GENERAL LAWS OF PRECISE ASYMPTOTICS FOR SUMS OF RANDOM VARIABLES

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ABSTRACT. In this paper, we obtain two general laws of precise asymptotics for sums of i.i.d random variables, which contain general weighted functions and boundary functions and also clearly show the relationship between the weighted functions and the boundary functions. As corollaries, we obtain Theorem 2 of Gut and Spătaru [A. Gut and A. Spătaru, *Precise asymptotics in the law of the iterated logarithm*, Ann. Probab. **28** (2000), no. 4, 1870–1883] and Theorem 3 of Gut and Spătaru [A. Gut and A. Spătaru, *Precise asymptotics in the Baum-Katz and Davids laws of large numbers*, J. Math. Anal. Appl. **248** (2000), 233–246].

1. Introduction and main results

Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables and $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$. Let $\varphi(x)$ and $f(x)$ be positive functions defined on $[n_0, \infty)$, $n_0 \in Z^+$, and $\sum_{n=n_0}^{\infty} \varphi(n) = \infty$, $f(x) \uparrow \infty$, $x \to \infty$. Set

$$P(\varphi, f, \epsilon) = \sum_{n=n_0}^{\infty} \varphi(n) P\{|S_n| \geq \epsilon f(n)\},$$

$\varphi(x)$ and $f(x)$ are called weighted function and boundary function, respectively.

Since Hsu and Robbins [10] introduced the concept of complete convergence, there has been research in two directions. One is to discuss the moment conditions, from which it follows that $P(\varphi, f, \epsilon) < \infty$, $\epsilon > 0$. For results of such aspect, one can refer to Hsu and Robbins [10], Erdös [4, 5] and Baum and Katz [1], etc. They, respectively, studied the cases in which $\varphi(n) = 1$, $f(n) = n$ and $\varphi(n) = n^{r/p-2}$, $f(n) = n^{1/p}$, where $0 < p < 2$, $r \geq p$.

Another aspect of research concerns about the convergence rate and limit value of $P(\varphi, f, \epsilon)$ as $\epsilon \searrow a$, $a \geq 0$. The first result in this direction was Heyde

[9], which proved that

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} P\{|S_n| \geq \epsilon n\} = EX^2,$$

whenever $EX = 0$ and $EX^2 < \infty$. For more analogous results, please refer to [7], [8], [11], [12], [14], etc, which are all on specific weighted functions and boundary functions. Such research of this field is called the precise asymptotics. In this field, people are prone to concern about more general weighted functions and boundary functions, and study the relationship between them. Some scholars have done research in this area, please refer to [3], [16], [17], etc. In this paper, we also obtain two general laws of precise asymptotics for sums of i.i.d random variables, which contain general weighted functions and boundary functions and also clearly show the relationship between the weighted functions and the boundary functions.

Below, let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function F, mean 0 and positive, finite variance σ^2. And let N be the standard normal random variable. C denotes positive constant, possibly varying from place to place, and $[x]$ denotes the largest integer $\leq x$. The notion $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

Theorem 1.1. *Suppose $h(x)$ is a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞ and $n_0 \in Z^+$. Assume that the following conditions are satisfied*:

(a) *there exists $n_1 \geq n_0$ such that $h^\delta(x)h'(x)$ is monotone nonincreasing on $[n_1, \infty)$, where $0 \leq \delta < 1$;*
(b) $\lim_{x \to \infty} \frac{h(x-1)}{h(x)} = 1$;
(c) $xh'(x)$ *is monotone nonincreasing.*

Then

$$(1.1) \quad \lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq n_0} h^\delta(n)h'(n)P\{|S_n| \geq \epsilon\sqrt{nh(n)}\} = \frac{1}{\delta+1}E|N|^{2\delta+2}\sigma^{2\delta+2}.$$

Setting $h(x) = \log \log x$, we can get the following result.

Corollary 1.1.
(1.2)
$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq 3} \frac{(\log \log n)^\delta}{n \log n}P\{|S_n| \geq \epsilon\sqrt{n \log \log n}\} = \frac{1}{\delta+1}E|N|^{2\delta+2}\sigma^{2\delta+2}.$$

Remark 1.1. Letting $\delta = 0$, (1.2) is Theorem 2 of Gut and Spătaru [7].

