6.98	431	3.58	447	2.02	421	1.29
	4	350	14.65	393	8.40	329	3.61	326	1.94	321	1.27
1.2	1	1054	11.97	1360	7.82	1035	3.09	1215	2.06	1271	1.63
	2	1142	24.69	1016	11.13	1116	6.29	1101	3.39	1086	2.37
	3	663	21.17	643	10.26	641	5.30	469	2.12	606	1.91
	4	529	22.29	479	10.22	533	5.81	540	3.18	483	1.94
1.3	1	947	11.02	1029	5.93	1247	3.71	1406	2.39	1464	1.89
	2	1840	39.79	2177	23.80	1818	10.20	2074	6.36	2283	4.94
	3	757	24.13	886	14.30	918	7.58	916	4.10	840	2.60
	4	1077	45.38	1165	24.80	1038	11.28	1163	6.82	1307	5.02

parallel preconditioner of Krylov subspace method is quite successful. Notice that the optimal value of ω reported in this paper is not the exact one, but the best one out of numerical experiments for 7 different values of ω. If we use the

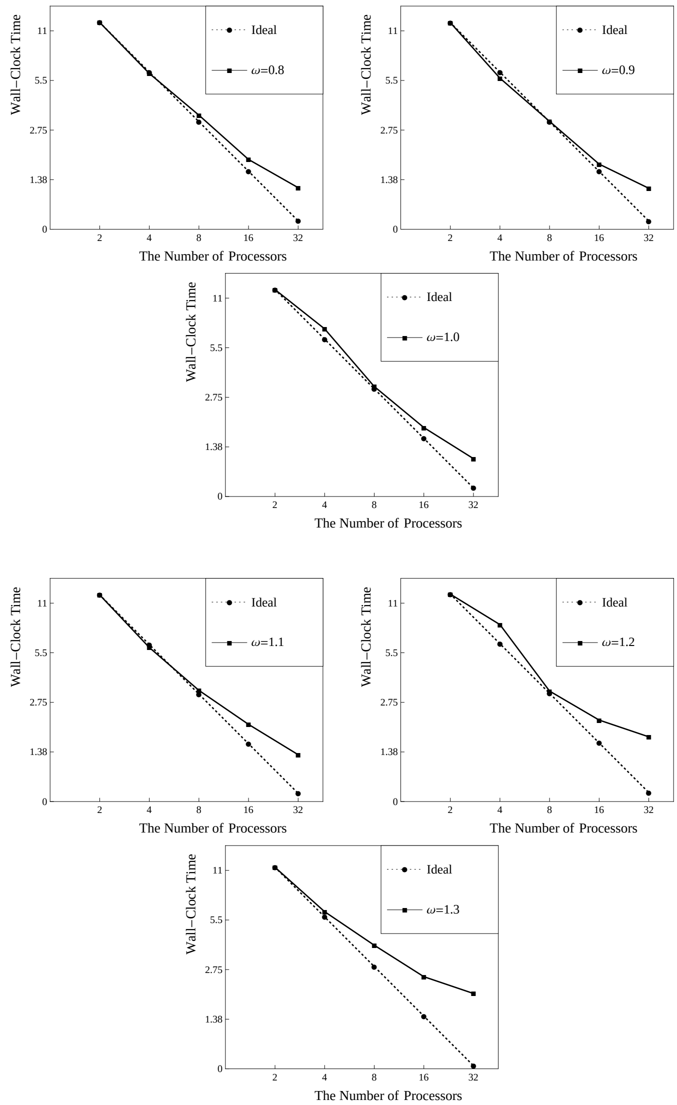

FIGURE 4. Scaling behaviors of BiCGSTAB using preconditioner P_2 for Example 4.2 with $n = 263169$

exact optimal value of ω, then performance results will be better than those reported in this paper.

Even though the multisplitting method with preweighting has a lot of parallism and its parallel efficiency is quite good, its performance is too slow as compared with BiCGSTAB with the parallel preconditioner P_s (see Tables 1 to 6). Therefore, the multisplitting method with preweighting itself is not recommended for use, but it is recommended for use as a parallel preconditioner of Krylov subspace method in order to solve large sparse linear systems.

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