If we set $h(x) = \log x$, we can get Theorem 3 of Gut and Spătaru [8] as follow.

Corollary 1.2.

$$(1.3) \quad \lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq 3} \frac{(\log n)^\delta}{n} P\{|S_n| \geq \epsilon\sqrt{n \log n}\} = \frac{1}{\delta+1} E|N|^{2\delta+2} \sigma^{2\delta+2}.$$

Theorem 1.2. *Under the conditions of Theorem 1.1, setting*

$$Q_n = \max_{1 \leq k \leq n} |S_k|,$$

we have

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq n_0} h^\delta(n) h'(n) P\{Q_n \geq \epsilon\sqrt{nh(n)}\}$$

$$(1.4)$$

$$= \frac{2}{\delta+1} E|N|^{2\delta+2} \sigma^{2\delta+2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2\delta+2}}.$$

2. Some lemmas

To prove the main results, the following lemmas are needed.

Lemma 2.1 ([15], Stout, p. 120). *Let $\{a_{ni}\}$ be a matrix of real numbers and $\{x_i\}$ be a sequence of real numbers satisfying $x_i \to 0$ as $i \to \infty$. Then*

$$\sum_{i=1}^{\infty} |a_{ni}| \leq M < \infty \text{ for all } n \geq 1$$

and

$$a_{ni} \to 0 \text{ as } n \to \infty \text{ for each } i \geq 1$$

imply that

$$\sum_{i=1}^{\infty} a_{ni} x_i \to 0 \text{ as } n \to \infty.$$

The following lemma we need is [14, Lemma 2] which is based on an inequality by Fuk and Nagaev [6].

Lemma 2.2. *Assume that $E|X|^\beta < \infty$, where $1 < \beta \leq 2$. For $x, y > 0$, we have*

$$P(|S_n| \geq x) \leq nP(|X| \geq y) + 2e^{x/y} \Big(\frac{nE|X|^\beta}{nE|X|^\beta + xy^{\beta-1}} \Big)^{x/y}$$

$$\leq nP(|X| \geq y) + 2n^{x/y} \Big(\frac{eE|X|^\beta}{xy^{\beta-1}} \Big)^{x/y}.$$

Lemma 2.3 ([2], Billingsley, pp. 79–80). *Let $\{W(t); t \geq 0\}$ be a standard Wiener process. Then for all $x > 0$,*

$$P\Big\{ \sup_{0 \leq s \leq 1} |W(s)| \geq x \Big\} = 1 - \sum_{k=-\infty}^{\infty} (-1)^k P\{(2k-1)x \leq N \leq (2k+1)x\}$$

$$= 4 \sum_{k=0}^{\infty} (-1)^k P\{N \geq (2k+1)x\}$$

$$= 2 \sum_{k=0}^{\infty} (-1)^k P\{|N| \geq (2k+1)x\}.$$

In particular,

$$P\left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq x \right\} \sim 2P(|N| \geq x) \sim \frac{4}{\sqrt{2\pi}x} e^{-x^2/2} \quad as \quad x \to \infty.$$

It is easy to get the following result from Theorem 3 of Shao [13].

Lemma 2.4. *For any $x > 0$, $y > 0$, we have*

$$P\left\{ \max_{1 \leq k \leq n} |S_k| \geq x \right\}$$

$$\leq 2nP\{|X| \geq y\} + 4\exp\left\{ -\frac{x^2}{8n\sigma^2} \right\} + 4\left(\frac{n\sigma^2}{4(xy + n\sigma^2)} \right)^{x/(12y)}.$$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 will be carried out by two steps. First, F itself will be assumed to be normal, after which the general case is treated.

a. *F is normal.* Obviously, it is sufficient to prove the conclusion for the case $\sigma^2 = 1$. We thus assume that F is the standard normal distribution function Φ, and set $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$, $x \geq 0$.

Proposition 3.1.

$$(3.1) \qquad \lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq n_0} h^\delta(n)h'(n)P\{|S_n| \geq \epsilon\sqrt{nh(n)}\} = \frac{1}{\delta+1} E|N|^{2\delta+2}.$$

Proof. Note that

$$\sum_{n \geq n_1} h^\delta(n)h'(n)P\{|S_n| \geq \epsilon\sqrt{nh(n)}\} = \sum_{n \geq n_1} h^\delta(n)h'(n)P\{|N| \geq \epsilon\sqrt{h(n)}\}$$

$$(3.2) \qquad\qquad\qquad\qquad = \sum_{n \geq n_1} h^\delta(n)h'(n)\Psi(\epsilon\sqrt{h(n)}).$$

Since $h^\delta(x)h'(x)$ is monotone nonincreasing on $[n_1, \infty)$, then

$$h^\delta(x)h'(x)\Psi(\epsilon\sqrt{h(x)})$$

is monotone nonincreasing on $[n_1, \infty)$. Hence we can get

$$\int_{n_1}^{\infty} h^\delta(x)h'(x)\Psi(\epsilon\sqrt{h(x)})dx \leq \sum_{n \geq n_1} h^\delta(n)h'(n)\Psi(\epsilon\sqrt{h(n)})$$

$$(3.3) \qquad\qquad\qquad\qquad \leq \int_{n_1-1}^{\infty} h^\delta(x)h'(x)\Psi(\epsilon\sqrt{h(x)})dx.$$

Note that for any positive constant C,

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \int_C^\infty h^\delta(x)h'(x)\Psi(\epsilon\sqrt{h(x)})dx = \lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \int_{h(C)}^\infty y^\delta \Psi(\epsilon y^{1/2})dy$$

$$= \lim_{\epsilon \searrow 0} 2 \int_{\epsilon\sqrt{h(C)}}^\infty u^{2\delta+1}\Psi(u)du$$

(3.4)
$$= \frac{1}{\delta+1}E|N|^{2\delta+2}.$$

Noting that

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq n_0} h^\delta(n)h'(n)P\{|S_n| \geq \epsilon\sqrt{nh(n)}\}$$

$$= \lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq n_1} h^\delta(n)h'(n)P\{|S_n| \geq \epsilon\sqrt{nh(n)}\},$$

and by (3.2)–(3.4), we can get (3.1). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

b. *The general case.* Now we assume that $X, X_1, X_2, \ldots$ are i.i.d. random variables with mean 0 and variance 1. Put $d(\epsilon) = h^{-1}(M/\epsilon^2)$, where $M > 2$.

Proposition 3.2. *We have*

(3.5) $\quad \lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \leq d(\epsilon)} h^\delta(n)h'(n)\left|P\{|S_n| \geq \epsilon\sqrt{nh(n)}\} - \Psi(\epsilon\sqrt{h(n)})\right| = 0.$

Proof. Set $\Delta_n = \sup_x |P\{|S_n| \geq \sqrt{n}x\} - \Psi(x)|$. Noting $\Delta_n \to 0$ as $n \to \infty$ and by Lemma 2.1, we can get

(3.6) $\qquad\qquad \dfrac{1}{h^{\delta+1}(m)} \sum_{n=1}^m h^\delta(n)h'(n)\Delta_n \to 0 \quad \text{as} \quad m \to \infty.$

So

$$\epsilon^{2\delta+2} \sum_{n \leq d(\epsilon)} h^\delta(n)h'(n)\left|P\{|S_n| \geq \epsilon\sqrt{nh(n)}\} - \Psi(\epsilon\sqrt{h(n)})\right|$$

$$\leq \epsilon^{2\delta+2} \sum_{n \leq [d(\epsilon)]} h^\delta(n)h'(n)\Delta_n$$

$$= \epsilon^{2\delta+2}h^{\delta+1}([d(\epsilon)]) \frac{1}{h^{\delta+1}([d(\epsilon)])} \sum_{n \leq [d(\epsilon)]} h^\delta(n)h'(n)\Delta_n$$

$$\leq \epsilon^{2\delta+2}(M/\epsilon^2)^{\delta+1} \frac{1}{h^{\delta+1}([d(\epsilon)])} \sum_{n \leq [d(\epsilon)]} h^\delta(n)h'(n)\Delta_n$$

(3.7)
$$= M^{\delta+1} \frac{1}{h^{\delta+1}([d(\epsilon)])} \sum_{n \leq [d(\epsilon)]} h^\delta(n)h'(n)\Delta_n \to 0 \quad \text{as} \quad \epsilon \searrow 0.$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$$

Proposition 3.3. *We have, uniformly with respect to all sufficiently small* $\epsilon > 0$,

$$(3.8) \qquad \lim_{M \to \infty} \epsilon^{2\delta+2} \sum_{n > d(\epsilon)} h^{\delta}(n) h'(n) \Psi(\epsilon\sqrt{h(n)}) = 0.$$

Proof. Note that for sufficiently small $\epsilon > 0$,

$$\epsilon^{2\delta+2} \sum_{n > d(\epsilon)} h^{\delta}(n) h'(n) \Psi(\epsilon\sqrt{h(n)}) \le \epsilon^{2\delta+2} \int_{[d(\epsilon)]}^{\infty} h^{\delta}(x) h'(x) \Psi(\epsilon\sqrt{h(x)}) dx$$

$$(3.9) \qquad\qquad = 2 \int_{\sqrt{\epsilon^2 h([d(\epsilon)])}}^{\infty} y^{2\delta+1} \Psi(y) dy.$$

Since $h(d(\epsilon) - 1) \le h([d(\epsilon)]) \le h(d(\epsilon)) = M/\epsilon^2$ and $\lim_{x \to \infty} \frac{h(x-1)}{h(x)} = 1$, we have

$$\lim_{\epsilon \searrow 0} \epsilon^2 h([d(\epsilon)]) = M.$$

So, for sufficiently small $\epsilon > 0$,

$$(3.10) \qquad\qquad \epsilon^2 h([d(\epsilon)]) \ge M/2.$$

Thus,

$$(3.9) \le 2 \int_{\sqrt{M/2}}^{\infty} y^{2\delta+1} \Psi(y) dy \to 0 \quad \text{as} \quad M \to \infty. \qquad\qquad \square$$

Proposition 3.4. *We have, uniformly with respect to all sufficiently small* $\epsilon > 0$,

$$(3.11) \qquad \lim_{M \to \infty} \epsilon^{2\delta+2} \sum_{n > d(\epsilon)} h^{\delta}(n) h'(n) P\{|S_n| \ge \epsilon\sqrt{nh(n)}\} = 0.$$

Proof. Lemma 2.2 with $x = \epsilon\sqrt{nh(n)}$, $y = \epsilon\sqrt{nh(n)}/3$ and $\beta = 2$ yields

$$\sum_{n > d(\epsilon)} h^{\delta}(n) h'(n) P\{|S_n| \ge \epsilon\sqrt{nh(n)}\}$$

$$\le \sum_{n > d(\epsilon)} h^{\delta}(n) h'(n) n P\{|X| \ge \epsilon\sqrt{nh(n)}/3\}$$

$$(3.12) \qquad\qquad + \sum_{n > d(\epsilon)} h^{\delta}(n) h'(n) 2n^3 \left(\frac{eE|X|^2}{\epsilon^2 nh(n)/3} \right)^3$$

$$=: I_1 + I_2.$$

By using (3.10), for sufficiently small $\epsilon > 0$, we have

$$\epsilon^{2\delta+2} I_2 \le C\epsilon^{2\delta-4} \sum_{n > d(\epsilon)} h^{\delta-3}(n) h'(n)$$

$$\le C\epsilon^{2\delta-4} \int_{[d(\epsilon)]}^{\infty} h^{\delta-3}(x) h'(x) dx$$

$$= C\epsilon^{2\delta-4} \int_{h([d(\epsilon)])}^{\infty} y^{\delta-3} dy$$

$$= C\epsilon^{2\delta-4} \frac{1}{2-\delta} \left(h([d(\epsilon)]) \right)^{\delta-2}$$

$$= \frac{C}{2-\delta} \left(\epsilon^2 h([d(\epsilon)]) \right)^{\delta-2}$$

$$(3.13) \qquad \leq \frac{C}{2-\delta} M^{\delta-2} \to 0 \quad \text{as} \quad M \to \infty.$$

By condition (c) of Theorem 1.1 and noting that $xh'(x) \geq 0$, it is easy to obtain that for sufficiently large $x > 0$, there exists a positive constant l such that

$$xh'(x) \leq l.$$

So, for sufficiently small $\epsilon > 0$ and any positive constant α,

$$\sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)nP\{|X| \geq \alpha\epsilon\sqrt{nh(n)}\}$$

$$\leq l \sum_{n>d(\epsilon)} h^{\delta}(n)P\{|X| \geq \alpha\epsilon\sqrt{nh(n)}\}$$

$$= l \sum_{n>d(\epsilon)} h^{\delta}(n) \sum_{k\geq n} P\{\alpha\epsilon\sqrt{kh(k)} \leq |X| < \alpha\epsilon\sqrt{(k+1)h(k+1)}\}$$

$$= l \sum_{k>d(\epsilon)} P\{kh(k) \leq \alpha^{-2}\epsilon^{-2}X^2 < (k+1)h(k+1)\} \sum_{d(\epsilon)<n\leq k} h^{\delta}(n)$$

$$(3.14) \qquad \leq l \sum_{k>d(\epsilon)} kh^{\delta}(k)P\{kh(k) \leq \alpha^{-2}\epsilon^{-2}X^2 < (k+1)h(k+1)\}.$$

Since $k > d(\epsilon) = h^{-1}(M/\epsilon^2)$, we have $h(k) > M/\epsilon^2$, and consequently

$$(3.15) \qquad\qquad (h(k))^{\delta-1} \leq (M/\epsilon^2)^{\delta-1}.$$

Thus

$$(3.14) \leq l(M/\epsilon^2)^{\delta-1} \sum_{k>d(\epsilon)} kh(k)P\{kh(k) \leq \alpha^{-2}\epsilon^{-2}X^2 < (k+1)h(k+1)\}$$

$$\leq l(M/\epsilon^2)^{\delta-1}\alpha^{-2}\epsilon^{-2}EX^2$$

$$(3.16) \quad = l\alpha^{-2}\epsilon^{-2\delta}M^{\delta-1}EX^2.$$

So, for sufficiently small $\epsilon > 0$,

$$(3.17) \qquad\qquad \epsilon^{2\delta+2}I_1 \leq 9l\epsilon^2 M^{\delta-1}EX^2 \to 0 \quad \text{as} \quad M \to \infty.$$

From (3.12), (3.13) and (3.17), we can get (3.11). $\qquad\square$

Theorem 1.1 now follows from the propositions and the triangle inequality.

4. Proof of Theorem 1.2

Without loss of generality, we assume that $\sigma^2 = 1$. Let $d(\epsilon)$ be as in Section 3. We prove Theorem 1.2 by the following propositions.

Proposition 4.1.

$$
\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq n_0} h^\delta(n) h'(n) P\Big\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon \sqrt{h(n)} \Big\}
$$

(4.1)

$$
= \frac{2}{\delta+1} E|N|^{2\delta+2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2\delta+2}}.
$$

Proof. By Lemma 2.3, it follows that for any $m \geq 1$ and $x > 0$,

$$
2 \sum_{k=0}^{2m+1} (-1)^k P\{|N| \geq (2k+1)x\}
$$

$$
\leq P\Big\{ \sup_{0 \leq s \leq 1} |W(s)| \geq x \Big\} \leq 2 \sum_{k=0}^{2m} (-1)^k P\{|N| \geq (2k+1)x\}.
$$

Hence, it is sufficient to prove that for any $q > 0$,

(4.2) $$ \lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \geq n_0} h^\delta(n) h'(n) P\{|N| \geq q\epsilon\sqrt{h(n)}\} = q^{-(2\delta+2)} \frac{E|N|^{2\delta+2}}{\delta+1}. $$

Similarly to the proof of (3.1), we can get (4.2). Thus the proposition is now proved. $\square$

Proposition 4.2. *We have*

(4.3)

$$
\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n \leq d(\epsilon)} h^\delta(n) h'(n) \Big| P\{Q_n \geq \epsilon\sqrt{nh(n)}\} - P\Big\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon\sqrt{h(n)} \Big\} \Big| = 0.
$$

Proof. Set $\Delta'_n = \sup_x |P\{Q_n \geq \sqrt{n}x\} - P\{\sup_{0 \leq s \leq 1} |W(s)| \geq x\}|$. Note that $\Delta'_n \to 0$ as $n \to \infty$, the rest proof is the same as that of Proposition 3.2, and we omit the details. $\square$

Proposition 4.3. *We have, uniformly with respect to all sufficiently small $\epsilon > 0$,*

(4.4) $$ \lim_{M \to \infty} \epsilon^{2\delta+2} \sum_{n > d(\epsilon)} h^\delta(n) h'(n) P\Big\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon\sqrt{h(n)} \Big\} = 0. $$

Proof. By Lemma 2.3, it is easy to obtain that for sufficiently large $x > 0$,

$$
P\{ \sup_{0 \leq s \leq 1} |W(s)| \geq x \} \leq CP\{|N| \geq x\}.
$$

So, for sufficiently small $\epsilon > 0$,

$$
\epsilon^{2\delta+2} \sum_{n > d(\epsilon)} h^\delta(n) h'(n) P\Big\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon\sqrt{h(n)} \Big\}
$$

$$\leq C\epsilon^{2\delta+2} \sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)P\{|N| \geq \epsilon\sqrt{h(n)}\}.$$

The rest proof is the same as that of Proposition 3.3. $\qquad\square$

Proposition 4.4. *We have, uniformly with respect to all sufficiently small* $\epsilon > 0$,

$$(4.5) \qquad \lim_{M\to\infty} \epsilon^{2\delta+2} \sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)P\{Q_n \geq \epsilon\sqrt{nh(n)}\} = 0.$$

Proof. By using Lemma 2.4 with $x = \epsilon\sqrt{nh(n)}$, $y = \frac{x}{12(\delta+2)}$, we have

$$\sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)P\{Q_n \geq \epsilon\sqrt{nh(n)}\}$$

$$\leq 2 \sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)nP\left\{|X| \geq \frac{\epsilon\sqrt{nh(n)}}{12(\delta+2)}\right\}$$

$$+ 4 \sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)\exp\left\{-\frac{\epsilon^2 h(n)}{8\sigma^2}\right\}$$

$$+ 4 \sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)\left(\frac{\sigma^2}{\frac{\epsilon^2 h(n)}{3(\delta+2)} + 4\sigma^2}\right)^{\delta+2}$$

$$=: I_3 + I_4 + I_5.$$

From (3.14)–(3.16), we can obtain that for sufficiently small $\epsilon > 0$,

$$(4.6) \qquad \epsilon^{2\delta+2}I_3 \to 0 \quad \text{as} \quad M \to \infty.$$

By using (3.10), for sufficiently small $\epsilon > 0$, we have

$$\epsilon^{2\delta+2}I_4 \leq 4\epsilon^{2\delta+2} \int_{[d(\epsilon)]}^{\infty} h^{\delta}(x)h'(x)\exp\left\{-\frac{\epsilon^2 h(x)}{8\sigma^2}\right\}dx$$

$$= 4\int_{\epsilon^2 h([d(\epsilon)])}^{\infty} u^{\delta}\exp\left\{-\frac{u}{8\sigma^2}\right\}du$$

$$(4.7) \qquad \leq 4\int_{M/2}^{\infty} u^{\delta}\exp\left\{-\frac{u}{8\sigma^2}\right\}du \to 0 \quad \text{as} \quad M \to \infty,$$

and

$$\epsilon^{2\delta+2}I_5 \leq C\epsilon^{2\delta+2} \sum_{n>d(\epsilon)} h^{\delta}(n)h'(n)\left(\frac{\sigma^2}{\epsilon^2 h(n)}\right)^{\delta+2}$$

$$\leq C\epsilon^{-2} \sum_{n>d(\epsilon)} h^{-2}(n)h'(n)$$

$$\leq C\epsilon^{-2} \int_{[d(\epsilon)]}^{\infty} h^{-2}(x)h'(x)dx$$

$$= C\epsilon^{-2} \int_{h([d(\epsilon)])}^{\infty} y^{-2} dy$$

$$= C\{\epsilon^2 h([d(\epsilon)])\}^{-1}$$

(4.8) $$\leq \frac{2C}{M} \to 0 \quad \text{as} \quad M \to \infty.$$

Thus, by (4.6)–(4.9), we get (4.5). $\qquad\square$

Theorem 1.2 now follows from the propositions and the triangle inequality.

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